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Information entropy and squeezing of quantum fluctuations

Arkadiusz Orłowski
Instytut Fizyki, Polska Akademia Nauk, Aleja Lotników 32/46, 02-668 Warszawa, Poland
and Department of Physics, Harvard University, Cambridge, Massachusetts 02138
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Quantum-mechanical entropies of position and momentum operators in a given state are shown to be reasonable and sensitive measures for squeezing of quantum fluctuations. It is shown to be true not only for states having Gaussian wave functions but also for more general, both pure and mixed, quantum states. A simple proof that the squeezing exhibited by the variance is always accompanied by a corresponding entropy reduction below the entropy vacuum level is given. These results show that the information entropy is not only a theoretically satisfactory concept but can also be useful as a tool for more practical quantum-optics applications. [S1050-2947(97)02609-7]

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In quantum mechanics two noncommuting observables cannot be simultaneously measured with arbitrary precision. This fact, often called the Heisenberg *uncertainty principle*, is a fundamental restriction that is related neither to imperfections of the existing real-life measuring devices nor to the experimental errors of observation [1]. It is rather the intrinsic property of the quantum state itself. Paradoxically enough, the uncertainty principle provides the only way to avoid many interpretational problems. It can also be used to make qualitative predictions in atomic physics, e.g., the size of the ground-state energy of an atom and the spread of the ground-state wave function [2]. The uncertainty principle specified for given pairs of observables finds its mathematical manifestation as the *uncertainty relations*. The first rigorous derivation of the uncertainty relation from the quantum-mechanical formalism applied for the basic noncommuting observables, i.e., for the position and momentum ($[\hat{x},\hat{p}]=i;\hbar=1$), is due to Kennard [3] (see also the work of Robertson [4]). This derivation, repeated in most textbooks on quantum mechanics ever since, leads to the celebrated inequality

$$\Delta\hat{x}\Delta\hat{p}\geqslant\frac{1}{2}. \tag{1}$$

In fact, it can be considered as a simple consequence of the properties of the Fourier transform that connects the wave functions of the system in the position and momentum representation.

In the above expression the fundamental quantum uncertainty inherently tied to the pair of noncommuting observables is measured by the variance of the corresponding Hermitian operators. For example, for $\hat{x}=\hat{x}^\dagger$,

$$(\Delta\hat{x})^2=\langle(\hat{x}-\langle\hat{x}\rangle)^2\rangle, \tag{2}$$

where $\langle\ \rangle$ denotes the averaging with respect to a given state. It should be noted, however, that the variance is not the only measure of quantum uncertainty that can be used to express the uncertainty principle. Being just the second central moment of the probability distribution, it gives only a rough characterization of the probability distribution that is not of the Gaussian shape. It is, of course, possible to introduce higher moments [5], but all of them considered separately still contain only a restricted amount of information about the spreading of the values around the mean value. It is now commonly recognized that in many cases the variances (or standard deviations) are not appropriate measures of the quantum uncertainty. There exist many physically interesting situations where using variances leads to inadequate descriptions. A much more satisfactory measure of quantum uncertainty is given by the information entropy of the given probability distribution. The advantages of the entropic approach have been thoroughly scrutinized [6].

In quantum mechanics the probability distribution of position (for a pure state) is given by the squared modulus of the wave function $P(x)=|\psi(x)|^2$. The probability distribution of momentum is given by a similar expression $P(p)=|\tilde{\psi}(p)|^2$, where the wave function in the momentum representation $\tilde{\psi}(p)$ is known to be the Fourier transform of the wave function $\psi(x)$. Following Shannon's ideas, we can define the entropies of position $S_x=-\int P(x)\ln P(x)dx$ and momentum $S_p=-\int P(p)\ln P(p)dp$, respectively. There exists a very deep and interesting inequality satisfied by the sum of the above-mentioned position and momentum entropies

$$S_x+S_p=-\int|\psi(x)|^2\ln|\psi(x)|^2dx-\int|\tilde{\psi}(p)|^2\ln|\tilde{\psi}(p)|^2dp \geqslant 1+\ln\pi. \tag{3}$$

A conjecture that the wave function and its Fourier transform should satisfy this relation was made almost 40 years ago by Everett in his work devoted to many-worlds interpretation of quantum mechanics [7] and independently by Hirschman [8] for some mathematical reasons. The proof, however, is very difficult and was presented for the first time by Beckner [9] and by Białynicki-Birula and Mycielski [10]. This inequality is often called the entropic uncertainty relation (EUR). There also exist various generalizations (e.g., to phase-space based EURs [11]). It is interesting to note that Eq. (3) is more general than the standard formulation of the Heisenberg uncertainty principle by Kennard. In fact, the latter can be easily *derived* from the EUR (but not vice versa). All the above discussion provides good *theoretical* (and aesthetic) reasons to prefer entropy rather than variance as a measure of the quantum uncertainty. The point, however, is that variances (being to some extent measurable quantities) are used to define some very important purely quantum-mechanical effects such as squeezing of quantum fluctuations and sub-Poissonian statistics. Thus there is a problem connected with the physical applicability of the information entropy and its usefulness in solving practical problems. One of the most important problems is, of course, the ability to predict the above-mentioned quantum-noise reduction.

To see this point more clearly, let me consider the one-dimensional harmonic oscillator modeling a single mode of the quantized electromagnetic field. It is still possible to keep my previous notation and use the terms ''position'' and ''momentum,'' simply remembering that now the $\hat{x}$ and $\hat{p}$ operators should be interpreted as the quadrature operators of the radiation field. There is a class of states, called coherent states, leading to the equality in the uncertainty relations. They are considered to be as close as possible to the classical states. For these states quantum uncertainties $\Delta\hat{x}$ and $\Delta\hat{p}$ are equal to each other and equal to the uncertainties of the vacuum state (the ground state) of the harmonic oscillator $\Delta\hat{x}=\Delta\hat{p}=1/\sqrt{2}$. A state is called squeezed state if the quantum noise of one of the quadrature components of the annihilation operator $[\hat{a}=(\hat{x}+i\hat{p})/\sqrt{2}]$ is less than the noise corresponding to the vacuum state

$$\Delta\hat{x}<\frac{1}{\sqrt{2}}$$

or

$$\Delta\hat{p}<\frac{1}{\sqrt{2}}. \tag{4}$$

If, in addition, we still have equality in the relation (1), such states are called the minimum uncertainty squeezed states. Of course, because of the Heisenberg uncertainty relation (1), it is impossible to have both quadratures squeezed simultaneously. Squeezing of quantum fluctuations as well as generation, detection, and properties of squeezed states of light still draws an increasing amount of attention [12]. It is stimulated by many nontrivial applications in the detection of gravitational radiation (where the signal is comparable to the quantum noise [13]) and in the modern low-noise optical communication systems [14]. Therefore, if we want to introduce the information entropy not only as a mathematical tool that allows more convenient interpretations but as a satisfactory equivalent of variances as measures of uncertainty, we should be able to prove that the entropy can be useful in the description of squeezing. It is the main objective of this paper to show not only that EUR is *stronger* than the standard uncertainty relation but also that the entropy of the single observable can be an as equally good and sensitive measure of squeezing of quantum fluctuations as the variance. First I will show that it *must* be the case for the states possessing the Gaussian probability distributions. Then I will give a nontrivial example showing that the usefulness of the entropy as a measure of squeezing is not restricted to the Gaussian case. At the end I will *prove* that the squeezing exhibited by one of the variances is *always* accompanied by the corresponding entropy reduction below the vacuum level.

Let me start from a simple generalization of the above-presented EUR. In the above form (3) it can be used only for systems in pure states. If the density matrix $\hat{\rho}$ denotes an arbitrary state, not necessarily pure, the more general EUR is satisfied:

$$-\int\langle x|\hat{\rho}|x\rangle\ln\langle x|\hat{\rho}|x\rangle dx-\int\langle p|\hat{\rho}|p\rangle\ln\langle p|\hat{\rho}|p\rangle dp \geqslant 1+\ln\pi. \tag{5}$$

The most general Gaussian probability distribution for the position, being the marginal distribution of the corresponding Wigner function, can be written as [15]

$$\langle x|\hat{\rho}|x\rangle=\frac{1}{\sqrt{2\pi(\tau-\mu-\mu^*)}}\exp\left(-\frac{(x-x_0)^2}{2(\tau-\mu-\mu^*)}\right). \tag{6}$$

Parameters μ, ν, and τ are functions of the second moments of the creation and annihilation (or position and momentum) operators. For example, squeezing in the $\hat{x}$ quadrature is possible only if $(\Delta\hat{x})^2=\tau-\mu-\mu^*<1/2$. Depending on the values of these parameters, we recognize the distribution corresponding to some well-known states. For example, for $\mu=0,\tau=\frac{1}{2}$, we have coherent states; for $\tau^2-4|\mu|^2=\frac{1}{4}$, we have minimum uncertainty squeezed states; for $\mu=x_0=0$, we have a thermal state. The entropies of position and momentum corresponding to the coherent states (and to the vacuum state) are equal to each other: $S_x=S_p=\frac{1}{2}(1+\ln\pi)\approx1.072\,36$. Now let me consider the position entropy of the above general Gaussian state. It is clear that because the Gaussian state is fully determined by the mean value and the variance, the entropy also will depend only on these parameters. Indeed, the information entropy of position for the above most general Gaussian distribution is given by

$$S_x=\frac{1}{2}(1+\ln\pi)+\frac{1}{2}\ln[2(\tau-\mu-\mu^*)]. \tag{7}$$

Thus it is clear that the behavior of entropy is equivalent to the behavior of variance in the case of Gaussian states. Therefore, there is no entropic squeezing for the thermal states.

What about more general quantum states? First consider the photon-number (Fock) states that exhibit no quadrature squeezing. That there is also no entropic squeezing can be

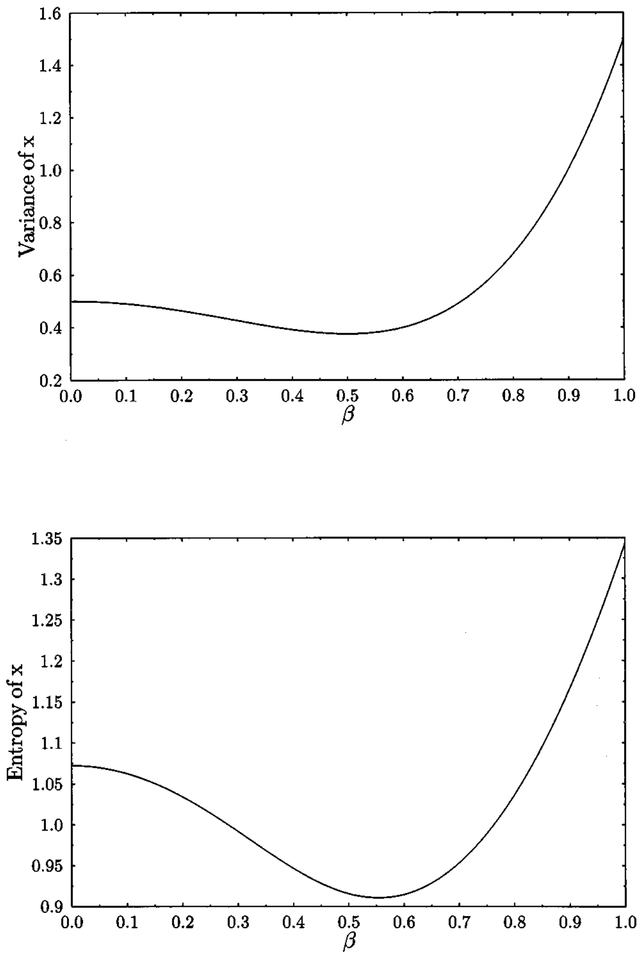

FIG. 1. Variance $(\Delta\hat{x})^2$ (top) and entropy S_x (bottom) of the quadrature $\hat{x}$ as functions of the parameter β.

seen from a very simple argument. The Wigner function of Fock's states is symmetrical with respect to both variables and the marginal distributions have the same form. As a consequence, the corresponding entropies are equal to each other. Therefore, because of the EUR, both must be greater than the vacuum level. This argument is valid for any state that has the symmetrical Wigner function: there is no entropic squeezing in such a case.

The usefulness of the entropy as a measure of squeezing is not restricted to any special class of states. To see this let us consider a simple nontrivial example of the arbitrary two-element superposition of the vacuum and the one-photon state

$$|\psi\rangle=\alpha|0\rangle+\beta|1\rangle. \tag{8}$$

I assume for simplicity that α and β are real; thus the normalization condition is simply $\alpha^2+\beta^2=1$. It is known that this state exhibits squeezing of the $\hat{x}$ quadrature for some values of β (because of the normalization, the variance of $\hat{x}$ can be considered as a function of β only) [16]. The quadrature $\hat{x}$ is squeezed for $\beta<1/\sqrt{2}$ and the maximum quantum noise reduction $[(\Delta\hat{x})^2=3/8]$ is obtained for $\beta=1/2$. The behavior of $(\Delta\hat{x})^2$ as a function of β is presented in Fig. 1 (top). In the same figure (bottom) we see the behavior of the entropy S_x. Although these two parameters follow each other quite closely, we can see that the entropy is more sensitive. For $\beta=0.75$, the variance is already greater than $\frac{1}{2}$, but entropy still exhibits a small amount of squeezing: $S_x\approx 0.989\,164$. This greater sensitivity of entropy can also be seen in the case of the Schrödinger cat state, i.e., for the superposition of two coherent states (often called the even coherent state). Very similar results can also be obtained for other nontrivial superpositions, including various finite superpositions of photon-number states [17] as well as investigating some dynamical processes [18].

Now let me prove generally that if the variance is less than the vacuum level the corresponding entropy is also less than the entropy vacuum level for an *arbitrary* (pure or mixed) quantum state $\hat{\rho}$. Normalizable quantum states with a finite variance $(\Delta\hat{x})^2$ are considered. The latter assumption implies that for such a state also the first moment $\langle x\rangle$ is finite. The above conditions can formally be written as the following constraints: $\int P(x)dx=1$ and $\int(x-\langle x\rangle)^2P(x)dx=(\Delta\hat{x})^2$, where $P(x):=\langle x|\hat{\rho}|x\rangle$. Now the task is to find such a state $\hat{\rho}_{\max}$ [or, more precisely, its probability distribution $P_{\max}(x)$] that the corresponding entropy S_x takes on the maximum value possible under these constraints. This problem is equivalent to finding the extremum of the functional

$$-\int P(x)\ln P(x)dx+\lambda\left(\int P(x)dx-1\right)+\gamma\left(\int (x-\langle x\rangle)^2P(x)dx-(\Delta\hat{x})^2\right), \tag{9}$$

where λ and γ are Lagrange multipliers to be determined. Calculating the variation of this functional with respect to $P(x)$ and equating the result to zero we obtain equation

$$-\ln P(x)-1+\lambda+\gamma(x-\langle x\rangle)^2=0, \tag{10}$$

which must be satisfied by distributions that provide the extremum of the functional (9). In this case the extremum is clearly a maximum. The distribution $P(x)$ solving this equation is obviously a Gaussian depending on (still unknown) parameters λ and γ. Using now the conditions of normalization and fixed (finite) variance to eliminate the Lagrange multipliers, we immediately see that among all states with variance $(\Delta\hat{x})^2$, the maximum entropy is attained for the state having the Gaussian probability distribution

$$\langle x|\hat{\rho}|x\rangle_{\max}=\frac{1}{\Delta\hat{x}\sqrt{2\pi}}\exp\left(-\frac{(x-\langle x\rangle)^2}{2(\Delta\hat{x})^2}\right). \tag{11}$$

Straightforward computation shows that the corresponding (maximum) entropy $S_{\max}$ is equal to $\frac{1}{2}\ln[2\pi e(\Delta\hat{x})^2]$. Because it is the maximum entropy possible under the considered conditions (normalization and a given variance), the entropy S_x of *any* other state satisfying the same conditions must obey the inequality

$$S_x \leqslant S_{\max} = \frac{1}{2}\ln[2\pi e(\Delta\hat{x})^2] = \frac{1}{2}(1+\ln\pi) + \frac{1}{2}\ln[2(\Delta\hat{x})^2]. \tag{12}$$

The term $\frac{1}{2}(1+\ln\pi)$ is the vacuum level for entropy. Thus, if there is a quadrature squeezing [meaning $(\Delta\hat{x})^2 < \frac{1}{2}$] then the term $\frac{1}{2}\ln[2(\Delta\hat{x})^2]$ becomes negative and the entropy S_x is also reduced below its vacuum-level value. The same reasoning can be performed for the momentum p. Note that the trick used here is exactly the same as that used for deriving the standard uncertainty relation from the EUR [10].

In conclusion, I have proved that the information entropy, being a superior measure of the quantum uncertainty from the formal point of view, is also a remarkably good practical measure of squeezing of quantum fluctuations. For all quantum states that exhibit quantum noise reduction below the vacuum level according to the variance, the corresponding information entropy is also less than the value of entropy corresponding to the vacuum level. It is possible, however, that for some nonclassical states the entropy is still less than the vacuum level, whereas the variance exhibits no more squeezing. The reason is that entropy is sensitive not only to squeezing but also to other nonclassical properties of a given state.

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