Phase and transition-amplitude holonomy in optics

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The Pancharatnam phase in two-photon experiments is interpreted in terms of a pair of paths in the product space of polarization states. Vector-length holonomy in Weyl space is identified with the transition amplitude for a two-level orthogonally polarized light system. It is shown that the geometry of the Poincaré sphere admits Weyl structure. [S1050-2947(97)03407-0]

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I. INTRODUCTION

Phase holonomy in quantum mechanics [1] has led to intense activity in this field, and stimulated rediscovering Reetov-Vladimirskii [2] and Pancharatnam phases [3] in optics. Underlying geometrical and topological aspects have been known to be significant, however their relationship with the phase holonomy is not easy and straightforward. Experiments with classical light beams and with single-photon light fields have been reported, demonstrating the existence of geometrical phases in optics, however their origin, i.e., whether classical or quantal, has been controversial. Intensity and state of polarization describe a classical light beam; polarization ellipse and degree of polarization define the state of polarization. In 1892, Poincaré proposed a geometrical representation for perfectly polarized light in terms of the points on a spherical surface, noting that the intensity I and the Stokes parameters (M,C,S) satisfied the relation $I^2 = M^2 + C^2 + S^2$. The Pancharatnam phase is a manifestation of the geometry of the Poincaré sphere. Naturally, one might ask whether this is the most general geometry. Two recently reported experiments in optics seem to be important for answering this question. Moreover, they might have significant implications on fundamental physics. In one of the experiments, Brendel *et al.* [4] measure the Pancharatnam phase for a two-photon light field, and find that for an identically polarized photon pair it is twice the Pancharatnam phase corresponding to a single-photon light field, and for orthogonal states of polarization of the pair it is zero. This experiment confirms Klyshko's result [5], which showed the equivalence of a one-photon field geometric phase to the classical Pancharatnam phase, and n times of this phase for n identically polarized photons per mode of the light field. The question of geometrical structure is not addressed, and a simple approach using a rotationally symmetric Hamiltonian as a function of spin is used to interpret this result in [4]. In the next section we discuss this experiment and explain the results in terms of a pair of paths in a product state space of the Poincaré sphere.

Another class of experiments [6] studies transitions in an optical two-level system driven by time-varying optical elements. This work is inspired by the Landau-Zener transitions in a quantum system, and recently proposed modifications in the transition probability due to Berry phase [7] for suitably chosen Hamiltonian curves. Authors of [6] have given an elegant Jones matrix formulation in optics corresponding to the quantum-mechanical description of a two-level transition amplitude. Obviously the geometry of the Poincaré sphere is inadequate to describe the transition probability. Noting that the multivalued energy eigenvalue is crucial for the Hamiltonian curves considered in [6,7], we propose a generalized Poincaré sphere to admit Weyl structure in Sec. III to interpret the transition amplitude as vector-length holonomy. This idea [8] seems quite natural since the geometry of the Poincaré sphere restricts the Weyl connection to be closed, i.e., the curvature 2-form is zero. This implies that only for multiply connected space can the transitions occur.

Multiphoton experiments such as those in [4], and the quantum analog of the observed transition probability in two-state systems in optics [6], are important to understand the nonclassical nature of light, if any. Of equal importance is the problem of the physical mechanism responsible for these effects. Since polarization of the light beams is made to change in both experiments, the role of angular momentum exchange could be crucial to affect such changes. In the final section we present a discussion on this aspect, together with concluding remarks.

II. PHASE AND GEOMETRY OF PATHS

In order to make the discussion self-contained, we first give a brief description of the experiment reported by Brendel *et al.* [4]. A time-correlated photon pair is generated by down-conversion in a beta barium borate crystal. Interferograms are recorded in a Michelson interferometer setup using avalanche photodiodes as photon detectors. The geometric phase is introduced using two quarter-wave plates placed in one arm of the interferometer. Interference patterns for second-order interference and fourth-order interference are measured by adjusting the path difference in relation to the coherence length of the light. In one set of experiments, a fourth-order interference pattern is observed to be the product of single-photon interferences. Using large path differences and a time-resolved coincidence detection scheme, the singe-photon effect is excluded in another set of experiments. Following the famous statement of Dirac, Brendel, Dultz, and Martienssen [4] say that "a photon pair interferes with itself." It is in this set of experiments that a geometric phase twice that of the classical value for parallel polariza

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tion or zero for orthogonal polarization of a two-photon state is observed.

The authors of [4] insist that the observed interference is quantum-optical for a two-photon state. In an earlier paper [9] Brendel, Mohler, and Martienssen argued that although fourth-order interference did not necessarily imply a quantum origin, with observed visibility above 50% by eliminating background intensity in the time-resolved detection, entangled temporal correlation of the photons in the pair determined interferogram. The same argument is used in [4], however the simultaneous assumption of an independent photon in the pair is used to explain the first set of experiments. How can an entangled two-photon state allow for uncorrelated independent photon states?

Assuming quantum-optical interference, the question arises whether the observed phase is a quantum-optical phase. It is known that in second quantized field theory in which the number operator makes sense, a conjugate Hermitian phase operator does not exist. Recent constructions of Hermitian phase operators in finite-dimensional state space also do not relate quantum optics measurements with the phase operators. Since phase and number operators do not commute, one usually adopts a complementary description of the quantum field. Essentially the interference pattern is an intensity measurement, and a quasiclassical approach (phase space or operational in the categorization of Barnett and Dalton [10]) is implicit in [4].

Returning to the problem of geometric phase, by its very definition this phase has a sort of absolute characteristic. The state space of polarization cycles is the Poincaré sphere, S^2. A vector parallel transported around a given curve on S^2 is rotated through an angle equal to the solid angle subtended by the area enclosed by the curve. The parallel transport law for a vector $\vec{s}$ is given by

$$\frac{ds^i}{dt}+\Gamma^i_{jk}\frac{dx^j}{dt}s^k=0, \tag{1}$$

where $x^i(t)$, $a\leqslant t\leqslant b$ defines a curve on S^2 and Γ^i_{jk} is the Christoffel symbol. This much is known from differential geometry.

In the case of the Poincaré sphere, any point on the surface represents a definite state of polarization of monochromatic light. Pancharatnam's original derivation [3] is based on the physical transmission of polarized light through analyzer P and interference phenomenon, using spherical trigonometry to calculate the net phase difference between the polarization states P_1 and P_2. Pancharatnam's theorem can be stated as follows: the geometrical phase depends on the solid angle Ω of the triangle P_1P_2P on the Poincaré sphere given by

$$\Gamma=-\tfrac{1}{2}\Omega(P_1P_2P). \tag{2}$$

Derivation of Eq. (2) using the Hermitian polarization matrix

$$H=\hat{r}\cdot\vec{\sigma}, \tag{3}$$

where $\vec{\sigma}$ is the Pauli spin matrix and $\hat{r}$ is a unit vector parametrized by polar angles (θ,ϕ), has been given by Berry [11] analogous to the derivation of the Berry phase in quantum mechanics. The appearance of a factor of $\frac{1}{2}$ in Eq. (2) is explained in terms of the two states of the polarization on the Poincaré sphere. But, identification of SU(2) group transformations on the Poincaré sphere is not proper, despite the fact that one has a two-state polarization system of light. In fact, the Jones matrix is an appropriate description for this, and one gets the correct phase relation (2) in this approach [12]. It is known that SU(2) has a 2:1 homomorphism with the rotation group SO(3) in real space, while the Poincaré sphere represents polarization states such that one direction, usually the z axis, is fixed. Therefore, for a spin-half particle, e.g., electron or neutron, the SU(2) group is quite natural, but not in the case of light. Rotation group SO(3) naturally leads to phase holonomy of Reetov-Vladimirskii for light.

This problem of a factor of $\frac{1}{2}$ in Eq. (2) is resolved by noting that the Poincaré sphere has the spherical polar coordinates $r,\theta,\phi,$ where

$$r=I,\qquad \theta=2\zeta,\qquad \phi=2\arctan\eta. \tag{4}$$

ζ is the azimuth of the polarization ellipse and $(1-\eta)$ is the ellipticity. The radius vector to a point on the sphere is called the polarization eigenvector. To derive the Pancharatnam phase we proceed in two steps: (i) calculate the phase holonomy in the complex structure, and (ii) since the phase corresponds to the electric field vector, take the square root of the phase factor $\exp(-i\Omega)$, since this corresponds to the complex representation of intensity. This gives the correct value (2). Since Pancharatnam used intensity and phase for a light beam in the standard way, this anomaly did not arise in his derivation.

The experimental result in [4] is formally written as $\frac{1}{2}\sigma\Omega(c)$, where σ=2 for parallel and σ=0 for orthogonal states of polarization are identified with the measured values. We have explained that the geometry of paths on S^2 unambiguously gives rotation equal to $\Omega(c)$, and the role of spin does not arise for the Poincaré sphere. Therefore, a geometrical generalization is to consider a pair of paths in a product space $S^2\times S^2$. A physically motivated definition for any polarization correlated light beams traversing circuits C_1 and C_2 can be stated as

$$\Gamma=\tfrac{1}{2}[\Omega^2(C_1)+\Omega^2(C_2)+2\Omega(C_1)\Omega(C_2)\cos 2\alpha]^{1/2}, \tag{5}$$

where α is the angle between the polarization vectors of photons. In the usual homotopy theory of topological spaces, products of loops and paths can be defined that are not generally commutative, however this cannot be adopted directly in our case since we have a pair of paths in a product space. In fact, Aitchison and Wanelik [13] define a complex geometric phase in terms of a pair of paths in a single state space. Equation (5) can be understood by first recalling that on the Poincaré sphere the polar angle θ is equal to twice the azimuth of the polarization ellipse; see Eq. (4). Therefore, the product space has one of the spheres rotated 2α with respect to the other one. Parallel transport of the polarization vectors completing circuits C_1 and C_2 on these spheres gives direction holonomy, the composite of which can be formally written as a scalar product giving Eq. (5).

Equation (5) is proposed to be valid for arbitrary polarization of photons in the pair, and since $\alpha=0$ ($\pi/2$) for parallel (orthogonal) polarizations, we get the experimental values $\Omega(C)$ (0) reported in [4]. Klyshko gave a quantum generalization of the Jones matrix calculus using the idea of a multiphoton polarization vector [5]. In general, for an n-photon state the projection space of the polarization vector is S^{2n}, therefore classical correspondence is not obvious. If all n photons have the same polarization, then the geometric phase is n times the classical phase. Experiments in [4] are specialized to such a situation, thus being indistinguishable from a classical interpretation. In contrast, definition (5) is more general for a two-photon light field, and does not require quantum-optical description. Though Pancharatnam used great circles on the Poincaré sphere to derive his theorem, recent calculations by Berry and Klein [14] show that the geometrical phase for arbitrary paths (including small circles) obeying Eq. (2) is valid in crystal optics. Geometrical origin of Eq. (5) thus ensures its general validity and experimental testability.

III. GEOMETRY OF TRANSITION PROBABILITY

An optical analog of the transition probability for a twisted Landau-Zener model has recently been demonstrated [6]. This is an interesting experiment for two reasons. First, a classical optical system exhibits tunneling effects, and second, local aspects of the geometrical phase can be studied. The two-level system is formed out of two orthogonal polarization states of a single longitudinal mode of an optical ring cavity. Time-dependent voltages applied to three electro-optic modulators simulate a Hamiltonian curve, for which the authors of [6] choose a twist function for the Gaussian twisted Landau-Zener model. Laser light with definite polarization is injected into the ring cavity, which is tuned in resonance with it. For a specific intracavity intensity level an acousto-optic modulator switches off the injection light. Time-dependent optical elements drive this state to an orthogonal polarization state within the cavity decay time.

The correspondence of the Jones matrix formalism for a two-level optical system and the Schrödinger formulation for a two-level quantum system driven by a time-dependent real symmetric 2×2 matrix Hamiltonian is the basis of their theoretical description. The authors are careful to point out the classical nature of their experiments. We ask the following question: is there a geometric description of the level transition amplitude? We will attempt here to show that there is.

The geometric rendition of the transition probability amplitude is inspired by the key role of the multivalued energy eigenvalue in this process. Intuitively, a multiply connected state space seems appropriate. In order to appreciate the significance of these observations, a brief discussion on Dykhne's formula is first outlined.

Let us consider a two-state nondegenerate quantum system described by the time-dependent Schrödinger equation with a time-dependent real symmetric 2×2 matrix Hamiltonian, $H(t)$. If the time variation of the Hamiltonian is very slow, the adiabatic theorem shows that the system initially in a state $|\Psi_i\rangle$ remains in this state for all time during its evolution. Dykhne's formula [15] gives a finite nonzero transition probability to the state $|\Psi_f\rangle$ having another eigenvalue of H. The transition amplitude does not depend on the nonadiabatic coupling responsible for the transition. Only the energy phase integrals near the crossing point of potential curves in the complex time plane are involved. For this reason, Hwang and Pechukas [15] remarked that ''Dykhne's formula is very simple, and very mysterious.'' Davis and Pechukas [15] have rigorously proved this formula, and Hwang and Pechukas prove a generalization of the adiabatic theorem in a complex time plane, and derive directly a nonadiabatic amplitude along the real time axis.

Joye *et al.* [7] expand on this by considering the Berry phase for a loop in the complex plane around the eigenvalue crossing. In a significant geometrical approach they introduce a metric constructed with the eigenvalues of H using the theory of Teichmueller spaces. This metric is shown to be useful for deciding the question of eigenvalue crossings. Berry [7] also obtained a geometric adiabatic amplitude for a complex Hermitian Hamiltonian. In Sec. III of his paper [7], it is shown that by a suitable transformation, the complex Hamiltonian is transformed into a real symmetric one, and Dykhne's formula is applicable. This point has been noted in Ref. [6] for optical experiments. Let us state our proposition [16]: the Weyl vector-length holonomy corresponds to the transition amplitude, and the Weyl space is a natural state space.

In the Weyl geometry one has the gauge transformations defined by

$$ds \rightarrow ds' = \Lambda ds, \tag{6a}$$

$$A_i \rightarrow A_i' = A_i + \partial_i(\ln\Lambda), \tag{6b}$$

where the metric is

$$ds^2 = g_{ij} dx^i dx^j \tag{7}$$

and $A_i dx^i$ is a linear ground form. Unlike the Riemannian space, in this space a vector under parallel displacement from point x^i to $x^i + dx^i$ undergoes changes in both direction and length, the length change for a closed path being given by

$$L' = L \exp\left(\oint A_i dx^i \right). \tag{8}$$

The proposition stated above reinterprets Eq. (8) by identifying L and L' to be the lengths of the state vector in state 1 and state 2, respectively. In quantum theory, the parent Hilbert state of the quantum states is $(N+1)$-dimensional complex vector space, and the physical state for the equivalence class of states under phase transformations is the projective Hilbert space isomorphic to the complex projective space CP^N. Since CP^N is Kähler, the natural generalization incorporating gauge transformations leads to the Weyl-Kähler geometry. For a simply connected Weyl-Kähler space, there is no vector-length holonomy, however non-simply connected Weyl-Kähler space admits such a structure; see [16] for details.

For a two-level optical system, let $|E\rangle$ and $|E'\rangle$ represent orthogonal polarization states. The transition amplitude using Eq. (8) is

$$\langle E|E\rangle = \exp\left(-\oint A_i dx^i\right)\langle E'|E'\rangle. \tag{9}$$

We first show that the polarization state space, i.e., the Poincaré sphere, does admit the Weyl structure. The natural metric on S^2 is induced by the Euclidean metric on R^3,

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2). \tag{10}$$

Introducing gauge transformations (6),

$$r = \exp(S), \tag{11a}$$

$$\Lambda = \exp(-S), \tag{11b}$$

$$A_i = -\partial_i S. \tag{12}$$

The 2-form F calculated from Eq. (12) is zero. Therefore, only the multivalued scalar function S will give a nonzero transition amplitude.

As an illustrative example, the multiply connected space is constructed from two hemispherical regions ($0 \leqslant \theta \leqslant \pi/2 + \epsilon$) and ($\pi/2 - \epsilon < \theta \leqslant \pi$), the metrics being related the scale transformation (11) such that $S = -B\phi$, with B a positive constant. Using Eq. (12), the length holonomy is calculated to be

$$\oint A_i dx^i = 2\pi BN. \tag{13}$$

N is an integer, and the transition probability becomes

$$P = \exp(-4\pi BN). \tag{14}$$

This result can be understood in terms of the analytic continuation of phase $e^{i\phi}$ near the crossing point at $\phi = \pi$ to the orthogonal polarization state, which lives on the second hemisphere, i.e., $|E\rangle \to |E'\rangle$. Paths in one region only give the Pancharatnam phase, and the scale transformation does not change the topology.

One can examine the Landau-Zener model and its variations geometrically. The Hamiltonian vector considered by Berry [7] is

$$H(\tau) = (\Delta\ \cos\phi(\tau), \Delta\ \sin\phi(\tau), A\tau). \tag{15}$$

The dynamical and geometrical parts in the exponent of P are treated separately. The Landau-Zener model has $\phi=0$. For this as well as for $\phi=\tau$, calculations show that the geometrical contribution to P is zero. The geometry of the Hamiltonian curve is, however, not the same. We distinguish between these two cases by considering the Weyl structure for S^1. The metric for S^1,

$$ds^2 = dr^2 + r^2 d\theta^2, \tag{16}$$

admits the Weyl structure, and the length holonomy in this case is also similar to Eq. (13) using $S = -B\theta$. Hamiltonian curve (15) for $\phi=0$ is topologically the same as S^1, while the uniform helix is locally $S^1 \times Z$. That both S^1 and S^2 are not simply connected topological spaces is the reason why one gets the transition amplitude holonomy. A useful reference on the topology of manifolds is [17]. An important point regarding the Weyl space needs to be understood since the objects appearing in Eqs. (9) and (13) look similar to the fiber bundles [17], and might cause confusion. Weyl's original theory was reinterpreted as a circle bundle over a Lorentzian manifold, so that instead of the gauge transformations one had complex phase transformations compatible with quantum mechanics. It is this version which has been used in non-Abelian generalizations and unified gauge theories of fundamental interactions. In the present paper, we have used the original Weyl space to allow vector-length changes under parallel transportation.

IV. DISCUSSION AND CONCLUSION

Experiments on the two-photon Pancharatnam phase [4] and transitions in two-level optical systems [6] have been analyzed in detail. Basically, the phenomenon regarding the changes in the polarization state of light is common to both sets of the experiments. Geometrically the Poincaré sphere represents the polarization state, and therefore a unified description has to be based on the geometry of S^2. We have introduced the idea of a pair of paths in a product Poincaré space to account for the result on the Pancharatnam phase reported in [4]. Equation (5) suggests coupling of the phases acquired in different paths for arbitrary polarization correlation, and contains the results obtained in [5] as special cases for identical and orthogonal polarizations.

In the present paper an interpretation of the Weyl connection representing the transition amplitude is proposed, and the geometry of the Poincaré sphere is shown to admit Weyl structure. In this Poincaré-Weyl space, the curvature 2-form is zero, therefore only for multi-valued scalar fields does one get nontrivial multiply connected space allowing transitions from one space to another. A simple illustrative example is given to obtain Eq. (14). The constant B is not determined by geometry, rather it is fixed by a physical problem. In the case of the Landau-Zener model, Eq. (15), the scalar field is calculated to be $S = \frac{1}{2}\ \ln(\Delta^2 + z^2)$ using Eq. (11a) and transforming Eq. (15) to a spherical coordinate system. Joye *et al.* [7] have given several geometries for which investigating Weyl structure would be interesting.

During the past few years, an entangled two-photon state has been used to study the foundational problems of quantum mechanics. In the final section of their paper [4], Brendel, Dultz, and Martienssen have given a tentative suggestion that their work on the geometric phase of two-photon light fields could be used for testing Bell's inequality and quantum nonlocality. In the present paper we have argued that the geometric phase for a two-photon state also has a geometrical interpretation rendering quantum-mechanical description unnecessary. In the optical level transitions, Bouwmeester and co-workers [6] indicate that that precise time evolution of the optical wave function is measurable in contrast to the problem of the collapse of quantum-mechanical state evolution. However, they note that Planck's constant in the description of the optical dynamics is a superfluous constant. In fact, by its very nature the state function in [6] is classical, therefore it cannot give information regarding histories of adiabatic quantum transitions. In quantum optics there is a problem in distinguishing a quantum state from a classical one [18]. In another context, Suter [19] demonstrated an optical analog of a ''quantum time-translation machine'' experimentally and explained it classically. Bouwmeester *et al.* [6] are careful to point out that although the Landau-Zener transition is supposed to be a quantum-mechanical effect, classical waves can exhibit tunneling. It is reasonable to conclude from this discussion that apparently both experiments [4,6] invoke quantum mechanics, but do not yield an unambiguous quantum-mechanical effect.

Instead of the approach adopted to study fundamental questions on quantum mechanics, we suggest that the role of angular momentum exchange in these experiments may be crucial [18]. Indeed, it is surprising that the mean value of the angular momentum's projection, Eq. (9) of Ref. [5], arises in the description of the geometric phase, but its role has not been sufficiently stressed. Van Enk calculated the Pancharatnam phase for transformations of Gaussian light beams [20] and confirmed the suggestion of angular momentum exchange [18]. Further plausibility argument can be given by an analogy. Provost and Vallee [21] considered the Riemannian structure of the quantum state space, and calculated metrics for some illustrative examples. The metric for the atomic coherent states having an angular momentum component equal to J in the (θ,ϕ) direction is given by

$$ds^2=\frac{J}{2}[d\theta^2+(\sin^2\theta)d\phi^2]. \qquad (17)$$

It has a Riemannian structure of S^2 with scalar curvature equal to $2/J$. Comparing with Eq. (10), a gauge field appears to affect angular momentum transfer. In optical experiments involving geometric phases, polarization cycles indicate angular momentum transfer to the optical elements [8,18], while in level transitions, time-dependent optical modulators [6] may transfer angular momentum from one state of light to another. It may be noted that there is a renewed interest in the problem of the angular momentum of light; see [22] for further references. The physical significance of the separation of spin and orbital parts of the angular momentum of light beams, and the meaning of the spin of photons, are being debated. Any attempt to visualize a light beam in terms of constituent photons necessarily leads to the following questions: Is a light field some kind of photon fluid? Does the internal structure of a photon make sense? Post has emphasized the role of angular momentum quantization in the early developments of quantum theory [23]. It has also been pointed out that rather than action, it would be more useful to treat Planck's constant as an angular momentum unit. An interesting work on the de Broglie wavelength of light beams [24] shows that the wavelength depends on the internal structure of the Bose condensate of photons. A possible approach to model the photon as an extended space-time object with internal structure has also been outlined [25]. It seems that the questions related to the optical phase holonomy and the phase of a photon could be significant for assessing such speculations.

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