Nonclassical maximum-entropy states

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We investigate specific maximum-entropy states of a single harmonic-oscillator mode that arise when only number states (Fock states) differing by a multiple of a certain integer k ($k\geqslant 1$) are allowed to be occupied. For $k=2$ the number-probability distribution of the even-number maximum-entropy state has a close resemblance to that of a squeezed-vacuum state. These maximum-entropy states can be obtained as the stationary solutions of a master equation which takes into account k-quantum absorption as well as k-quantum emission processes only. The steady-state solution of this master equation depends on the initial conditions. For the vibrational motion of a trapped ion such nonclassical maximum-entropy states could be produced with the help of the recently proposed method of laser-assisted quantum reservoir engineering. [S1050-2947(97)02008-8]

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I. INTRODUCTION

It is well known that the density operator describing a single-mode quantized radiation field that is in thermal equilibrium with its surrounding can be obtained by maximizing the entropy of the quantum field under the constraint that the mean energy (or the mean photon number, respectively) is kept constant (see [1]). The expression for the thermal density operator also holds for the density operator of a chaotic radiation field [2] that results from the contributions of many statistically independent classical light sources, i.e., from light sources the radiation of which can be described by a positive-definite P representation. (In the multimode case, the chaotic or thermal density operator factorizes into the products of the corresponding single-mode operators [2].) In this contribution, we generalize the concept of the maximum-entropy state of a single quantized harmonic oscillator of frequency ω to nonclassical k-quantum maximum-entropy states ($k = 2,3,\dots$) that arise when only energy eigenstates are allowed to be occupied, the energy of which differs by a multiple of $k\hbar\omega$. These states do not possess a positive-definite P representation, i.e., they are intrinsically nonclassical.

In order to generate the nonclassical maximum-entropy states, one might think of radiation fields that result from the contributions of many statistically independent k-photon emitters ($k\geqslant 2$), in analogy to the classical chaotic radiation where $k=1$. For $k=2$ the photon-number distribution of such a field is expected to oscillate similarly to the number distribution of a squeezed-vacuum field [3] which arises from spontaneous degenerate parametric down conversion (see, e.g., [4]). In both cases two photons are always emitted simultaneously, the probability for the presence of an odd number of photons, therefore, is zero. However, even if a radiation source which consists of a large number of independent atoms undergoing spontaneous two-photon emission could be built, the detection of the resulting even-odd oscillating photon-number distribution of the total field would be practically impossible due to the different emission directions involved in each spontaneous transition.

A more realistic possibility for the generation of nonclassical k-quantum maximum-entropy states becomes obvious when the concepts of quantum optics are adopted to the description of the vibrational center-of-mass motion of a trapped ion in a harmonic-oscillator potential [5]. When the ion is irradiated by laser light which is tuned to a definite vibrational sideband of the internal electronic transition, the center-of-mass motion of the ion is influenced in a specific way due to the recoil experienced in each cycle of absorption and subsequent spontaneous emission. By changing the laser frequencies and intensities it is possible to design different kinds of couplings between the vibrational energy levels and the surrounding reservoir of the vacuum modes of the electomagnetic field [6]. In particular, this coupling can give rise to k-phonon transitions in the energy of the center-of-mass motion. In analogy to the classical thermal equilibrium states for $k=1$, the nonclassical k-phonon maximum-entropy states ($k\geqslant 2$) turn out to be the equilibrium states which result, under certain initial conditions, when k-phonon absorption and k-phonon emisson processes act simultaneously.

II. PROPERTIES OF THE MAXIMUM-ENTROPY STATES

We consider the steady-state density operator ρ of a single harmonic-oscillator mode, the energy of which can be changed due to k-quantum absorption and k-quantum emission processes only. In order to introduce the maximum-entropy states we assume that only number states differing by a multiple of k are allowed to be occupied. In the stationary state all nondiagonal elements ρ_{mn} in the Fock representation vanish. We can discriminate k different kinds of k-quantum maximum-entropy states, which we label by the integer parameter q, where $0\leqslant q\leqslant k-1$. The diagonal density-matrix elements obey the equation

$$\rho_{nn}^{(k,q)}=\begin{cases}0 & \text{for } n\neq mk+q\\ p_{mk+q} & \text{for } n=mk+q,\end{cases} \qquad (1)$$

where $m=[n/k]$ is the largest integer that does not not exceed n/k. The von Neumann entropy, which takes its maximum value in the equilibrium state, can be written as

$$S^{(k,q)}=-\sum_{n=0}^{\infty}\rho_{nn}^{(k,q)}\ln(\rho_{nn}^{(k,q)})=-\sum_{m=0}^{\infty}p_{mk+q}\ln(p_{mk+q}). \tag{2}$$

In order to obtain the number-probability distribution of the maximum-entropy states, we have to maximize expression (2) under the constraints $\Sigma_m p_{mk+q}=1$ and $\Sigma_m(mk+q)p_{mk+q}=\overline{n}$, with $\overline{n}$ being the mean number of quanta. Using the method of Lagrange multipliers we find that the entropy $S^{(k,q)}$ takes its maximum value when

$$p_{mk+q}=\frac{k}{\overline{n}-q+k}\left(\frac{\overline{n}-q}{\overline{n}-q+k}\right)^{m}. \tag{3}$$

The structure of Eq. (3) is analoguous to that of a thermal photon-number distribution [2] with mean photon number $(\overline{n}-q)/k$. (Note that $\overline{n}-q=k\Sigma_n m p_{mk+q}>0$.) The special case $k=1$ corresponds to the thermal state.

For the maximum-entropy states all higher-order quantum-number moments can be reduced to expressions containing the mean number of quanta only. For this purpose use has to be made of the relation

$$\overline{\frac{n-q}{k}\left(\frac{n-q}{k}-1\right)\cdots\left(\frac{n-q}{k}-l+1\right)}=l!\left(\frac{\overline{n}-q}{k}\right)^{l}, \tag{4}$$

($l=1,2,\ldots$) that can be derived from Eqs. (1) and (3), in analogy to the expression valid for a thermal field where $k=1$ and $q=0$. The quantum-number variance reads

$$\Delta n^2=\overline{(n-\overline{n})^2}=(\overline{n}-q)(\overline{n}-q+k). \tag{5}$$

From Eq. (5) we find that the k-quantum maximimum-entropy states exhibit sub-Poissonian statistics, with $\Delta n^2<\overline{n}$, provided that the inequality

$$q<\overline{n}<\left(q-\frac{k-1}{2}\right)+\left[\left(q-\frac{k-1}{2}\right)^2+q(k-q)\right]^{1/2} \tag{6}$$

is fulfilled. Obviously, for $q=0$ the statistics is always super-Poissonian. However, for $k\geqslant 2$ the maximum-entropy states are, nevertheless, highly nonclassical because of the oscillatory distribution of the number probabilities $\rho_{nn}^{(k,q)}$ [cf. Eq. (1)] which prevents the existence of a positive-definite P distribution.

In the special case $k=2$ we obtain the variances $\Delta n^2=(\overline{n}-1)(\overline{n}+1)$ for $q=1$, i.e., when only odd-number states are occupied, and $\Delta n^2=\overline{n}(\overline{n}+2)$ for $q=0$, i.e., for the occupation of even-number states only. It is interesting to compare the number-probability distribution of an even-number two-quantum maximum-entropy state, which is characterized by the recursion relation $p_{2(m+1)}/p_{2m}=\overline{n}/(\overline{n}+2)$, with that of a squeezed-vacuum state. From the analytical formula for the number-probability distribution [3,4] of the squeezed vacuum we obtain the recursion relation $p_{2(m+1)}^{sv}/p_{2m}^{sv}=[(2m+1)\overline{n}]/[(2m+2)(\overline{n}+1)]$. As becomes obvious from Fig. 1, the resulting number-probability distributions are remarkably similar. Hence if an oscillating distribution of the kind shown in Fig. 1 is detected, a considerable accuracy is required to make sure that the oscillator mode, indeed, is in a squeezed-vacuum state [7].

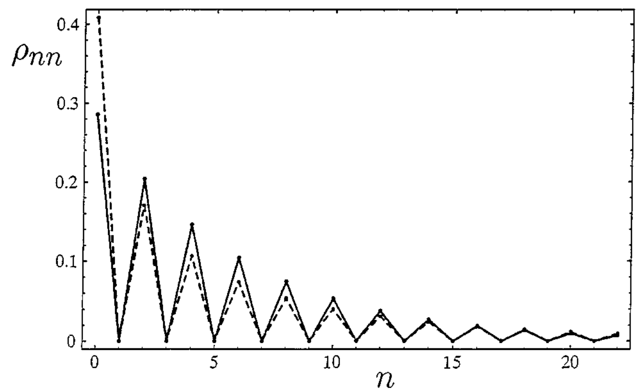

FIG. 1. Number-probability distribution ρ_{nn} of an even-number two-quantum maximum-entropy state with mean number of quanta $\overline{n}=5$ and of a squeezed-vacuum state with the same value of $\overline{n}$. For clearness, the separate dots are connected by a full line (two-quantum maximum-entropy state) and by a dashed line (squeezed vacuum).

For completeness we also calculate the quasiprobability distributions of the two-quantum maximum-entropy states which can be given in simple analytical terms. Making use of Eqs. (1) and (3) we obtain for the the Q function $Q(\alpha)=\langle\alpha|\rho|\alpha\rangle/\pi$ of the even-number two-quantum maximum-entropy state ($q=0$) the expression

$$Q^{(2,0)}(\alpha)=\frac{1}{\pi}\frac{2e^{-|\alpha|^2}}{\overline{n}+2}\cosh\left(|\alpha|^2\left[\frac{\overline{n}}{\overline{n}+2}\right]^{1/2}\right). \tag{7}$$

The Q function of the odd-number state ($q=1$) reads

$$Q^{(2,1)}(\alpha)=\frac{1}{\pi}\frac{2e^{-|\alpha|^2}}{\sqrt{\overline{n}^2-1}}\sinh\left(|\alpha|^2\left[\frac{\overline{n}-1}{\overline{n}+1}\right]^{1/2}\right) \tag{8}$$

(cf. Fig. 2). The corresponding Wigner functions can be calculated by inverting the relation $Q(\alpha)=(2/\pi)\int W(\beta)e^{-2|\alpha-\beta|^2}d^2\beta$ [8]. We find for the even-number two-quantum maximum-entropy state

$$W^{(2,0)}(\alpha)=\frac{1}{\pi}\frac{2}{\sqrt{\overline{n}+2}}\left[\frac{\exp\left(-2|\alpha^2|\frac{\sqrt{\overline{n}+2}-\sqrt{\overline{n}}}{\sqrt{\overline{n}+2}+\sqrt{\overline{n}}}\right)}{\sqrt{\overline{n}+2}+\sqrt{\overline{n}}}+\frac{\exp\left(-2|\alpha^2|\frac{\sqrt{\overline{n}+2}+\sqrt{\overline{n}}}{\sqrt{\overline{n}+2}-\sqrt{\overline{n}}}\right)}{\sqrt{\overline{n}+2}-\sqrt{\overline{n}}}\right], \tag{9}$$

and for the odd-number state

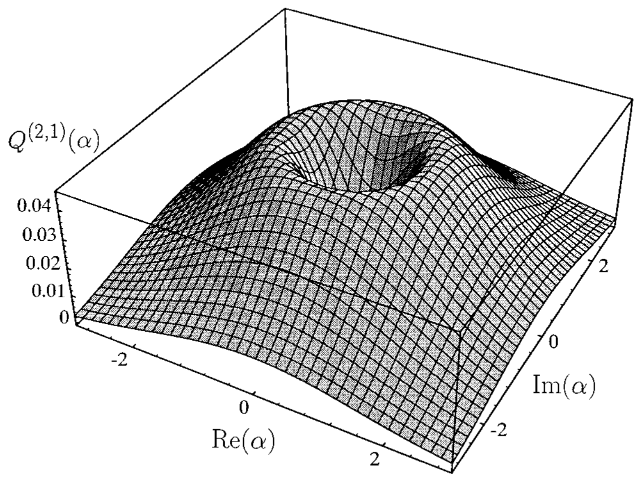

FIG. 2. Q function of an odd-number two-quantum maximum-entropy state with $\overline{n}=5$.

$$W^{(2,1)}(\alpha)=\frac{1}{\pi}\frac{2}{\sqrt{\overline{n}-1}}\left[\frac{\exp\left(-2|\alpha^2|\frac{\sqrt{\overline{n}+1}-\sqrt{\overline{n}-1}}{\sqrt{\overline{n}+1}+\sqrt{\overline{n}-1}}\right)}{\sqrt{\overline{n}+1}+\sqrt{\overline{n}-1}}-\frac{\exp\left(-2|\alpha^2|\frac{\sqrt{\overline{n}+1}+\sqrt{\overline{n}-1}}{\sqrt{\overline{n}+1}-\sqrt{\overline{n}-1}}\right)}{\sqrt{\overline{n}+1}-\sqrt{\overline{n}-1}}\right]. \tag{10}$$

Similarly to the case of the even and odd coherent superposition states [9], the Wigner function is positive for the even-number two-photon maximum entropy state and exhibits negative values in the odd-number case (see Fig. 3). These remarkably strong negativities are centered around $\alpha=0$.

III. GENERATION OF THE STATES

In order to discuss the generation of the maximum-entropy states we consider the interplay of k-quantum emission processes (with emission constant β_k) and k-quantum absorption processes (with absorption constant γ_k). Since we are interested in the equilibrium state which results from the interaction with a corresponding reservoir, both kinds of k-quantum processes have to be assumed to be unsaturated. (In contrast, e.g., to the case of a k-photon laser with m-photon losses ($k,m=1,2,\ldots$) [10], where the emission process is saturated.) We therefore start from the master equation

$$\dot{\rho}=-\frac{\beta_k}{2}(a^k a^{\dagger k}\rho-2a^{\dagger k}\rho a^k+\rho a^k a^{\dagger k})-\frac{\gamma_k}{2}(a^{\dagger k}a^k\rho-2a^k\rho a^{\dagger k}+\rho a^{\dagger k}a^k) \tag{11}$$

for the density operator ρ of the considered field mode. Here a and $a^\dagger$ are the boson annihilation and creation operators, respectively. From Eq. (11) the equation of motion for the diagonal elements $\rho_{nn}\equiv p_n$ is given by

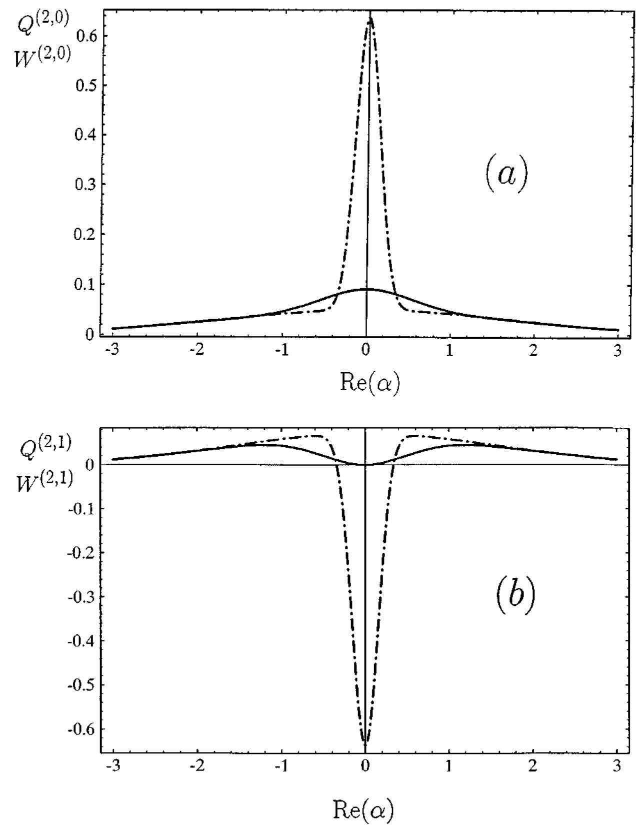

FIG. 3. Cut at Im$(\alpha)=0$ through the Q function (full line) and the Wigner function (dashed-dotted line) of an even-number (a) and an odd-number (b) two-quantum maximum-entropy state with $\overline{n}=5$.

$$\dot{p}_n=\frac{(n+k)!}{n!}(\gamma_k p_{n+k}-\beta_k p_n)-\frac{n!}{(n-k)!}(\gamma_k p_n-\beta_k p_{n-k}). \tag{12}$$

Invoking detailed-balance considerations, we immediately arrive at the recurrence relation $p_{n+k}/p_n=\beta_k/\gamma_k$ which has to be fulfilled in the equilibrium state, i.e., for $\dot{p}_n=0$. Since only number states which differ by a multiple of k are coupled due to the interaction with the reservoir, we have to discriminate k different interaction channels for $k\geqslant 2$ which we again characterize by the integer parameter q with $0\leqslant q\leqslant k-1$. Making use of the decomposition $n=mk+q$, where $m=[n/k]=1,2,\ldots,$ the steady-state number-probability distribution can be written as

$$p_{mk+q}=C_{q,k}(\beta_k/\gamma_k)^m. \tag{13}$$

We mention that for $k=1$, where $q=0$, the well-known master equation for the interaction with a thermal heat bath [1] can be regained from Eq. (11), when Eq. (13) is used to express β_1/γ_1 by the value of $\overline{n}$. For $k\geqslant 2$ the constants $C_{q,k}$ have to be determined from the number-probability distribution $p_n^{(i)}$ of the initial state using the relation

$$\frac{C_{q,k}}{C_{0,k}}=\frac{\Sigma_{m=0}^{\infty}p_{mk+q}^{(i)}}{\Sigma_{m=0}^{\infty}p_{mk}^{(i)}}. \tag{14}$$

Thus for $k \geqslant 2$, where the number of possible q values is larger than 1, the equilibrium state depends on the initial conditions. When the condition

$$p_{k[n/k]+q}^{(i)} = 0 \quad \text{for} \quad q \neq q_0 \tag{15}$$

is fulfilled for $k \geqslant 2$, the system evolves towards the k-quantum maximum-entropy state described by Eqs. (1) and (3) with $q = q_0$, where the mean number of quanta is given by the relation

$$\overline{n} = q_0 + k/(\gamma_k/\beta_k - 1), \tag{16}$$

which follows from Eqs. (13)–(15). From Eq. (16) we conclude that a steady state can only be reached when the absorption constant γ_k exceeds the emission constant β_k. This is obvious from physical reasons, since in the case $\beta_k = \gamma_k$ the mean number of quanta would already grow to infinity due to the excess caused by spontaneous k-quantum emission. The requirement (15), which is necessary for the generation of k-quantum maximum-entropy states, is satisfied, e.g., when the system is initially prepared in a number state.

Finally we address the question as to how the k-quantum maximum-entropy states could be produced in a real experiment. Recently it has been shown [6] that the system-reservoir coupling described by a master equation of the general form

$$\dot{\rho} = -\gamma(f^\dagger f \rho - 2 f \rho f^\dagger + \rho f^\dagger f), \tag{17}$$

with $f \equiv f(a, a^\dagger)$, can be experimentally realized for the motional state of a single ion trapped in a harmonic potential, with trap frequency ν, and being irradiated by a standing-wave laser field of a freqency $\omega_L = k_L c$, which is nearly in resonance with the frequency ω_0 of an internal electronic transition of the ion. The operator a is then given by $a = X(m\nu/2\hbar)^{1/2} + iP(2m\hbar\nu)^{1/2}$, where X, P, and m are the position, momentum operator, and the mass of the ion, respectively. It is assumed that the Lamb-Dicke parameter $\eta = (\hbar k_L^2/2m\nu)^{1/2}$ is small and that the spontaneous decay of the excited electronic level can be neglected during the oscillation period of the ion in the trap. Moreover, the intensity of the laser field has to be low enough to allow for the elimination of the excited-state population [6]. When the position dependence of the standing-wave field is expressed by $\sin(k_L X + \phi_0)$ and when this term is expanded in the Lamb-Dicke limit, it can be shown, with the help of the method outlined in [6], that a specific kind of system-reservoir coupling can be realized, which is described by a master equation of the form of Eq. (17), where $f \equiv a^k$, provided that $\omega_L = \omega_0 - k\nu$, and $f \equiv a^{\dagger k}$ for $\omega_L = \omega_0 + k\nu$. To this end ϕ_0 has to be chosen to be equal to zero for odd values of k and to be equal to $\pi/2$ for even values of k, which means that the trap center is in a node ($\phi_0 = 0$) or antinode ($\phi_0 = \pi/2$) of the standing wave, respectively [6]. The constant γ of Eq. (17) is proportional to the square of the Rabi frequency. When the ion is excited at the kth lower motional sideband ($\omega_L = \omega_0 - k\nu$), k phonons are annihilated during each absorption-spontaneous emission cycle of the ion, and for excitation at the kth upper sideband ($\omega_L = \omega_0 + k\nu$) we have the case of k-phonon emission [6].

When the ion is irradiated by two appropriately detuned incoherent lasers [6], a combination of k-phonon absorption and k-phonon emission resulting in the master equation (11) could be experimentally realized. The ratio γ_k/β_k would then be determined by the ratio of the squared Rabi frequencies or of the laser intensities, respectively. In particular, in order to prepare an even-number two-phonon maximum-entropy state the properties of which are shown in Figs. 1 and 3(a), one could start from the motional ground state of the ion since laser cooling to the zero-point energy of motion has already been achieved experimentally [11]. (We mention that recently a generation of even and odd coherent states of the motion of a trapped ion has been proposed via bichromatic excitation of both the electronic transition and the second vibrational sideband [12].)

In summary, we have introduced specific k-quantum maximum-entropy states which possess nonclassical properties and we have shown that these states could be produced in a real experiment.

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