Traveling and standing waves in a laser with an injected signal

Stefano Longhi
Dipartimento di Fisica, Politecnico di Milano, Piazza Leonardo da Vinci 32, Milano 20133, Italy
(Received 28 February 1997)

A complex order parameter description of the spatiotemporal dynamics close to threshold for a single-longitudinal-mode, large-aspect-ratio laser with a weak injected signal is presented in one transverse spatial dimension and for a longitudinal cavity frequency smaller than the atomic resonance frequency. It is shown that, when the frequency of the external signal is tuned close to the atomic resonance frequency, the spatiotemporal dynamics of the laser may be described by two coupled Ginzburg-Landau equations with parametric terms analogous to those derived in hydrodynamics to study excitation of parametric waves in oscillatory convection. As a result of the parametric terms, which physically arise by means of a four-wave-intercation process mediated by the external signal, the bifurcating modes which are superimposed to the forced (homogeneous) mode may be either traveling or standing waves. In particular, it is shown that for small frequency detunings of the injected signal standing waves are preferred to traveling waves close to threshold. Stability analysis of traveling and standing waves beyond threshold within the amplitude equations reveals the emergence of both phase (Eckhaus) and amplitude instabilities. [S1050-2947(97)01308-5]

PACS number(s): 42.65.Sf, 42.60.Mi

I. INTRODUCTION

Pattern formation in nonlinear optics has attracted much interest in recent years and a lot of theoretical and experimental work has been done in this field [1]. In particular, inclusion of diffraction in the nonlinear dynamics of laser systems has revealed the appearance of symmetry-breaking bifurcations, spontaneous pattern formation, and complex space-time behavior. Although in laser cavities with a small Fresnel number the nonlinear dynamics is strongly influenced by external boundaries (i.e., spherical mirrors) and involves few transverse cavity modes [2], a universal description of the transverse laser dynamics may be obtained in the large-aspect-ratio limit, where pattern-forming properties are boundary free and depend only upon the intrinsic symmetries of the laser equations [3]. In the latter case the nonlinear dynamics of the laser may be reduced, under suitable approximations, to that of simpler equations having a universal form (order parameter equations). The first derivation of an order parameter equation for the single-longitudinal-mode laser equations was given by Coullet *et al.* [4], who showed that the laser dynamics close to threshold can be described by a complex Ginzburg-Landau equation whenever the selected longitudinal frequency is larger than the atomic resonance frequency (negative detuning). Although in this case the bifurcating solution is homogeneous in space, the authors showed that optical vortices may exist when diffraction is included in the equation. This analysis was extended by Newell and Moloney in the positive detuning case, where the laser emission spontaneously occurs off-axis because of a tilted-wave mechanism [3,5–7]. In this case the authors obtained coupled Newell-Whitehead-Segel-type equations in the vicinity of the laser threshold. In the one-dimensional case, such equations reduce to two coupled Ginzburg-Landau equations describing the competition between two counterpropagating traveling waves [5]. As shown in Ref. [5], a single tilted (traveling) wave is able to dominate, whereas the superposition of two counterpropagating traveling waves (a standing wave) is always unstable, collapsing to one of the two traveling waves, which are global attractors of the laser equations. The effect of a weak injected signal on the transverse laser dynamics was investigated by Mandel *et al.* [8], who derived in the negative-detuning case a complex Ginzburg-Landau equation for the order parameter. The derivation of an order parameter equation in the positive-detuning case was not given in a closed form in that paper. In Ref. [9], a numerical analysis of the laser equations with a weak injected signal was given in the one-dimensional case and for a positive detuning. The authors showed that the excitation of a single tilted wave is likely also when a weak signal is injected into the laser cavity, with the only difference that in this case the traveling wave is superimposed to the spatially homogeneous state forced by the injected signal. That analysis, however, assumes implicitly that the frequency of the injected signal is detuned from the atomic resonance frequency, which is exactly the frequency of the emerging tilted wave. When the frequency of the injected signal is tuned close to the atomic resonance frequency, a four-wave interaction with conservation of both transverse photon momentum and energy may occur among the forced mode and two counterpropagating traveling waves. Such an interaction, which is forbidden when the external signal is detuned from the atomic resonance frequency, is expected to introduce new features in the laser dynamics. From a mathematical point of view, this corresponds to the appearance in the amplitude equations of new resonant terms. In this paper we present a complex order parameter analysis of the laser dynamics with a weak injected signal in the positive-detuning case when the frequency of the external signal is close to the atomic resonance frequency. Using a weakly nonlinear analysis of the laser equations close to threshold, in Sec. II we generalize the amplitude equations describing the competition between traveling- and standing-wave patterns in one transverse spatial dimension previously derived by Jakobsen *et al.* [5]. When the frequency of the external signal is slightly detuned from the atomic resonance fre-

1050-2947/97/56(2)/1553(11)/$10.00

quency, parametric terms in the amplitude equations appear as a result of a four-wave-mixing process among these waves and the forced mode induced by the external signal. The structure of the resulting amplitude equations is analogous to that derived in hydrodynamics to study oscillatory convection [10–12] and, more generally, parametric excitation of waves in spatially extended systems [12,13]. The effects of the injected signal on the laser dynamics are studied in Sec. III. It is shown that the amplitude equations have two families of solutions with one degree of freedom (the transverse wave number), which, with the terminology used in oscillatory convection, correspond to traveling waves (TW's) and standing waves (SW's). Because of the parametric interaction, the TW solution does not correspond, however, to a pure right- or left-traveling wave as in the case of a laser without an injected signal [5], but to a mixed-mode solution. While the TW solutions exist for any transverse wave number and their frequency is not locked to that of the external signal, the SW solutions exist only in a narrow interval of the transverse wave numbers and their frequency is locked to that of the external signal. Depending on the magnitude of the detuning between the atomic resonance frequency and the injected signal frequency, the bifurcating mode may be either a TW or a SW. When a SW is selected at the threshold, a secondary bifurcation to a TW state occurs as the pump parameter is increased. The stability of TW and SW solutions is investigated by use of standard linear stability methods, and it is shown that both amplitude and phase instabilities may appear. As a general rule, it turns out that the stability region of SW's in the parameter space lies below the curve of existence of TW's, and it shrinks as the frequency detuning between the external signal and the atomic resonance is increased due to the appearance of an amplitude instability. In particular, SW's are always unstable at high values of the frequency detuning. The stability region of TW's is unbounded in the parameter space and, away from the wave number region where SW's exist, it reduces to the usual parabolic Eckhaus domain.

II. WEAKLY NONLINEAR ANALYSIS OF THE LASER EQUATIONS AND DERIVATION OF THE AMPLITUDE EQUATIONS

The starting point of our analysis is provided by the set of the Maxwell-Bloch equations for a single-longitudinal-mode, homogeneously broadened two-level laser with flat mirrors, extended to include the injection in the cavity of a plane-wave field of frequency close to the atomic transition frequency [6–9]. In one transverse spatial dimension and using the same notations as in Ref. [7], they read

$$\partial_t e = i(\Omega-\nu)e + ia\partial_x^2 e - \sigma e + \sigma p + E, \qquad (1)$$

$$\partial_t p = -p - i\nu p + (r-n)e, \qquad (2)$$

$$\partial_t n = -bn + \tfrac{1}{2}(pe^* + p^* e). \qquad (3)$$

In these equations, e and p are the normalized slowly varying envelopes of the electric and polarization fields, respectively, n is proportional to the difference between the atomic inversion and the unsaturated inversion, a is the diffraction parameter, r is the pump parameter, σ and b are, respectively, the decay rates of the electric field (γ_0) and of the population inversion (γ_1), both scaled to the decay rate of the polarization (γ_2), t is the time variable scaled to $1/\gamma_2$, and E is proportional to the amplitude of the plane-wave field externally injected in the cavity. Without loss of generality, we will assume that the amplitude E is a real and positive number. In writing Eqs. (1)–(3), the reference frequency of the field-matter envelopes has been chosen coincident with that of the injected signal, and the detuning parameters Ω and ν are defined by

$$\Omega = \frac{\omega_A - \omega_C}{\gamma_2}, \qquad \nu = \frac{\omega_S - \omega_A}{\gamma_2}, \qquad (4)$$

where ω_A, ω_C, and ω_S are, respectively, the atomic resonance frequency, the longitudinal cavity frequency, and the injected signal frequency. As in Ref. [8], we consider the case of a weak injected signal by assuming in Eq. (1), $E = E_0\epsilon$, ϵ being a small parameter. At leading order in ϵ, Eqs. (1)–(3) reduce therefore to those studied in Refs. [5,14]. In this case, it is known that, assuming the pump parameter as a bifurcation parameter, for a longitudinal cavity frequency smaller than the atomic resonance frequency ($\Omega>0$) the laser threshold is reached at $r=1$ and the neutral modes are two counterpropagating TW's with transverse wave number $k = (\Omega/a)^{1/2}$; in the opposite case ($\Omega<0$), the threshold is reached at $r=(1+\Omega^2)^{1/2}$ and the bifurcating mode is homogeneous in space. Following the analysis of Ref. [8], the effect of a weak injected signal on the laser dynamics close to threshold may be investigated by using a multiple-scale perturbation expansion of the laser equations. The negative detuning case was investigated in Ref. [8] for different scalings of the signal detuning from resonance, and a complex Ginzburg-Landau equation for the order parameter was derived. As showed by Newell and Moloney [3], the derivation of the amplitude equations in a closed form is possible also in the positive detuning case, which was not considered in Ref. [8]. Here we concentrate therefore on the derivation of the amplitude equations in the positive detuning case. As we want to study the influence of the inejcetd signal on the nonlinear selection mechanism of TW's, we consider a pump parameter r close to its threshold value by assuming $r=1+\epsilon^2$. Such a dependence of r on ϵ ensures that the amplitude of the bifurcating modes and of the injected signal inside the cavity be of the same order of magnitude. The derivation of the amplitude equations is based on a standard weakly nonlinear analysis of Eqs. (1)–(3) similar to those developed in Refs. [6–8] and consists in looking for a solution of the laser equations as a power expansion in ϵ,

$$\mathbf{v} = \epsilon\mathbf{v}^{(1)} + \epsilon^2\mathbf{v}^{(2)} + \epsilon^3\mathbf{v}^{(3)} + \cdots, \qquad (5)$$

where $\mathbf{v} = (e,p,n)^T$. The introduction of slow space and time variables follows from a straightforward analysis of the linear problem, which is discussed in Refs. [3,6]. In the one transverse spatial dimension, the right scaling for the spatial coordinate is $x = X_0 + \epsilon X_1$, where the slow spatial coordinate X_1 is introduced in order to include in the dynamical equations the continuous band of modes that become active above threshold. A multiple scale for the time variable is then introduced by setting $t = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots$. As in Refs.

[5,6,8,9] we consider here a laser system belonging to the so-called class C, for which the decay rates γ_0, γ_1, and γ_2 for the cavity field, population inversion, and polarization are of the same order of magnitude. Using the previous scaling and the derivative rules $\partial_x^2=\partial_{X_0}^2+2\epsilon\partial_{X_0}\partial_{X_1}+\epsilon^2\partial_{X_1}^2$, $\partial_t=\partial_{T_0}+\epsilon\partial_{T_1}+\epsilon^2\partial_{T_2}+\cdots$, substitution of expansion (5) into Eqs. (1)–(3) yields a hierarchy of equations for successive corrections to $\mathbf{v}$. The amplitude equations then arise combining the solvability conditions in the perturbation expansion at orders ϵ^2 and ϵ^3. To further proceed in the perturbation analysis, two cases have to be distinguished depending on the order of magnitude of the detuning parameter ν for the external signal.

A. Detuned injected signal

In this case the detuning parameter ν is assumed to be of $O(1)$. The equations at $O(\epsilon)$ read

$$\mathcal{L}\mathbf{v}^{(1)}=(E_0,0,0)^T, \tag{6}$$

where the linear operator $\mathcal{L}$ is given by

$$\mathcal{L}=\begin{pmatrix} \partial_{T_0}-i(\Omega-\nu)-ia\partial_{X_0}^2+\sigma & -\sigma & 0 \\ -1 & 1+i\nu+\partial_{T_0} & 0 \\ 0 & 0 & -b \end{pmatrix}. \tag{7}$$

As the linear operator $\mathcal{L}$ is singular, the solutions of Eq. (6) are given by the superposition of the solution for the homogeneous problem and of the forced solution, and read

$$e^{(1)}=B_1\exp(ikX_0-i\nu T_0)+B_2\exp(-ikX_0-i\nu T_0)+B_0,$$

$$p^{(1)}=B_1\exp(ikX_0-i\nu T_0)+B_2\exp(-ikX_0-i\nu T_0)+\frac{B_0}{1+i\nu},$$

$$n^{(1)}=0,$$

where

$$B_0=\frac{(1+\nu^2)E_0}{\sigma\nu^2+i[\sigma\nu+(\nu-\Omega)(1+\nu^2)]}$$

is the amplitude of the forced mode, $B_{1,2}$ are the amplitudes (depending on slow space-time variables) of the counter-propagating TW's, and $k=(\Omega/a)^2$ is their transverse wave number. The equations at order ϵ^2 are

$$\mathcal{L}\mathbf{v}^{(2)}=\mathbf{g}^{(2)}, \tag{8}$$

where $\mathbf{g}^{(2)}$ depends on functions of previous approximation and is given explicitly by

$$\mathbf{g}^{(2)}=(-\partial_{T_1}e^{(1)}+2ia\partial_{X_0}\partial_{X_1}e^{(1)},-\partial_{T_1}p^{(1)},\tfrac{1}{2}(e^{(1)}p^{(1)*}+e^{(1)*}p^{(1)}))^T.$$

As the operator $\mathcal{L}$ is singular, to avoid secular terms in Eq. (8) the driving term $\mathbf{g}^{(2)}$ must be orthogonal to the two singular eigenvectors $\mathbf{u}_{1,2}=(1,\sigma,0)^T\exp(\pm ikX_0-i\nu T_0)$ of the adjoint problem. This yields the solvability conditions

$$(1+\sigma)\partial_{T_1}B_{1,2}=\mp 2ak\partial_{X_1}B_{1,2}, \tag{9}$$

and the solution at $O(\epsilon^2)$ may be chosen as

$$e^{(2)}=0,$$

$$p^{(2)}=\frac{1}{\sigma}\partial_{T_1}e^{(1)}-\frac{2ia}{\sigma}\partial_{X_0}\partial_{X_1}e^{(1)},$$

$$n^{(2)}=\beta_1+\beta_2\exp(i\nu T_0)+\beta_2^*\exp(-i\nu T_0),$$

where

$$\beta_1=\frac{1}{b}\left[|B_1|^2+|B_2|^2+\frac{|B_0|^2}{1+\nu^2}+B_1B_2^*\exp(2ikX_0)+B_1^*B_2\exp(-2ikX_0)\right],$$

$$\beta_2=\frac{2+i\nu}{2b(1+i\nu)}[B_1^*B_0\exp(-ikX_0)+B_2^*B_0\exp(ikX_0)].$$

At order ϵ^3 we obtain the equations

$$\mathcal{L}\mathbf{v}^{(3)}=\mathbf{g}^{(3)},$$

where the driving term is given by

$$\mathbf{g}^{(3)}=(-\partial_{T_2}e^{(1)}+ia\partial_{X_1}^2e^{(1)},-\partial_{T_1}p^{(2)}-\partial_{T_2}p^{(1)}+e^{(1)}-n^{(2)}e^{(1)},\tfrac{1}{2}(e^{(1)}p^{(2)*}+e^{(1)*}p^{(2)}))^T.$$

The solvability conditions at this order yield

$$(1+\sigma)\partial_{T_2}B_{1,2}=ia\partial_{X_1}^2B_{1,2}-\partial_{T_1}^2B_{1,2}\mp 2ak\partial_{T_1}\partial_{X_1}B_{1,2}+\sigma B_{1,2}-\frac{\sigma|B_0|^2}{2b}\left(\frac{4+\nu^2+i\nu}{1+\nu^2}\right)B_{1,2}-\frac{\sigma}{b}(|B_{1,2}|^2+2|B_{2,1}|^2)B_{1,2},$$

which using Eq. (9) read, explicitly,

$$(1+\sigma)\partial_{T_2}B_{1,2}=(\kappa+ia)\partial_{X_1}^2 B_{1,2}+\sigma B_{1,2} - \frac{\sigma|B_0|^2}{2b}\left(\frac{4+\nu^2+i\nu}{1+\nu^2}\right)B_{1,2} - \frac{\sigma}{b}(|B_{1,2}|^2+2|B_{2,1}|^2)B_{1,2}, \qquad (10)$$

where the diffusion coefficient κ is given by

$$\kappa=\frac{4\sigma\Omega a}{(1+\sigma)^2}. \qquad (11)$$

The total time derivative of the amplitudes B_1 and B_2 is obtained by collecting all terms and looks like $\partial_t B_{1,2} = \epsilon\partial_{T_1}B_{1,2}+\epsilon^2\partial_{T_2}B_{1,2}$. Reintroducing the original variables $E=\epsilon E_0$, $r=1+\epsilon^2$, redefining $\epsilon B_{1,2}$ as $B_{1,2}$, and observing that $\partial_x B_{1,2}=\epsilon\partial_{X_1}B_{1,2}$, from Eqs. (9) and (10) we finally obtain the amplitude equations

$$(1+\sigma)\partial_t B_1+2\sqrt{a\Omega}\,\partial_x B_1=(ia+\kappa)\partial_x^2 B_1+\sigma(r-1)B_1 - \frac{\delta}{2b}|E|^2 B_1-\frac{\sigma}{b} \times(|B_1|^2+2|B_2|^2)B_1, \qquad (12)$$

$$(1+\sigma)\partial_t B_2-2\sqrt{a\Omega}\,\partial_x B_2=(ia+\kappa)\partial_x^2 B_2+\sigma(r-1)B_2 - \frac{\delta}{2b}|E|^2 B_2-\frac{\sigma}{b} \times(|B_2|^2+2|B_1|^2)B_2, \qquad (13)$$

where

$$\delta=\frac{\sigma(1+\nu^2)(4+\nu^2+i\nu)}{\sigma^2\nu^4+[\sigma\nu+(\nu-\Omega)(1+\nu^2)]^2}.$$

The electric field in the cavity is then given by

$$e(x,t)=B_1\exp(ikx-i\nu t)+B_2\exp(-ikx-i\nu t) + \frac{(1+\nu^2)E}{\sigma\nu^2+i[\sigma\nu+(\nu-\Omega)(1+\nu^2)]}. \qquad (14)$$

The amplitude equations (12) and (13) are of Ginzburg-Landau type and they reduce to those obtained in Ref. [5] in the absence of the injected signal, i.e., for $E=0$. In this case it is known that stable solutions of the amplitude equations are pure TW's ($B_1=0$ and $B_2\neq 0$ or vice versa), SW's being always unstable collapsing to one of the two TW's. Furthermore, the stability of TW's within the amplitude equations is determined by the usual Eckhaus criterion [6,14]. The effect of an external signal detuned from the atomic resonance frequency is merely to modify the linear gain and the dispersion relation of TW's through the parameter δ in the amplitude equations. These new terms, however, do not influence the stability properties of TW's, which remain stable attractors of the amplitude equations. The emerging TW, however, is now detuned from the atomic resonance frequency by the amount $\mathrm{Im}(\delta)|E|^2/2b(1+\sigma)$, and it is superimposed to the forced mode which oscillates at a different frequency [see Eq. (14)]. These results are in agreement with the numerical simulations previously presented in Ref. [9]. It should be noted that, although in the amplitude equations singularities do not emerge when considering the resonant limit $\nu\to 0$, they fail to describe the nonlinear dynamics of the original equations as new secular terms appear in the perturbation expansion at $O(\epsilon^3)$. Physically, these terms originate as a four-wave interaction with conservation of both energy and photon momentum in the transverse plane is now allowed.

B. Near-resonant injected signal

We consider an injected signal whose frequency is close to the atomic resonance frequency by making the detuning ν of order ϵ, i.e., we assume $\nu=\epsilon\nu_0$. In this case, the problem at $O(\epsilon)$ in the perturbation expansion becomes

$$\mathcal{L}\mathbf{v}^{(1)}=(E_0,0,0)^T,$$

where the linear operator $\mathcal{L}$ is given by Eq. (7) where ν is set equal to zero. The solution at this order is given by

$$e^{(1)}=B_1\exp(ikX_0)+B_2\exp(-ikX_0)+B_0,$$

$$p^{(1)}=e^{(1)}, \quad n^{(1)}=0,$$

where $B_0=iE_0/\Omega$ is the amplitude of the forced mode and $B_{1,2}$ are the amplitudes of the two counterpropagating TW's. At $O(\epsilon^2)$ one obtains

$$\mathcal{L}\mathbf{v}^{(2)}=(-\partial_{T_1}e^{(1)}+2ia\partial_{X_1}\partial_{X_0}e^{(1)}-i\nu_0 e^{(1)},-\partial_{T_1}e^{(1)} -i\nu_0 e^{(1)},|e^{(1)}|^2)^T.$$

The solvability conditions at this order yield

$$(1+\sigma)\partial_{T_1}B_{1,2}=\mp 2ak\partial_{X_1}B_{1,2}-i(1+\sigma)\nu_0 B_{1,2}, \qquad (15)$$

and the solution at this order may be chosen as

$$e^{(2)}=0, \quad p^{(2)}=-\partial_{T_1}e^{(1)}-i\nu_0 e^{(1)}, \quad n^{(2)}=\frac{|e^{(1)}|^2}{b}.$$

Finally, the problem at $O(\epsilon^3)$ is

$$\mathcal{L}\mathbf{v}^{(3)}=\mathbf{g}^{(3)},$$

where

$$\mathbf{g}^{(3)}=(-\partial_{T_2}e^{(1)}+ia\partial_{X_1}^2 e^{(1)},-\partial_{T_2}e^{(1)}-\partial_{T_1}p^{(2)}-i\nu_0 p^{(2)} +e^{(1)}-e^{(1)}n^{(2)},\tfrac{1}{2}(p^{(2)}e^{(1)*}+p^{(2)*}e^{(1)}))^T.$$

The solvability conditions at this order read

$$(1+\sigma)\partial_{T_2}B_{1,2}=ia\partial_{X_1}^2 B_{1,2}+\sigma(i\nu_0+\partial_{T_1})(\partial_{T_1}B_{1,2}+i\nu_0 B_{1,2}) +\sigma B_{1,2}-\frac{\sigma}{b}(|B_{1,2}|^2+2|B_{2,1}|^2+2|B_0|^2)B_{1,2} -\frac{\sigma}{b}B_0^2 B_{2,1}^*. \qquad (16)$$

Using the solvability conditions at $O(\epsilon^2)$ [Eq. (15)], Eq. (16) may be cast in the form

$$(1+\sigma)\partial_{T_2}B_{1,2}=(ia+\kappa)\partial_{X_1}^2 B_{1,2}+\sigma B_{1,2} -\frac{\sigma}{b}(|B_{1,2}|^2+2|B_{2,1}|^2+2|B_0|^2)B_{1,2} -\frac{\sigma}{b}B_0^2 B_{2,1}^*, \quad (17)$$

where κ is defined by Eq. (11). The total time derivative of the amplitudes B_1 and B_2 is obtained by collecting all terms and looks like $\partial_t B_{1,2}=\epsilon\partial_{T_1}B_{1,2}+\epsilon^2\partial_{T_2}B_{1,2}$. Reintroducing the original variables $E=\epsilon E_0$, $r=1+\epsilon^2$, $\nu=\epsilon\nu_0$, the normalized amplitudes

$$A_{1,2}=\sqrt{\frac{\sigma}{b(1+\sigma)}}(r-1)B_{1,2},$$

and observing that $\partial_x B_{1,2}=\epsilon\partial_{X_1}B_{1,2}$, from Eqs. (15) and (17) we finally obtain the amplitude equations

$$\partial_t A_1+s\partial_x A_1=d\partial_x^2 A_1+(\alpha-i\nu)A_1+\mu A_2^* -(|A_1|^2+2|A_2|^2)A_1, \quad (18)$$

$$\partial_t A_2-s\partial_x A_2=d\partial_x^2 A_2+(\alpha-i\nu)A_2+\mu A_1^* -(|A_2|^2+2|A_1|^2)A_2, \quad (19)$$

where the coefficients s, d, α, and μ are defined by

$$s=\frac{2\sqrt{a\Omega}}{1+\sigma}, \quad (20)$$

$$d=\frac{ia+\kappa}{1+\sigma}, \quad (21)$$

$$\alpha=\frac{\sigma}{1+\sigma}\left(r-1-\frac{2|E|^2}{b\Omega^2}\right), \quad (22)$$

$$\mu=\frac{\sigma E^2}{b(1+\sigma)\Omega^2}. \quad (23)$$

Equations (18) and (19), which represent the main result of this section, describe the nonlinear dynamics of the laser equations near threshold when the weak injected field is quasiresonant with the atomic transition frequency. The main distinctive feature of the amplitude equations in this case is the appearance of parametric terms in the interaction between the two counterpropagating TW's. In their present form, these equations are generalized complex Ginzburg-Landau equations with parametric terms that describe quite generally parametric excitation of waves in spatially extended nonlinear systems [12,13]; in particular, in hydrodynamics these equations have been derived in oscillatory convection to study parametric excitation of surface waves on a horizontal layer of fluid vertically vibrated [10–12]. In our case, however, as we assume a fixed amplitude of the injected signal and we ask what happen when the pump parameter r is increased, the linear gain α (and not the parametric gain μ) plays the role of bifurcation parameter in the equations.

III. TRAVELING- AND STANDING-WAVE SOLUTIONS OF THE AMPLITUDE EQUATIONS

In this section we limit our analysis considering the case of a near-resonant injected signal. As previously observed, the detuned case does not introduce new features in the laser dynamics with respect to the case where no external signal is injected in the cavity [9]; therefore, we will not consider this case. In the near-resonant limit, the starting point of the analysis is provided by the amplitude equations (18) and (19). These equations have the trivial zero solution, which corresponds to the laser emitting in the forced mode. This solution becomes unstable as the pump parameter is increased and off-axis emission takes place. It should be noted that, since in the amplitude equations the two counterpropagating waves are always linearly coupled, a pure TW state does not exist and the bifurcating mode is always a mixed mode of the two TW's. In this section we study the linear stability of the forced mode solution, deriving the neutral stability curve for this mode in the wave-number space, and we show that there exist two exact families of solutions for the amplitude equations in the form of SW's and mixed TW's. The stability of these solutions is investigated by use of standard linear stability methods, and the emergence of both phase and amplitude instabilities is predicted.

A. Stability analysis of the forced mode: Neutral stability curve

Linear stability analysis of the force mode is easily done by linearizing the amplitude equations (18) and (19) around the zero solution and looking for exponential gwowth of the perturbations. The most general solution of the linearized problem is given by a superposition of solutions of the form

$$\begin{pmatrix} A_1 \\ A_2^* \end{pmatrix} \sim \exp(\lambda_\pm t+iQx), \quad (24)$$

where Q is the transverse wave number of the perturbation and the two eigenvalues $\lambda_\pm$ are given by

$$\lambda_\pm(Q)=-d_R Q^2+\alpha\pm\sqrt{\mu^2-(d_I Q^2+sQ+\nu)^2}, \quad (25)$$

where d_R and d_I are the diffusion and diffraction coefficients in the amplitude equations (i.e., the real and imaginary parts of d, respectively). From Eq. (25) it follows that, for small values of the pump parameter (i.e., of α), the two eigenvalues have a negative real part and the forced mode is linearly stable. As the pump parameter is increased, at least one of the two eigenvalues becomes unstable. In particular, for wave numbers Q satisfying the inequality

$$|d_I Q^2+sQ+\nu|<\mu, \quad (26)$$

the most unstable eigenvalues is λ_- and, as the pump parameter is increased, it leads to a *steady-state* bifurcation. In this case, the neutral stability curve, corresponding to $\lambda_-=0$, is given by

$$\alpha_N(Q)=d_R Q^2-\sqrt{\mu^2-(d_I Q^2+sQ+\nu)^2}. \quad (27)$$

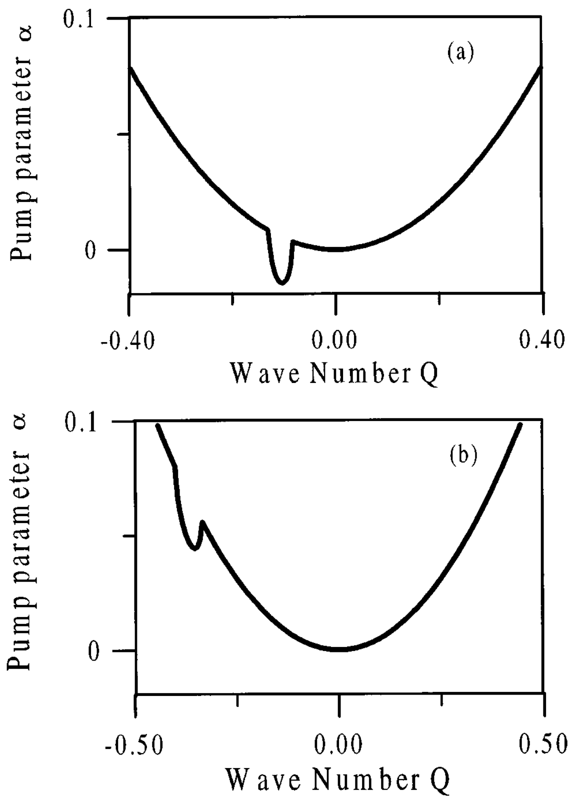

FIG. 1. Behavior of the neutral stability curve for (a) $\nu=0.1$ and (b) for $\nu=0.3$. The values of the other parameters are $\mu=0.02$, $a=1$, and $\sigma=1$. In (a) the bifurcating mode is a SW whose frequency is locked to that of the injected signal, whereas in (b) the emerging mode is a mixed TW which is not frequency locked to the signal.

On the other hand, for wave numbers Q which do not satisfy the inequality (26), as the pump parameter is increased, the forced mode loses its stability through a *Hopf* bifurcation corresponding to the vanishing of the real part of two complex conjugate eigenvalues. In this case the neutral stability curve, obtained by making $\mathrm{Re}\lambda_{\pm}=0$, is given by

$$\alpha_N(Q)=d_R Q^2 \tag{28}$$

and the frequency of the Hopf bifurcation is given by

$$\omega(Q)=\pm\sqrt{-\mu^2+(d_I Q^2+sQ+\nu)^2}. \tag{29}$$

The neutral stability curve for the forced mode is hence composed by two parts, one of which has the usual parabolic shape and extends at infinity in the wave-number space, the other one being limited in the wave-number region defined by the inequality (26). A typical behavior of the neutral stability curve is shown in Fig. 1. As the amplitude equations have been derived under the scaling $\alpha\sim\epsilon^2$, $\mu\sim\epsilon^2$, and $\nu\sim\epsilon$, it is easy to show that the inequality (26) is satisfied in a narrow interval of width of $O(\epsilon^2)$ around the wave number $Q_C\simeq-\nu/s$. The threshold for the emergence of off-axis waves is determined by the minimum of the neutral stability curve. It is easy to show that the most unstable wave number is $Q=0$ whenever the inequality

$$\nu^2>\frac{1+\sigma}{\sigma}\mu \tag{30}$$

is satisfied, and the threshold for instability is reached at $\alpha_{\mathrm{th}}=0$. The bifurcating mode is hence uniform in space, but is oscillating in time, i.e., its frequency is not locked to that of the injected signal. Furthermore, the solution of the linear problem shows that the two emerging TW's have different intensities (mixed-mode solution). On the contrary, when inequality (30) is not satisfied, a threshold lowering is predicted at the value $\alpha_{\mathrm{th}}=d_R(\nu/s)^2-\mu$, and the most unstable wave number is at $Q=Q_C$. As this is a steady-state bifurcation, the emerging mode is frequency locked to the injected signal. The intensities of the emerging TW's are in this case the same, and hence a SW pattern will emerge.

B. Standing and mixed traveling waves

The stability analysis of the forced-mode solution suggests that two different kinds of solutions for the amplitude equations should exist. The existence and stability of these solutions have been previously investigated in oscillatory convection when diffusion and diffraction terms are neglected in the amplitude equations [11]; in that context, such solutions were called standing and (mixed) traveling waves. Here we extend the analysis by including diffusion and diffraction effects. Let us look for a solution of Eqs. (18) and (19) of the form

$$\begin{pmatrix} A_1 \\ A_2^* \end{pmatrix}=\begin{pmatrix} R_1\exp(i\phi_1) \\ R_2\exp(-i\phi_2) \end{pmatrix}\exp(i\omega t+iQx), \tag{31}$$

where R_1, R_2, ϕ_1, and ϕ_2 are the amplitudes and phases of the counterpropagating TW's, Q is their wave number (which represents the family parameter), and $\omega=\omega(Q)$ is the dispersion relation of TW's which has to be determined. Inserting the ansatz (31) into Eqs. (18) and (19) and defining $\phi=\phi_1+\phi_2$, one finds the set of equations

$$\omega R_1+QsR_1=-d_I Q^2 R_1-vR_1-\mu R_2\sin\phi,$$

$$-d_R Q^2 R_1+\alpha R_1+\mu R_2\cos\phi=R_1(R_1^2+2R_2^2),$$

$$-\omega R_2+QsR_2=-d_I Q^2 R_2-\nu R_2-\mu R_1\sin\phi,$$

$$-d_R Q^2 R_2+\alpha R_2+\mu R_1\cos\phi=R_2(R_2^2+2R_1^2),$$

which may be solved with respect to R_1, R_2, ϕ, and ω. Eliminating ω from the second and fourth equations, one obtains

$$(R_1^2-R_2^2)(R_1R_2-\mu\cos\phi)=0, \tag{32}$$

so that two kinds of solutions, corresponding either to $R_1=R_2$ or $R_2R_1=\mu\cos\phi$, may exist.

1. Standing waves ($R_1=R_2$)

In this case one finds two branches of SW's given by

$$R^2=\frac{\pm\mu\cos\phi+\alpha-d_R Q^2}{3}, \tag{33}$$

$$\sin\phi = -\frac{\nu + d_I Q^2 + sQ}{\mu}, \tag{34}$$

where R denotes either R_1 or R_2 and we have assumed for definitness $\cos\phi > 0$ in Eq. (33); the dispersion relation for SW's is then given by

$$\omega(Q) = 0; \tag{35}$$

i.e., SW's are frequency locked to the external signal. From Eq. (34) it follows that the SW solutions exist only in the interval of the wave-number space defined by Eq. (26). In this case the boundary of existence for SW's for the upper branch [corresponding to the upper sign in Eq. (33)] coincides with the neutral stability curve given by Eq. (27), whereas the lower branch of SW's exists for $\alpha(Q) > \alpha_N(Q) + 2\sqrt{\mu^2 - (\nu + d_I Q^2 + sQ)^2}$. As will be shown below, the SW solutions corresponding to the lower branch are always unstable, and therefore we will not consider these solutions in the following.

2. Mixed traveling waves ($R_1 \neq R_2$)

In this case the amplitudes and phases of TW's are given by

$$R_{1,2}^2 = \frac{\alpha - d_R Q^2 \pm \sqrt{(\alpha - d_R Q^2)^2 - 4\Phi}}{2}, \tag{36}$$

$$\tan\phi = -2\,\frac{d_I Q^2 + sQ + \nu}{\alpha - d_R Q^2}, \tag{37}$$

where, in Eq. (36), the parameter Φ is defined by

$$\Phi = \mu^2 \cos^2\phi = \frac{\mu^2(\alpha - d_R Q^2)^2}{(\alpha - d_R Q^2)^2 + 4(d_I Q^2 + sQ + \nu)^2}.$$

The dispersion relation for the mixed TW solutions is then given by

$$\omega(Q) = \tfrac{1}{2}\tan\phi\,(R_1^2 - R_2^2). \tag{38}$$

It is easy to show that the mixed TW solution exists for any value of the transverse wave number Q and that, outside the interval defined by Eq. (26), the boundary of existence coincides with the neutral stability curve given by Eq. (28). In this case the frequency of TW's, given by Eq. (38), reduces to the frequency of the Hopf bifurcation given by Eq. (29) when α approaches α_N. On the contrary, in the interval of wave numbers around Q_C defined by Eq. (26), the existence curve of mixed TW solutions is given by

$$\alpha_E(Q) = d_R Q^2 + 2\sqrt{\mu^2 - (d_I Q^2 + sQ + \nu)^2} \tag{39}$$

and, on this curve, SW's and mixed TW's coincide. A typical behavior of the neutral stability curve and of the curves for existence of SW's and mixed TW's is shown in Fig. 2. Before further proceeding in the analysis, it is interesting to observe that, when considering the limit $\mu \to 0$, the domain of existence of SW's shrinks and disappears, whereas the mixed TW solution becomes a *pure* right or left TW, thus recovering the well-known scenario for the off-axis laser emission in absence of external signal [5].

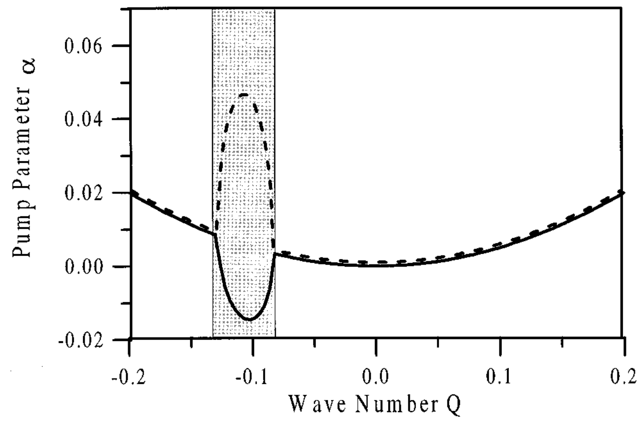

FIG. 2. Neutral stability curve (solid line), curve of existence of mixed TW's (dashed line), and domain of wave numbers satisfying inequality (26) (shaded area) for parameter values $\nu=0.1$, $\mu=0.02$, $a=1$, and $\sigma=1$. SW solutions exist above the neutral stability curve inside the shaded area, and they are frequency locked to the external signal. Note also that the neutral stability curve and the curve of existence of TW's coincide outside the shaded area.

C. Linear stability analysis of traveling and standing waves

In the previous subsections it has been shown that the amplitude equations (18) and (19) admit of traveling- and standing-wave solutions and that the emerging mode will depend on the signal detuning parameter. According to Eq. (30), the bifurcating mode is a mixed TW when Eq. (30) is satisfied and a SW in the opposite case. The central question is to determine the region in the plane (Q,α) of stable traveling and standing waves beyond the neutral stability curve, what is called the *Busse balloon*, using a hydrodynamic terminology [6]. In the case where no signal is injected in the laser cavity, the Busse balloon for the amplitude equations is merely determined by the usual Eckhaus criterion and instability of TW's is due only to long-wavelength (or phase) perturbations [5,6,14]. In this case a phase-diffusion equation is able to capture all sources of instabilities (at least near threshold, where the amplitude equations are valid). For the amplitude equations (18) and (19), the scenario is more complicated and it turns out that a global stability analysis of both traveling and standing waves has to be done by a full linearization of the amplitude equations around these solutions. We consider therefore perturbations of the steady-state solutions by setting

$$A_1 = (A_{1S} + \delta A_1)\exp(iQx + i\omega t),$$

$$A_2 = (A_{2S} + \delta A_2)\exp(-iQx - i\omega t) \tag{40}$$

where $A_{1S} = R_1\exp(i\phi_1)$ and $A_{2S} = R_2\exp(i\phi_2)$ denote the steady-state complex amplitudes, defined by Eqs. (33) and (34) for SW's and by Eqs. (36) and (37) for mixed TW's, respectively. Insertion of Eq. (40) into Eqs. (18) and (19) yields the following linearized equations for the perturbations δA_1 and δA_2:

$$\partial_t \delta A_1 = [-i(\omega + sQ + \nu + d_I Q^2) - d_R Q^2 + \alpha - 2|A_{1S}|^2 - 2|A_{2S}|^2]\delta A_1 + (2idQ - s)\partial_x \delta A_1 + d\partial_x^2 \delta A_1 + (\mu - 2A_{1S}A_{2S})\delta A_2^* - 2A_{1S}A_{2S}^* \delta A_2 - A_{1S}^2 \delta A_1^*, \quad (41)$$

$$\partial_t \delta A_2 = [i(\omega - sQ - \nu - d_I Q^2) - d_R Q^2 + \alpha - 2|A_{1S}|^2 - 2|A_{2S}|^2]\delta A_2 - (2idQ - s)\partial_x \delta A_2 + d\partial_x^2 \delta A_2 + (\mu - 2A_{1S}A_{2S})\delta A_1^* - 2A_{2S}A_{1S}^* \delta A_1 - A_{2S}^2 \delta A_2^*. \quad (42)$$

The most general solution of Eqs. (41) and (42) is given by a linear combination of solutions of the form

$$\delta A_1 = \xi_1 \exp(\lambda t + iqx) + \xi_2^* \exp(\lambda^* t - iqx),$$

$$\delta A_2 = \xi_3 \exp(\lambda t + iqx) + \xi_4^* \exp(\lambda^* t - iqx),$$

where q is the wave number of the perturbation and $(\xi_1, \xi_2, \xi_3, \xi_4)^T$ are the eigenvectors and $\lambda = \lambda(q)$ the corresponding eigenvalues of the fourth-order matrix $\mathcal{M} = [m_{i,k}]_{i,k=1,2,3,4}$ defined by

$$m_{11} = -\mu \frac{A_{2S}^*}{A_{1S}} - |A_{1S}|^2 + iq(2idQ - s) - dq^2,$$

$$m_{12} = -A_{1S}^2, \quad m_{13} = -2A_{1S}A_{2S}^*, \quad m_{14} = \mu - 2A_{1S}A_{2S},$$

$$m_{21} = m_{12}^*, \quad m_{22} = m_{11}^* - 2iq(s + 2iQd^*),$$

$$m_{23} = m_{14}^*, \quad m_{24} = m_{13}^*, \quad m_{31} = m_{13}^*, \quad m_{32} = m_{14},$$

$$m_{33} = -\mu \frac{A_{1S}^*}{A_{2S}} - |A_{2S}|^2 + iq(s - 2iQd) - dq^2,$$

$$m_{34} = -A_{2S}^2, \quad m_{41} = m_{14}^*, \quad m_{42} = m_{13}, \quad m_{43} = m_{34}^*,$$

$$m_{44} = m_{33}^* + 2iq(s + 2iQd^*).$$

Instability of steady-state solutions to the growth of a transverse modulation with wave number q arises when the real part of at least one of the matrix eigenvalues becomes positive. We can distinguish in general two kinds of instabilities arising from spatially *homogeneous* perturbations (corresponding to $q=0$) and from spatially *inhomogeneous* perturbations (corresponding to $q \neq 0$). Stability analysis for traveling- and standing-wave solutions with respect to homogeneous perturbations may be done analytically and the details of the calculations are given in the Appendix. It turns out that mixed TW solutions are always stable against the growth of homogeneous perturbations in the entire domain of existence, whereas there are two sources of instabilities for SW solutions. The first one is related to bistability of SW's and indicates that the lower branch [corresponding to the lower sign in Eq. (33)] is always unstable. The second source of instability appears above the neutral stability curve for the upper branch when α approaches the curve $\alpha_E(Q)$ defined by Eq. (39). Such an instability indicates the existence of a steady-state bifurcation at $\alpha = \alpha_E(Q)$, resulting in the transition from a SW to a mixed TW solution. Therefore the stability domains for traveling and standing waves are mutually exclusive. Stability analysis with respect to inhomogeneous perturbations (pattern-forming instabilities) is more involved and in general it requires numerical determination of the matrix eigenvalues. The translational invariance of the amplitude equations ensures the existence of a neutral mode with zero eigenvalue at $q=0$, and this is also directly shown in the Appendix. This allows classification of instabilities arising from spatially inhomogeneous perturbations in two classes: *phase instabilities*, arising from long-wavelength unstable growth bands emanating from this neutral mode, and *amplitude instabilities*, which occur at shorter wavelengths and correspond to modes that are stable at $q=0$ [6]. The numerical analysis of the matrix eigenvalues in the linearized problem shows that for both traveling- and standing-wave solutions there exist phase and amplitude instabilities. Furthermore, the stability properties for these solutions strongly depend on the signal detuning parameter ν. Figure 3 shows a typical behavior of the Busse ballon for both TW's and SW's in the case of a small signal detuning, for which inequality (30) is not satisfied and the bifurcating mode is a SW. The Busse ballon for SW's is represented by the black area in the figure, and it turns out that it is determined solely by the emergence of a long-wavelength phase instability (Eckhaus instability). For mixed TW's, the stability domain is delimited below by the thick dashed curve shown in the figure, and in this case either phase or amplitude instabilities determine the stability boundary. In particular, analysis of the matrix eigenvalues in the linearized problem indicates that for wave numbers Q either far away from Q_C or close to Q_C the stability boundary for TW's is limited

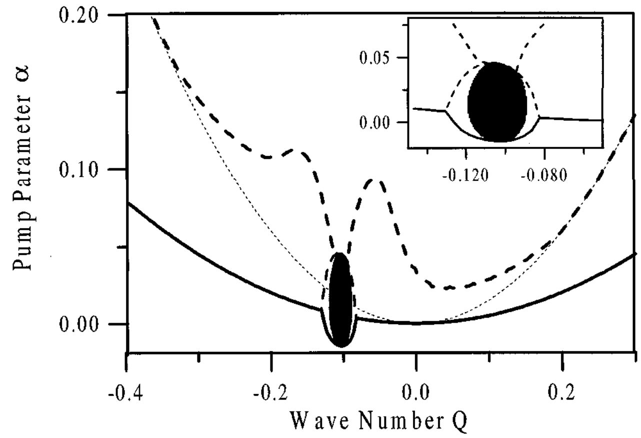

FIG. 3. Stability domains (Busse ballons) for traveling and standing waves as computed from the linear stability analysis for the same parameter values as in Fig. 2. The solid line is the neutral stability curve, the thin dashed line (partially obscured by the solid one) is the curve of existence for TW's, the black area is the stability domain for SW's, and the thick dashed line defines the boundary of stability for mixed TW's (TW's are stable above this line). The dotted line indicates the usual Eckhaus parabolic boundary for TW's when parametric terms are absent in the amplitude equations (18) and (19). The inset in the figure shows an enlargment of the stability domain for SW's. Here and in the following figures, we refer α as pump parameter.

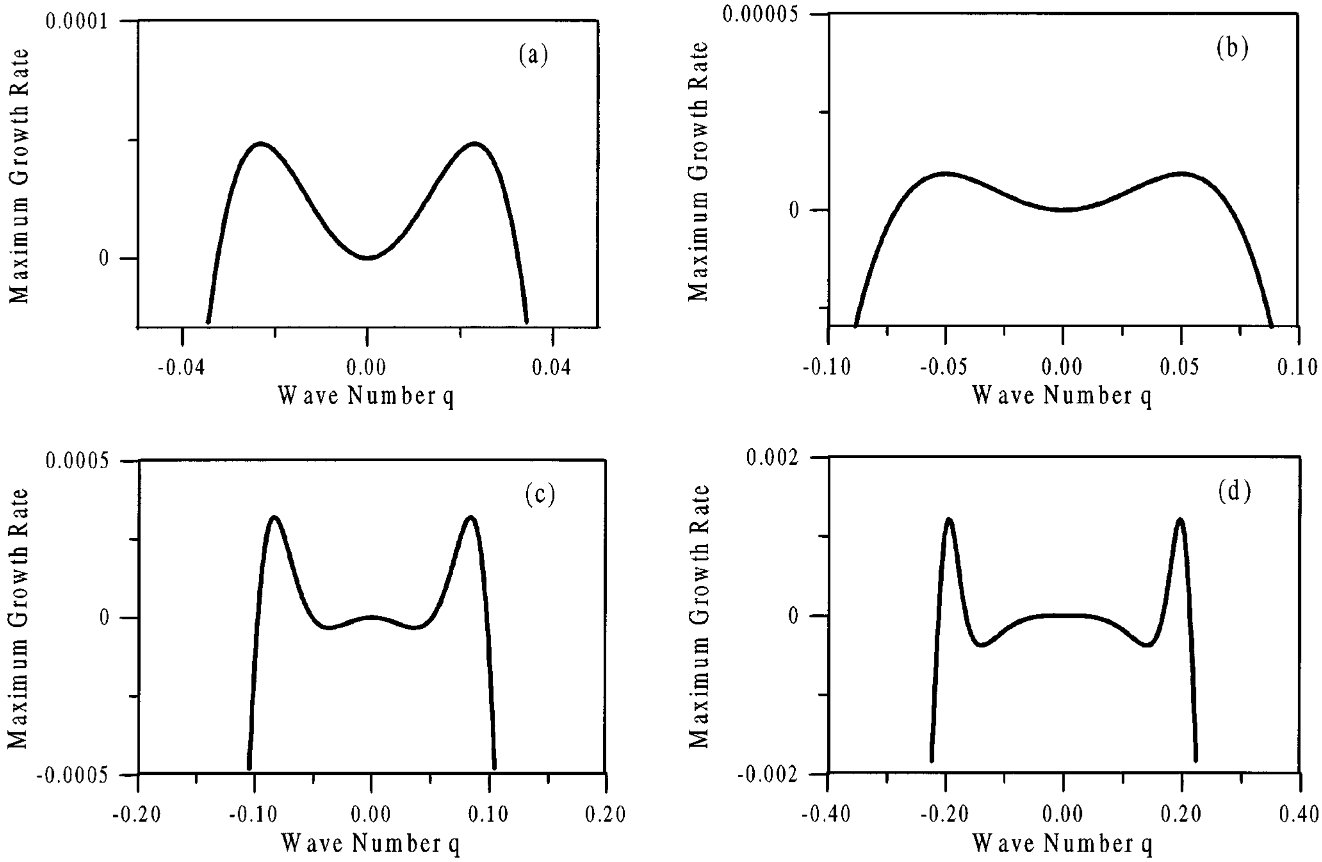

FIG. 4. Typical behavior of the largest real part of the marix eigenvalues in the linearized problem as a function of the perturbing wave number q for four TW's close to the instability boundary and for the same parameter values as in Fig. 3. (a) $Q=-0.15$ and $\alpha=0.1$, (b) $Q=-0.4$ and $\alpha=0.24$, (c) $Q=-0.2$ and $\alpha=0.102$, and (d) $Q=-0.3$ and $\alpha=0.14$. (a), (b) correspond to the emergence of a long-wavelength phase instability, whereas in (c) and (d) a short-wavelength amplitude instability takes place.

by the emergence of long-wavelength phase instabilities and it reduces to the usual parabolic Eckhaus boundary $\alpha=3d_RQ^2$ (shown in the figure with a dotted line) at large wave numbers [see Figs. 4(a) and 4(b)]. On the contrary, amplitude instabilities at intermediate values of wave numbers are most dangerous and they delimite the stability boundary in this region [see Figs. 4(c) and 4(d)]. It is interesting to note that the stability domains for SW's and TW's intersect around $Q=Q_C$, where a steady-state bifurcation from SW's to mixed TW's occurs. A limiting case is that of an exact resonant signal ($\nu=0$), which is shown in Fig. 5. In this case the stability domains for SW's and TW's become symmetric in the wave-number space and they have in common only one point at $Q=0$.

The stability properties change at larger signal detunings, for which inequality (30) is satisfied. In this case it turns out that the stability domain for SW's shrinks due to the appearance of an amplitude instability, whereas the stability domain for mixed TW's gets closer to the usual parabolic Eckhaus domain, as shown in Figs. 6 and 7. In particular, in Fig. 7, SW's are always unstable in their domain of existence as a consequence of a short-wavelength instability which invades the entire domain above the neutral stability curve. This instability is shown in Fig. 8, where the largest real part of the matrix eigenvalues in the linearized problem is plotted as a function of the perturbing wave number q.

We can physically understand the dynamical properties of traveling and standing waves by observing that, as the detuning parameter is zero or small, the SW's are preferred to TW's near threshold as the four-wave-interaction process mediated by the injected signal allows for a threshold lowering. Only at higher values of the pump parameter does the gain saturation mechanism typical of the laser equations select TW's instead of SW's and a bifurcation takes place. However, as the signal detuning is increased, the conserva-

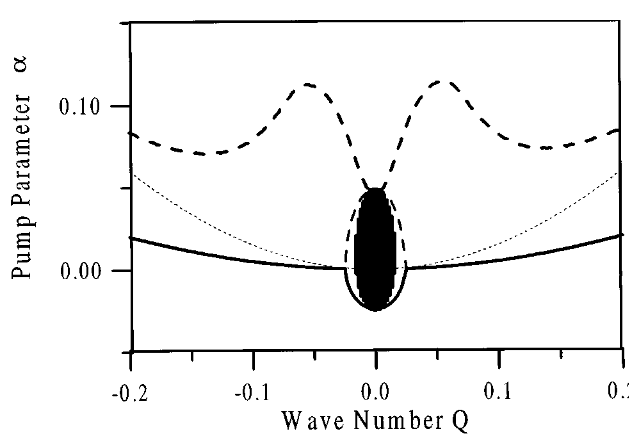

FIG. 5. Stability domains for traveling and standing waves in the resonant case $\nu=0$. The values of other parameters are $\mu=0.0242$, $a=1$, and $\sigma=1$. The meaning of the various curves is the same as in Fig. 3.

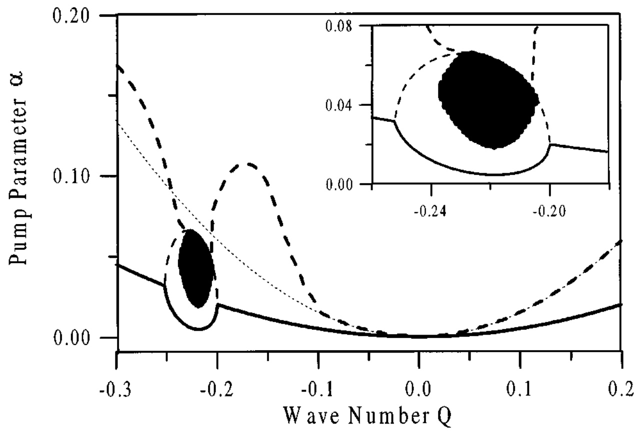

FIG. 6. Same as Fig. 3 for ν=0.2.

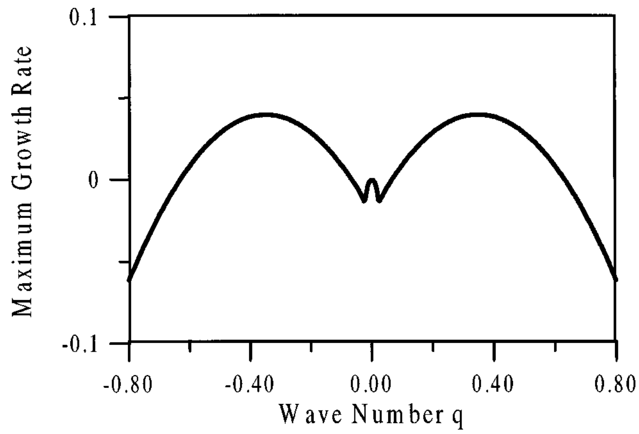

FIG. 8. Short-wavelength instability of SW's for large signal detunings. The figure shows the real part of the most unstable eigenvalue in the linearized problem as a function of the perturbing wave number q for the SW solution corresponding to $Q=-0.35$, α=0.06, and for the same parameter values as in Fig. 7.

tion of photon energy in the interaction process requested for the existence of frequency-locked SW's selects waves highly detuned from the atomic resonance frequency. When the inequality (30) is satisfied, the threshold lowering due to the frequency-locking mechanism becomes less effective and the laser dynamics prefers to ''ignore'' the external signal and to emit a frequency unlocked wave.

IV. CONCLUSIONS

In this paper we have investigated analytically the nonlinear dynamics close to the threshold describing the competition among traveling and standing waves in a single-longitudinal-mode, large-aspect-ratio laser with a weak injected signal in one transverse spatial dimension. By using a multiple scale perturbation method, we have derived the amplitude equations for the laser equations extending the analysis developed in Ref. [5] to the case where an external signal is injected in the laser cavity. It has been shown that, when the frequency of the external signal is tuned close to the resonance frequency of the two-level system, the competition among traveling and standing waves may be described by two coupled Ginzburg-Landau equations with parametric terms analogous to those derived in hydrodynamics to study paramteric excitation of waves in oscillatory convection. The presence of the parametric terms, which physically arise from a four-wave-intercation process with conservation of both photon momentum and photon energy, introduces new features in the laser dynamics even near threshold where the validity of the amplitude equations is restricted. We have shown that there exist two families of solutions for the amplitude equations corresponding to standing and mixed traveling waves, and we have studied the bifurcation properties among these solutions. In particular, when the signal detuning is small, SW's are preferred to TW's close to the threshold, but a secondary instability leading to an exchange of stability occurs at higher values of the pump parameter. Linear stability analysis of traveling and standing waves has also revealed the emergence of both amplitude and phase instabilities.

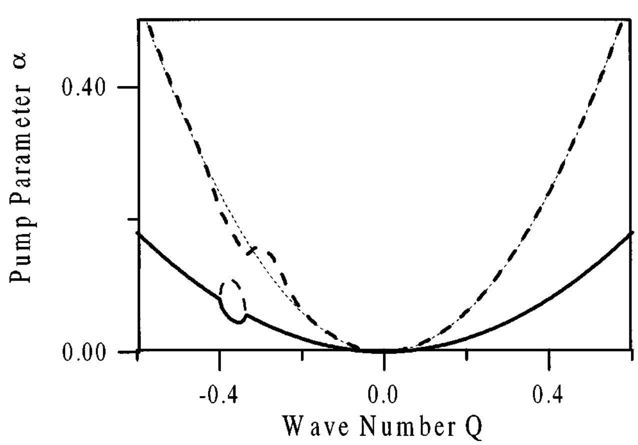

FIG. 7. Same as Fig. 3 for ν=0.3. Note that in this case SW's are always unstable.

APPENDIX: LINEAR STABILITY ANALYSIS OF TRAVELING AND STANDING WAVES AGAINST SPATIALLY HOMOGENEOUS PERTURBATIONS

In this appendix we derive analytically the stability conditions for both traveling- and standing-wave solutions of the amplitude equations considering the limiting case of spatially homogeneous perturbations. If this is the case, it may be shown that the characteristic polynomial of the matrix $\mathcal{M}$ in the linearized problem is given by

$$\lambda(c_0\lambda^3+c_1\lambda^2+c_2\lambda+c_3)=0, \tag{A1}$$

where

$$c_0=R_1^2R_2^2,$$

$$c_1=2R_1R_2(R_1^2+R_2^2)(\mu\cos\phi+R_1R_2),$$

$$c_2=\mu^2(R_1^2-R_2^2)^2+12R_1^3R_2^3(\mu\cos\phi-R_1R_2),$$

$$c_3=2\mu R_1^2R_2^2(R_1^2+R_2^2)(-\mu+6\mu\cos^2\phi-6R_1R_2\cos\phi)+2\mu^2(R_1^6+R_2^6).$$

An instability arises when at least one of the following conditions is violated (Routh-Hurwitz criterion):

$$c_1>0, \quad c_2>0, \quad c_3>0, \tag{A2}$$

$$H_2=c_1c_2-c_0c_3>0. \tag{A3}$$

1. Standing-wave solution

In this case, $R_1=R_2=R$ and the polynomial coefficients become

$$c_0=R^4,$$

$$c_1=4R^4(\mu\cos\phi+R^2),$$

$$c_2=12R^6(\mu\cos\phi-R^2)+8R^6\mu\cos\phi+4R^2\mu^2\cos^2\phi,$$

$$c_3=24R^6\mu\cos\phi(\mu\cos\phi-R^2).$$

Considering the lower branch of SW's [corresponding to the lower sign in Eq. (33)], it is easy to show that the two conditions $c_1>0$ and $c_2>0$ are incompatible each other; therefore, the lower branch is always unstable. For the upper branch of SW's, of conditions (A2) only the last one may be violated and this occurs when $\alpha>\alpha_E$, where α_E is the boundary of existence of mixed TW's defined by Eq. (39). It is easy to show that, once conditions (A2) are satisfied, also condition (A3) is satisfied as well. In conclusion, SW's are unstable in the phase space domain where mixed TW's exist.

2. Mixed traveling waves

In this case, $R_1R_2=\mu\cos\phi$ and the polynomial coefficients may be written as

$$c_0=R_1^2R_2^2,$$

$$c_1=4R_1^2R_2^2(R_1^2+R_2^2),$$

$$c_2=\mu^2(R_1^2-R_2^2)^2+4R_1^2R_2^2(R_1^4+R_2^4+R_1^2R_2^2),$$

$$c_3=2\mu^2(R_1^2+R_2^2)(R_1^2-R_2^2)^2.$$

Conditions (A2) are manifestely satisfied, whereas after some algebra the Hurwitz determinat H_2 may be written in the form

$$H_2=2R_1^2R_2^2(R_1^2+R_2^2)[\mu^2(R_1^2-R_2^2)^2 + 8R_1^2R_2^2(R_1^4+R_2^4+R_1^2R_2^2)],$$

which is always positive. Therefore mixed TW's are always stable in their domain of existence against the growth of spatially homogeneous perturbations.

[1] For a recent review on pattern formation in nonlinear optics, see, for instance, the editorial introduction of the special issue on Nonlinear Optical Structures, Patterns, Chaos, Chaos Solitons Fractals **4**, 1251 (1994).

[2] See, for instance, F. Prati, M. Brambilla, and L. A. Lugiato, Riv. Nuovo Cimento **17**, 1 (1994).

[3] A. C. Newell and J. V. Moloney, *Nonlinear Optics* (Addison-Wesley, Redwood City, CA, 1992).

[4] P. Coullet, L. Gil, and F. Rocca, Opt. Commun. **73**, 403 (1989).

[5] P. K. Jakobsen, J. V. Moloney, A. C. Newell, and R. Indik, Phys. Rev. A **45**, 8129 (1992).

[6] P. K. Jakobsen, J. Lega, Q. Feng, M. Staley, J. V. Moloney, and A. C. Newell, Phys. Rev. A **49**, 4189 (1994); J. Lega, P. K. Jakobsen, J. V. Moloney, and A. C. Newell, *ibid*. **49**, 4201 (1994).

[7] J. Lega, J. V. Moloney, and A. C. Newell, Physica D **83**, 478 (1995).

[8] Paul Mandel, M. Georgiou, and T. Erneux, Phys. Rev. A **47**, 4277 (1993).

[9] M. Georgiou and Paul Mandel, Chaos Solitons Fractals **4**, 1657 (1994).

[10] S. Douady, S. Fauve, and O. Thual, Europhys. Lett. **10**, 309 (1989).

[11] H. Riecke, J. D. Crawford, and E. Knobloch, Phys. Rev. Lett. **61**, 1942 (1988).

[12] H. Riecke, Europhys. Lett. **11**, 213 (1990).

[13] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[14] G. K. Harkness, W. J. Firth, J. B. Geddes, J. V. Moloney, and E. M. Wright, Phys. Rev. A **50**, 4310 (1994).