Effective two-level model for a three-level atom in the Ξ configuration

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(Received 28 February 1997; revised manuscript received 23 April 1997)

It is shown that the system of a three-level atom coupled to two modes of quantized cavity fields in the Ξ configuration with arbitrary detunings can be exactly reduced to a two-level system with an effective coupling which depends nonlinearly on the intensity of the two cavity fields. [S1050-2947(97)01209-2]

PACS number(s): 42.50.Hz, 42.50.Ar

Single-mode and multimode multiphoton (or multiphonon) processes involving one atom or ion with a few energy levels have recently been of increasing interest in the fields of cavity quantum electrodynamics [1–8] and laser trapping and cooling [9–13]. The investigation of these processes is of theoretical value in its own right and may have some potential applications in these two fields in the near future. On the other hand, this increasing interest is partly due to the fact that such nonlinear processes have experimentally been realized in the field of ion trapping [13]. In dealing with multiphoton (or multiphonon) transitions, one can consider a two-level atomic system in the single-mode case [1,2,4,11] and a three-level atomic system in the two-mode case [1,3–8,12]. In the latter case, one usually reduces it to an effective two-level problem on the assumption of large detuning(s) by the approximation of either the adiabatic elimination [5,12] or evaluating a unitary transformation perturbatively [6]. The effective two-level Hamiltonian thus obtained has the form of the usual Jaynes-Cummings model but with the single-mode field operators replaced by products of a field operator of one mode and a field operator of the other with the effective coupling parameter $\lambda \propto g_1 g_2/\Delta$, where g_j, Δ denote the coupling parameters and detuning, respectively [5–7]. Such a result obviously cannot be extrapolated to the situations of large g_j/Δ_j, and is singular for the zero-detuning case. As a matter of fact, the result may need modification in the situation when the field(s) is strong even if the ratios g_j/Δ_j are small [7]. One purpose of transforming three-level systems to two-level systems is to simplify the subsequent calculations of the dynamical and statistical properties of the atom and fields. What is more, such a transformation will be used to design two-level systems with modified desirable coupling parameter [7] and decay rate [12]. In view of the facts that the detunings are experimentally adjustable parameters that can be tuned to any values, that the coupling parameters can also easily be adjusted in the field of laser cooling of single trapped atoms or ions [10], and that nonlinear interactions are important when the field modes are relatively strong, it is thus desirable to obtain effective two-level models which describe the situations where coupling constants have arbitrary relations to the detunings.

In this paper we will show that the system of a three-level atom interacting with two quantized modes in the Ξ configuration with arbitrary detunings can, just as one of us did for the Λ configuration [7], be exactly reduced to an effective two-level model with an effective Raman coupling, which depends nonlinearly on the intensity of the two quantized field modes. An exact transformed Hamiltonian will be obtained in which one of the three levels is decoupled and the effective coupling does not show any singularity for any ratios of coupling parameters to detunings including the zero-detuning case.

We consider a three-level system of energies E_1, E_2, and E_3 in the Ξ configuration interacting with two quantized cavity modes 1 and 2 as shown in Fig. 1 [4,9]. The Hamiltonian of the system is written as [4]

$$H=\sum_{i=1}^{3} E_i\sigma_{ii}+\hbar\omega_1 b_1^\dagger b_1+\hbar\omega_2 b_2^\dagger b_2+\hbar g_1(b_1\sigma_{31}+b_1^\dagger\sigma_{13}) + \hbar g_2(b_2\sigma_{23}+b_2^\dagger\sigma_{32}), \tag{1}$$

where symbols $b_j (j=1,2)$ represent the field operators of modes 1 and 2, $\sigma_{ii}=|i\rangle\langle i|$ are the level occupation numbers, and $\sigma_{ij}=|i\rangle\langle j|$ $(i\neq j)$ are the transition operators from levels j to i. Levels 3 and 1 (2) are coupled by a dipole-coupling constant $g_1(g_2)$. There is no direct coupling between levels 1 and 2. The quantities Δ_1 and Δ_2 in Fig. 1 denote detunings given by $\Delta_j=(E_3-E_j)/\hbar-\omega_j$, $j=1,2$. Note that we have changed some notation with respect to the previous literature, and in particular have interchanged the numbering of levels 2 and 3. One of the purposes for such change is that we want to make full use of the corresponding derivations of the Λ configuration in Ref. [7] and facilitate the comparison of the results in these two configurations.

We introduce the unitary transformation

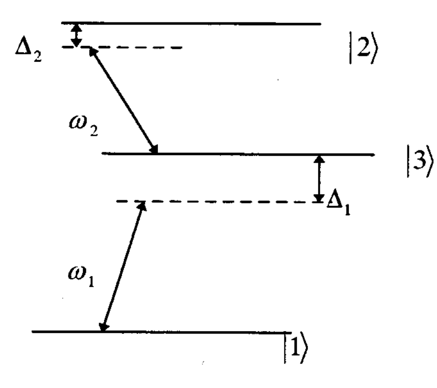

FIG. 1. Three-level atom in the Ξ configuration.

$$X'=\exp(S)X\ \exp(-S), \tag{2}$$

where X' denotes the transformed atomic and photon variables, and

$$S=\alpha(b_1\sigma_{31}-b_1^\dagger\sigma_{13})+\beta(b_2\sigma_{23}-b_2^\dagger\sigma_{32}), \tag{3}$$

where α and β are transformation parameters. By the same routine as what we did previously [7], we obtain the exact transformed Hamiltonian

$$H'=E_0+\hbar\omega_1N_1+\hbar\omega_2N_2+\tfrac{1}{2}\hbar\eta\sigma_{33}+\hbar\lambda(b_1b_2\sigma_{21}+b_1^\dagger b_2^\dagger\sigma_{12})+\tfrac{1}{2}\hbar\omega(\sigma_{22}-\sigma_{11}), \tag{4}$$

where

$$N_1=b_1^\dagger b_1+1-\sigma_{11},\quad N_2=b_2^\dagger b_2+\sigma_{22}, \tag{5}$$

$$\eta=\Delta_1-\Delta_2+\frac{3\sin2\xi}{\xi}\left(\frac{g_1}{\alpha}\overline{\alpha^2}-\frac{g_2}{\beta}\overline{\beta^2}\right)-\frac{3(\Delta_1\overline{\alpha^2}-\Delta_2\overline{\beta^2})}{2\xi^2}(1-\cos2\xi), \tag{6a}$$

$$E_0=\frac{1}{2}(E_1+E_2-\hbar\omega_1-\hbar\omega_2)+\frac{\hbar}{6}(\Delta_1-\Delta_2-\eta), \tag{6b}$$

$$\begin{aligned}\lambda=&\frac{\alpha\beta(1-\cos\xi)}{\xi^4}[(\Delta_2\overline{\alpha^2}-\Delta_1\overline{\beta^2})+(\Delta_2\overline{\beta^2}-\Delta_1\overline{\alpha^2})\cos\xi]\\&+\frac{\alpha\beta\sin\xi}{\xi^3}\left[\left(\frac{g_1}{\alpha}+\frac{g_2}{\beta}\right)(\overline{\beta^2}-\overline{\alpha^2})+2\left(\frac{g_1}{\alpha}\overline{\alpha^2}-\frac{g_2}{\beta}\overline{\beta^2}\right)\cos\xi\right],\end{aligned} \tag{6c}$$

$$\begin{aligned}\omega=&\Delta_1+\Delta_2-\frac{(1-\cos\xi)}{\xi^4}[2(\Delta_1+\Delta_2)\overline{\alpha^2}\overline{\beta^2}+(\Delta_1\overline{\alpha^2}+\Delta_2\overline{\beta^2})\xi^2+(\overline{\alpha^2}-\overline{\beta^2})\\&\times(\Delta_1\overline{\alpha^2}-\Delta_2\overline{\beta^2})\cos\xi]+\frac{2\sin\xi}{\xi^3}\left[2\overline{\alpha^2}\overline{\beta^2}\left(\frac{g_1}{\alpha}+\frac{g_2}{\beta}\right)+(\overline{\alpha^2}-\overline{\beta^2})\right.\\&\left.\times\left(\frac{g_1}{\alpha}\overline{\alpha^2}-\frac{g_2}{\beta}\overline{\beta^2}\right)\cos\xi\right],\end{aligned} \tag{6d}$$

where $\overline{\alpha}=\alpha\sqrt{N_1}$, $\overline{\beta}=\beta\sqrt{N_2}$, and $\xi=\sqrt{\overline{\alpha^2}+\overline{\beta^2}}$.

The transformation parameters α and β are determined by the two equations

$$\begin{aligned}&-(\Delta_1+\Delta_2)\frac{\alpha\overline{\beta^2}}{\xi^3}\sin\xi+\frac{\alpha\overline{\beta^2}}{\xi^2}\left(\frac{g_1}{\alpha}+\frac{g_2}{\beta}\right)\cos\xi-\frac{\alpha(\Delta_1\overline{\alpha^2}-\Delta_2\overline{\beta^2})}{2\xi^3}\sin2\xi\\&+\left(\frac{g_1}{\alpha}\overline{\alpha^2}-\frac{g_2}{\beta}\overline{\beta^2}\right)\frac{\alpha}{\xi^2}\cos2\xi=0,\end{aligned} \tag{7a}$$

$$\begin{aligned}&-(\Delta_1+\Delta_2)\frac{\beta\overline{\alpha^2}}{\xi^3}\sin\xi+\frac{\beta\overline{\alpha^2}}{\xi^2}\left(\frac{g_1}{\alpha}+\frac{g_2}{\beta}\right)\cos\xi+\frac{\beta(\Delta_1\overline{\alpha^2}-\Delta_2\overline{\beta^2})}{2\xi^3}\sin2\xi\\&-\left(\frac{g_1}{\alpha}\overline{\alpha^2}-\frac{g_2}{\beta}\beta^2\right)\frac{\beta}{\xi^2}\cos2\xi=0,\end{aligned} \tag{7b}$$

which generally need to be solved by numerical computation to obtain the two parameters α and β. However, these two parameters can be obtained analytically in the particularly interesting case considered previously [4], that is, as the detunings satisfy the relation $\Delta_1=-\Delta_2\equiv\Delta$ which implies an exact two-photon resonant condition $E_2-E_1=\hbar(\omega_1+\omega_2)$. Equation (7) is satisfied in this case if we choose

$$\alpha=\frac{g_1}{2\sqrt{\overline{g_1^2}+\overline{g_2^2}}}\arctan\left(\frac{2\sqrt{\overline{g_1^2}+\overline{g_2^2}}}{\Delta}\right), \tag{8a}$$

$$\beta=-\frac{g_2}{2\sqrt{\overline{g_1^2}+\overline{g_2^2}}}\arctan\left(\frac{2\sqrt{\overline{g_1^2}+\overline{g_2^2}}}{\Delta}\right), \tag{8b}$$

where $\overline{g_j}=g_j\sqrt{N_j}$, and N_j, $j=1,2$, are given by Eq. (5). We then find, after some manipulations, that the complicated expressions for the parameters in Eq. (6) are greatly simplified and have the forms

$$\eta=2\Delta+3\tau,\quad \tau=\left(\sqrt{\left(\frac{\Delta}{2}\right)^2+\overline{g_1^2}+\overline{g_2^2}}-\frac{|\Delta|}{2}\right)\mathrm{sgn}\Delta, \tag{9a}$$

$$E_0=\frac{1}{2}(E_1+E_2-\hbar\omega_1-\hbar\omega_2-\hbar\tau), \tag{9b}$$

$$\omega=\frac{\overline{g_1^2}-\overline{g_2^2}}{\overline{g_1^2}+\overline{g_2^2}}\tau,\quad \lambda=-\frac{g_1g_2}{\overline{g_1^2}+\overline{g_2^2}}\tau, \tag{9c}$$

where ''sgn'' represents the signum function. Substituting Eq. (9) into Eq. (4), we explicitly arrive at the exact transformed Hamiltonian in which level 3 decouples from the other two levels. Equation (4), together with its parameters determined by Eqs. (6) and (7) [or by Eq. (9) in the particular case $\Delta_1=-\Delta_2\equiv\Delta$], is the central result of this paper.

Let us now discuss this result. First, the λ and ω terms in the transformed Hamiltonian [Eq. (4)] obviously only produce transitions between levels 1 and 2, while the other terms do not cause any transitions among the three levels.

This means that level 3 can be exactly decoupled and does not contribute to the population dynamics. Consequently, we can set $\sigma_{33}=0$ in the transformed Hamiltonian to obtain an effective two-level model with levels 1 and 2 subject to an effective intensity-dependent coupling, i.e., the parameter λ depends on photon numbers of the two modes. We have therefore shown that the system of a three-level atom coupled to two modes of quantized cavity fields in the Ξ configuration with arbitrary detunings Δ_j and coupling parameters g_j can be exactly reduced to a two-level system with an effective intensity-dependent coupling λ. Secondly, the intensity-dependent coupling occurs naturally here while previous studies usually introduce it phenomenologically. Thirdly, the effective coupling λ is valid and nonsingular for any ratios of coupling parameters g_j to detunings Δ_j. It is easily seen from Eq. (9) that the effective coupling $\lambda \approx -g_1 g_2/\Delta$ when $\Delta^2 \gg \overline{g_1^2 - g_2^2}$, and as $\Delta^2 \ll (\overline{g_1^2 + g_2^2})$, it becomes $\lambda \approx -g_1 g_2/\sqrt{\overline{g_1^2+g_2^2}}$ which remains finite as $\Delta \rightarrow 0$. Fourthly, the absolute value of the effective coupling λ is easily seen from Eq. (9) to be a monotonically decreasing function of the detuning Δ, which means that the smaller the detuning, the stronger the effective coupling between levels 1 and 2. This is a reasonable result that can be anticipated physically since the smaller the detuning, the stronger the direct couplings between levels 1 and 3 and between levels 2 and 3, and also the effective coupling between levels 1 and 2. This result suggests that zero detuning or small detuning would be more desirable by taking into account the fact that stronger coupling is better for realizing all kinds of nonclassical phenomena such as collapse and revivals, squeezed and trapped states experimentally [10]. It is emphasized that our results here are still valid in these small detuning cases where those of the adiabatic elimination cease to be so. Finally, we point out that since we have obtained the exact transformed Hamiltonian where one of the three levels decouples from the other two levels, we can easily derive its exact eigenvalues and eigenvectors, and the corresponding dynamics, by the unified and standarized formulas we developed [8(b)], and we can also derive the statistical properties of the atom and the fields.

It is interesting to compare Eqs. (4) and (9) describing the Ξ-type system in the exact two-photon resonant case with those describing the Λ-type system in the exact two-photon resonant case. The transformed Hamiltonian for the Λ-type system in the exact two-photon resonant case is [the exact two-photon resonant condition for the Λ-type system reads [7] $\Delta_1 = \Delta_2 \equiv \Delta$, i.e., $E_2 - E_1 = \hbar(\omega_1 - \omega_2)$; note that we have here chosen level 3 in between levels 1 and 2 while level 3 is chosen as the highest level in Ref. [7] for the Λ-type system]

$$H' = E_0 + \hbar\omega_1 N_1 + \hbar\omega_2 N_2 + \tfrac{1}{2}\hbar\eta\sigma_{33} + \hbar\lambda(b_1 b_2^\dagger \sigma_{21} + b_1^\dagger b_2 \sigma_{12}) + \tfrac{1}{2}\hbar\omega(\sigma_{22} - \sigma_{11}), \tag{10}$$

where the expressions of parameters E_0, η, λ, and ω can still be determined by Eq. (9) here except that N_j, $j=1,2$ are different [7] [there are minor errors in Eq. (31) of Ref. [7] for the expressions of the parameters E_0, η, λ, and ω as the detuning Δ is negative]. Comparing Eq. (4) with Eq. (10), we see that by intentionally choosing different level numberings and different N_j, the transformed Hamiltonians for the Ξ-type and Λ-type systems, respectively, turn out to have nearly the same form except for different effective interaction terms.

In summary, we have investigated the system of a three-level atom interacting with two quantized cavity modes (the two modes can have either different or identical frequencies) in Ξ configuration, we have obtained the exact transformed Hamiltonian and shown that one of the three levels (level 3) can be made to decouple from the other two levels, and hence, can be eliminated from the exact transformed Hamiltonian to obtain an effective two-level Hamiltonian with a nonsingular intensity-dependent coupling between levels 1 and 2. The effective two-level Hamiltonian is, within the framework of the original Hamiltonian, valid for any magnitudes of the ratios of the coupling constants to detunings including the zero-detuning case. These results can be utilized to investigate the dynamics and statistics of atomic and field quantities, and the effects of the naturally occurring intensity-dependent coupling on them particularly in the situations of strong couplings (large g), small detunings, and intense fields, where the adiabatic elimination ceases to be valid. In addition, the results and the approach used here can also be utilized to design two-level systems with desirable effective coupling and effective decay rate for ion or atom sideband cooling [12] particularly in the above-mentioned situations.

One of the authors (Y.W.) acknowledges helpful discussions with A. Douglas Stone. This work is partially supported by Chinese National Science Foundation, Chinese National Education Committee, and the National Laboratory of MRAMP at Wuhan Institute of Physics and Mathematics of the Chinese Academy of Sciences.

[1] S. M. Barnett, P. Filipowicz, J. Javanainen, P. L. Knight, and P. Meystre, in *Frontiers in Quantum Optics*, edited by E. R. Pike and S. Sarkar (Hilger, London, 1986); E. A. Hinds, Adv. At. Mol. Opt. Phys. **28**, 237 (1990); P. Meystre, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1992), Vol. 30; R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).

[2] B. Buck and C. V. Sukumar, Phys. Lett. **81A**, 132 (1981); C. V. Sukumar and B. Buck, *ibid.* **83A**, 211 (1981); P. L. Knight, Phys. Scr. **T12**, 51 (1986); S. J. D. Phoenix and P. L. Knight, J. Opt. Soc. Am. B **7**, 116 (1990); M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. A **36**, 3771 (1987); I. Ashraf, J. Gea-Banacloche, and M. S. Zubairy, *ibid.* **42**, 6702 (1990).

[3] Fam Le Kien, G. M. Meyer, M. O. Scully, H. Walther, and S. Y. Zhu, Phys. Rev. A **49**, 1367 (1994); A. S. Shumovsky, E. I. Aliskenderov, Fam Le Kien, and N. D. Vinh, J. Phys. A **19**, 3607 (1986).

[4] H.-I. Yoo and J. H. Eberly, Phys. Rep. **118**, 239 (1985).

[5] C. C. Gerry and J. H. Eberly, Phys. Rev. A **42**, 6805 (1990); D. A. Cardimona, V. Kovanis, M. P. Sharma, and A. Gavrielides, *ibid.* **42**, 3710 (1991).

[6] M. Alexanian and S. K. Bose, Phys. Rev. A **52**, 2218 (1995).

[7] Y. Wu, Phys. Rev. A **54**, 1586 (1996).

[8] (a) Y. Wu, Phys. Rev. A **54**, 4534 (1996); (b) Xiaoxue Yang, Y. Wu, and Yuanjie Lee, *ibid.* **55**, 4545 (1997).

[9] J. I. Cirac, R. Blatt, A. S. Parkins, and P. Zoller, Phys. Rev. Lett. **70**, 762 (1995); Phys. Rev. A **49**, 1202 (1994); J. I. Cirac, A. S. Parkins, R. Blatt, and P. Zoller, Phys. Rev. Lett. **70**, 556 (1993); Opt. Commun. **97**, 353 (1993); C. D'Helon and G. J. Milburn, Phys. Rev. A **52**, 4755 (1995); J. I. Cirac, R. Blatt, A. S. Parkins, and P. Zoller, *ibid.* **49**, 1202 (1994).

[10] J. I. Cirac, R. Blatt, P. Zoller, and W. D. Phillips, Phys. Rev. A **46**, 2668 (1992); Ying Wu and Xiaoxue Yang, Phys. Rev. Lett. **78**, 3086 (1997).

[11] R. L. de Matos Filho and W. Vogel, Phys. Rev. A **50**, R1988 (1994).

[12] I. Marzoli, J. I. Cirac, R. Blatt, and P. Zoller, Phys. Rev. A **49**, 2771 (1994).

[13] C. Monroe *et al.*, Phys. Rev. Lett. **75**, 4011 (1995); D. M. Meekhof *et al.*, *ibid.* **76**, 1796 (1996).