Geometric phase, quantum measurements, and the de Broglie–Bohm model

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An approach to the measurement induced geometric phase based on the de Broglie–Bohm hidden-variable model is developed. The analysis involves an evolving geometric phase connecting the initial and final states of an individual experimental run. As an illustration the geometric phase produced by a cyclic sequence of spin-$\frac{1}{2}$ filtering experiments is considered. It is argued, in this case, that the evolving geometric phase can be made close to the Samuel-Bhandari phase. [S1050-2947(97)01808-8]

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The existence of the geometric phase in the case of quantum measurements was first pointed out by Samuel and Bhandari [1]. The essence of their analysis was to realize that the phase of the Bargmann invariant [2] associated with a cyclic sequence of measurements of the first kind is geometric in the sense of the Pancharatnam connection [3]. It was subsequently shown [4,5] that the unitary geometric phase [6,7] is obtained if the sequence of projections is dense. Furthermore, the appearance of a non-Abelian geometric phase [8] in the case of incomplete measurements was demonstrated [9] and the restriction to measurements of first kind was relaxed [10]. Recently the geometric phase in the context of the quantum Zeno effect [11] was discussed [12]. Experimental verifications of the measurement induced geometric phase have also been carried out [13,14].

Various models have been developed in order to give a physical account of the measurement process which goes beyond the usual minimal instrumentalist interpretation of quantum theory [15]. Among the most well established are the models based on environment induced decoherence [16–18], quantum state diffusion [19,20], and de Broglie–Bohm hidden variables [21,22]. A common feature of these ''nonminimal'' approaches is that insights into the measurement process are provided which cannot otherwise be obtained. One expects this should be true also in the case of measurement induced geometric phases. The purpose of the present report is to analyze this in the framework of the de Broglie–Bohm model [21–25] which we first briefly review.

A physical system is assumed to consist of a wave and a particle. The wave is described by the wave function $\Psi(q,t)$ which is a solution of the time dependent Schrödinger equation. The particle follows a trajectory $Q(t)$ in configuration space given by the equation

$$\dot{Q}=\frac{J^{\Psi}}{\rho^{\Psi}}\bigg|_{q=Q}. \tag{1}$$

Here J^{Ψ} is the quantum-mechanical current and ρ^{Ψ} the normalized probability density expressing the impossibility to actually observe the trajectory.

Now with the partitioning $q=(v,w)$ of the system we can write

$$\dot{V}=\frac{J^{\Psi_W}}{\rho^{\Psi_W}}\bigg|_{v=V}, \tag{2}$$

where

$$\Psi_{W(t)}(v,t)=\Psi(v,W(t),t), \tag{3}$$

is the conditional wave function of the v system [26]. In other words the wave function governing the dynamics of the v system is the total wave function, given the condition that the w system evolves according to $W(t)$. The measurement theory of the de Broglie–Bohm model can now be developed on the basis of the conditional wave function. Suppose an observable O with the associated nondegenerate normalized eigenfunctions $\psi_n(v)$ and eigenvalues o_n is being measured for a system described by the normalized wave function

$$\psi(v)=\sum_n c_n\psi_n(v), \tag{4}$$

where c_n are complex expansion coefficients. The interaction between the measured system (v) and the apparatus (w) may result in the superposition

$$\Psi(v,w,t)=\sum_n c_n\psi_n(v)\Phi_n(w,t), \tag{5}$$

where the apparatus wave packets Φ_n eventually get macroscopically separated. In this case the reduction

$$\Psi_{W(t)}(v,t)\rightarrow\Psi'_{W(t)}(v,t)\approx c_{n'}\psi_{n'}(v)\Phi_{n'}(W(t),t) \tag{6}$$

occurs in a given individual run of the experiment if W evolves into the support of $\Phi_{n'}$. Assuming the Φ_n's are normalized, one can show (see, e.g., Ref. [23], Sec. 8.3.3) that the probability for Eq. (6) is $|c_{n'}|^2$. The measured value $o_{n'}$ of O can be inferred by observing the ''pointer'' W and, furthermore, the result is repeatable here provided that the subsequent measurement of O is made immediately afterwards (i.e., measurement of the first kind).

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Next we define the geometric phase for Eq. (3). The dynamical phase change $\delta\eta(t)$ at t along the Hilbert space trajectory $\Psi_{W(t)}(t)$ is given by

$$\delta\eta(t)=\arg[\langle\Psi_{W(t)}(t)|\Psi_{W(t+\delta t)}(t+\delta t)\rangle] \approx -i\left\langle\frac{\Psi_{W(t)}(t)}{\|\Psi_{W(t)}(t)\|}\left|\frac{d}{dt}\right|\frac{\Psi_{W(t)}(t)}{\|\Psi_{W(t)}(t)\|}\right\rangle\delta t, \qquad (7)$$

where $\langle\ \cdot\ \rangle$ denotes integration over v and $\|\cdot\| = \langle\ \cdot\ |\ \cdot\ \rangle^{1/2}$. The geometric phase $\gamma[\Psi_W]$ associated with the evolution $\Psi_{W(0)}(0)\rightarrow\Psi_{W(T)}(T)$ is then defined by removing the accumulated dynamical phase from the total phase [27,28]

$$\gamma[\Psi_W]=\arg[\langle\Psi_{W(0)}(0)|\Psi_{W(T)}(T)\rangle] + i\int_0^T\left\langle\frac{\Psi_{W(t)}(t)}{\|\Psi_{W(t)}(t)\|}\left|\frac{d}{dt}\right|\frac{\Psi_{W(t)}(t)}{\|\Psi_{W(t)}(t)\|}\right\rangle dt. \qquad (8)$$

The geometric phase $\gamma[\Psi_W]$ is real, reparametrization invariant and projective geometric. It depends only on the curve in projective Hilbert space $\mathcal{P}$ (the space of rays in Hilbert space $\mathcal{H}$ of the v system) corresponding to the evolution of Ψ_W. Furthermore, it can be demonstrated that $\gamma[\Psi_W]$ reduces to the Aharonov-Anandan formula [7] for cyclic wave functions decoupled from the environment W.

We are now prepared to discuss the geometric phase for measurements. Consider three consecutive Stern-Gerlach spin-$\frac{1}{2}$ filtering experiments in the x, n_ϕ, and z direction, where n_ϕ lies in the x-y plane and makes an angle ϕ with the x axis. The sequence of postselected spin states is given by the projections $P_{\uparrow_x}=|\uparrow_x\rangle\langle\uparrow_x|$, $P_{\uparrow_\phi}=|\uparrow_\phi\rangle\langle\uparrow_\phi|$, and $P_{\uparrow_z}=|\uparrow_z\rangle\langle\uparrow_z|$. If the initial spin state is $|\uparrow_z\rangle\langle\uparrow_z|$ then the sequence is cyclic and the Samuel-Bhandari geometric phase γ_{SB} becomes the phase of the Bargmann invariant $\langle\uparrow_z|P_{\uparrow_\phi}P_{\uparrow_x}|\uparrow_z\rangle$. It follows that $\gamma_{SB}=-\frac{1}{2}\phi$, which is minus half the solid angle enclosed by the geodesic triangle connecting the points $\Pi(|\uparrow_z\rangle)$, $\Pi(|\uparrow_x\rangle)$, and $\Pi(|\uparrow_\phi\rangle)$ in $\mathcal{P}$. This phase can be observed as an interference oscillation by varying ϕ.

The de Broglie–Bohm analysis of this experiment goes as follows (for details concerning the de Broglie–Bohm model of Stern-Gerlach experiments, see Refs. [29,30]). Assume the spin measurements take place at times t_x, t_ϕ, and t_z ($t_x<t_\phi<t_z$). After the particle has left the first Stern-Gerlach field the spatial (‘‘apparatus’’) wave function splits into two separating wave packets A_x^{up} and A_x^{down}. The subsequent ($t>t_x$) evolution of the conditional wave function is given by

$$|\Psi_{\mathbf{R}(t)}(t)\rangle=A(Y(t),Z(t))[A_x^{up}(X(t),t)P_{\uparrow_x} + A_x^{down}(X(t),t)P_{\downarrow_x}]|\uparrow_z\rangle, \qquad (9)$$

where $\mathbf{R}(t)=(X(t),Y(t),Z(t))$ is the particle trajectory and $A(y,z)$ is the (y,z)-dependent part of the spatial wave function.

The measurement is completed at time T_x, say, when A_x^{up} and A_x^{down} are macroscopically separated. Then X is postselected in the support of A_x^{up} yielding

$$\gamma[\Psi_{\mathbf{R}}]=\arg(\langle\uparrow_z|P_{\uparrow_x}|\uparrow_z\rangle) + \arg[A_{tot}(X(T_x),Y(T_x),Z(T_x),T_x)] + i\int_{t_x}^{T_x}\left\langle\frac{\Psi_{\mathbf{R}(t)}(t)}{\|\Psi_{\mathbf{R}(t)}(t)\|}\left|\frac{d}{dt}\right|\frac{\Psi_{\mathbf{R}(t)}(t)}{\|\Psi_{\mathbf{R}(t)}(t)\|}\right\rangle dt, \qquad (10)$$

where $A_{tot}=AA_x^{up}$. Clearly the first term on the right-hand side (RHS) is zero. The other terms are $\mathbf{R}$ dependent and therefore random valued, because the $\mathbf{R}$ value is randomly distributed according to $|\Psi(\mathbf{r},T_x)|^2$. Similarly, by postselecting X and Y in support of A_ϕ^{up} and Z in support of A_z^{up}, the geometric phase for the complete sequence becomes

$$\gamma[\Psi_{\mathbf{R}}]=\arg(\langle\uparrow_z|P_{\uparrow_\phi}P_{\uparrow_x}|\uparrow_z\rangle) + \arg[A_{tot}(X(T),Y(T),Z(T),T)] + i\int_{t_x}^{T}\left\langle\frac{\Psi_{\mathbf{R}(t)}(t)}{\|\Psi_{\mathbf{R}(t)}(t)\|}\left|\frac{d}{dt}\right|\frac{\Psi_{\mathbf{R}(t)}(t)}{\|\Psi_{\mathbf{R}(t)}(t)\|}\right\rangle dt = -\frac{1}{2}\phi+\gamma(\mathbf{R}), \qquad (11)$$

where T is the final time. The first term on the RHS of Eq. (11) is the Samuel-Bhandari phase γ_{SB}. The phase $\gamma(\mathbf{R})$ is a random contribution to the geometric phase. Generally for a cyclic sequence of measurements of first kind we have

$$\gamma[\Psi_W]=\gamma_{SB}+\gamma(W), \qquad (12)$$

where both γ_{SB} and $\gamma(W)$ are projective geometric and are therefore physical quantities of the measured system. The appearance of γ_{SB} in Eq. (12) can be understood as being a consequence of the emergent ‘‘collapse’’ of the system wave function as depicted by Eq. (6).

As an illustration it is sufficient to consider the x measurement above. Take the spatial wave function as Gaussian of fixed width σ (spreading is neglected) and use the impulse approximation during the passage of the particle through the Stern-Gerlach field yielding

$$\dot{X}=u\ \tanh\left(\frac{utX}{\sigma^2}\right), \qquad (13)$$

where $2u$ is the speed of the center of A_x^{up} relative the center of A_x^{down}. The solutions $X(t)=X(t;X(0))$ of Eq. (13) determine the total (γ_{tot}) and dynamical (γ_{dyn}) phases

$$\gamma_{tot}(t)=\arctan\left[\tan[u(X(t)-X(0))]\tanh\left(\frac{utX(t)}{2\sigma^2}\right)\right],$$

$$\gamma_{dyn}(t)=u^2\int_0^t\tanh^2\left(\frac{ut'X(t')}{\sigma^2}\right)dt', \qquad (14)$$

where we have put $\hbar=1$ and $t_x=0$. First we choose $(\sigma,u)=(0.1,1)$. In Fig. 1 γ_{tot} and γ_{dyn} associated with the trajectory ending at $X(0.4)=0.4$ [center of $A_x^{up}(x,0.4)$], together with the geometric phase $\gamma[\Psi_{\mathbf{R}}]=\gamma_{tot}-\gamma_{dyn}$, are shown as a function of t. We note that $\gamma[\Psi_{\mathbf{R}}]$ tends to an asymptotic value. This follows from the emergent reduction

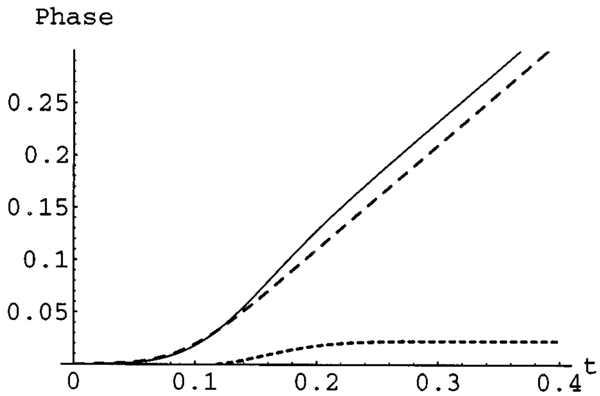

FIG. 1. Total (solid), dynamical (dashed), and geometric (dotted) phases as a function of time associated with the trajectory ending at $X(0.4)=0.4$ and $(\sigma,u)=(0.1,1)$.

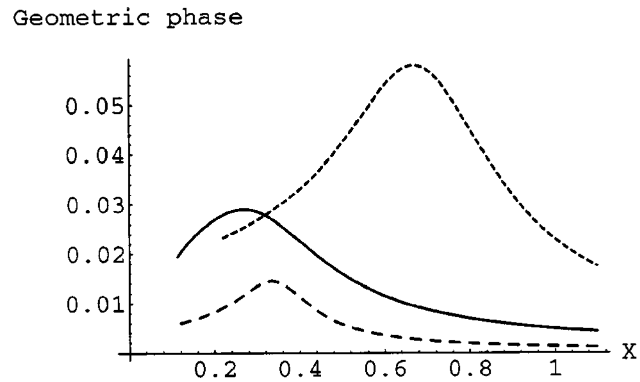

FIG. 2. Geometric phase as a function of postselected X value. The chosen parameter values are $(\sigma,u)=(0.1,1)$ (solid), $(\sigma,u)=(0.05,1)$ (dashed), and $(\sigma,u)=(0.1,2)$ (dotted).

of the wave function: In the limit where A_x^{up} and A_x^{down} are macroscopically separated ($ut \gg \sigma$) and X is in the support of A_x^{up}, which together implies $utX(t)/(2\sigma^2) \gg 1$, we obtain $\dot{\gamma}_{tot} \rightarrow u\dot{X} \approx u^2$ and $\dot{\gamma}_{dyn} = u^2 \arctan[utX(t)/\sigma^2] \rightarrow u^2$, which implies $\dot{\gamma}[\Psi_{\mathbf{R}}] \rightarrow 0$. In Fig. 2 the asymptotic geometric phase as a function of the postselected X value is shown for three different values of σ and u. The dependence on the random variable X is clearly seen. Furthermore, Fig. 2 shows that the size of the X-dependent contribution to the geometric phase depends on the experimental parameters (σ,u). In fact it appears that the difference between $\gamma[\Psi_{\mathbf{R}}]$ and γ_{SB} can be made small by narrowing the incoming wave packet.

To summarize we will make the following points.

(1) The Samuel-Bhandari phase can be identified as the nonrandom part of the geometric phase for the conditional wave function in the case of measurements. As the above calculation indicates, we expect the random part $\gamma(W)$ to be small if the experiment utilizes narrow apparatus wave packets.

(2) The unpredictable dynamics of the measurement process, which plays no role in the Samuel-Bhandari analysis (the nonrandom contribution to the total phase equals the geometric phase in their analysis), is explicitly taken into account.

(3) The geometric phase associated with the physical evolution of the conditional wave function during the measurement process does not coincide with the geodesic closure rule (Pancharatnam connection) applied to the sequence of postselected states.

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