Renormalization group methods in quantum optics

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The velocity-dependent spontaneous emission of a two-level atom in a Fabry-Pérot cavity in the strong-coupling regime and the deflection of a beam of two-level atoms in a classical standing wave inside a cavity are discussed using a renormalization group approach. In this way we are able to renormalize the leading-order solutions for both problems through calculations of the corrections at first order. In fact, the first-order terms are not bounded for large times and no sense can be attached to this higher-order correction unless small times are considered. These are like the divergences of quantum field theory. To make them harmless, the condition for the Raman-Nath regime is recovered. The renormalization group methods permit one to eliminate those divergences generating a renormalized leading-order wave function without any condition of applicability. For the spontaneous emission of a two-level atom in a Fabry-Pérot cavity in the strong regime, using a Hamiltonian without losses, it is shown that the unperturbed levels are shifted by a term proportional to the zeroth-order Bessel function with an argument yielded by the ratio of the Rabi frequency and the Doppler-shifted frequency of the mode of the cavity. When the detuning is zero, the correction to the leading-order wave function is not present and known results are recovered. For the beam of two-level atoms in a classical standing wave, when the detuning is much larger than the Rabi frequency, it is shown that the renormalization group equation, which gives the correction for the renormalized leading-order wave function, is a time-dependent Schrödinger equation for a free particle that induces a spreading of the initial Gaussian wave packet. [S1050-2947(97)01108-6]

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I. INTRODUCTION

In recent years an important way to apply the renormalization group has been devised [1]. The method originated from the well-known fact that ordinary small perturbation theory can give rise to higher-order corrections that could not be bounded in the limit of very large times. This problem was born quite a long time ago in celestial mechanics and the name of secularities was attached to such terms as their effect, in astronomy, is apparent on a time scale of a century and can generally be neglected. But small perturbation theory is ubiquitous and these divergent terms can give rise to problems. In quantum mechanics this question was faced for the very first time in [2] where a unified theory of quantum resonance for a system with a discrete spectrum was given. It was shown there that increasing the strength of the perturbation could take the rotating-wave approximation to show its limits: Resonance equations should apply instead. The problem of secularities was then solved through a multiple-scale analysis [3] that has been until now a standard perturbation approach to obtain global solutions, that is, solutions useful for any time, not just small times. The approach discussed in Ref. [1] improves the multiple-scale analysis, showing that it gives equivalent results to a more general renormalization group method. Besides, the renormalization group approach makes very simple the application of perturbation schemes to obtain global solutions and has been recently applied in quantum mechanics [4].

A lot of interesting problems in quantum optics could require the application of the renormalization group [5]. We examine the problem of the velocity-dependent spontaneous emission of a two-level atom in a Fabry-Pérot cavity discussed in Ref. [6] where the ''bare'' Rabi frequency largely overcomes the rate of spontaneous emission and the cavity losses and, more generally, the problem of the deflection of a beam of two-level atoms in a classical standing wave [7], a typical problem of atomic optics that is normally considered in the Raman-Nath regime where the kinetic energy term of the atom can be neglected due to the small interaction time. The latter condition is quite easily understood, as we will see, as a device to neglect secular terms. While this works fine, experiments could be devised where an increasing interaction time is considered.

In quantum mechanics we can study the above systems by the method described in Ref. [8], but this approach gives rise to secularities to higher orders; that is, we have terms that increase without bound with increasing time and so a serious limitation appears in the method. The renormalization group permits us to eliminate that problem. One obtains that the strong-coupling limit of spontaneous emission of a two-level atom in a Fabry-Pérot cavity has the levels of the unpertubed part of the Hamiltonian shifted due to interaction with the cavity, an effect that disappears when the detuning is zero, recovering known results [6]. A two-level atom beam in a classical standing wave, when the detuning is much greater than the ''bare'' Rabi frequency, undergoes a spreading from its initial Gaussian form, described by the free-particle time-dependent Schrödinger equation. When this spreading is neglected, well-known results are recovered [7].

The paper is organized as follows. In Sec. II we discuss the method and apply it to a toy model whose solution is known; the renormalization group method is introduced here. In Sec. III we analyze the problem of a two-level atom in a Fabry-Pérot cavity without losses. In Sec. IV we derive the condition for the Raman-Nath regime and find the correction to the leading-order wave function for a beam of two-level atoms in a classical standing wave. In Sec. V the conclusions are given.

II. DESCRIPTION OF THE METHOD

In order to apply the renormalization group method we consider a perturbation scheme for an infinitely large perturbation. This approach was developed in Ref. [8] and the following results were obtained. Let us consider a Hamiltonian

$$H=H_0+\lambda V, \tag{1}$$

with $\lambda\rightarrow\infty$, H_0 the unperturbed part of the Hamiltonian, and V the perturbation. A perturbation series for the equation

$$(H_0+\lambda V)|\psi\rangle=i\hbar\frac{\partial}{\partial t}|\psi\rangle \tag{2}$$

is given, after rescaling the time as $t\rightarrow\lambda t$, by

$$|\psi(\lambda t)\rangle=U(\lambda t,\lambda t_0)\Bigg[I-\frac{1}{\lambda}\frac{i}{\hbar}\int_{\lambda t_0}^{\lambda t}dt'\,U^\dagger(t',\lambda t_0)H_0U(t',\lambda t_0)+\left(-\frac{1}{\lambda}\frac{i}{\hbar}\right)^2\int_{\lambda t_0}^{\lambda t}dt'\int_{\lambda t_0}^{t'}dt''U^\dagger(t',\lambda t_0)H_0U(t',\lambda t_0)U^\dagger(t'',\lambda t_0)H_0U(t'',\lambda t_0)+O\left(\frac{1}{\lambda^3}\right)\Bigg]|\psi(\lambda t_0)\rangle, \tag{3}$$

$U(\lambda t,\lambda t_0)$ being the time evolution operator solution of the equation, for $\lambda=1$,

$$V(t)U(t,t_0)=i\hbar\frac{\partial}{\partial t}U(t,t_0). \tag{4}$$

This perturbation scheme can give divergent results in the limit $t\rightarrow\infty$ that can be evaded taking $\lambda\rightarrow\infty$, and so it does not seem very useful. It is at this point that the renormalization group method can be useful to eliminate the divergences. In order to see how the method works we consider a toy model, that is, a spin-1/2 particle in a constant magnetic field with two components. The reason to consider such a trivial example is that it gives problems to both the small and strong perturbation schemes, as we are going to see. The Hamiltonian for such a simple system can be written as

$$H=X\sigma_1+Z\sigma_3, \tag{5}$$

σ_1 and σ_3 being the Pauli matrices and X,Z the two components of the magnetic field taken to be $X\gg Z$; that is, X is a large perturbation for the other part of the Hamiltonian.

The solution of that problem is straightforward. We have

$$|\psi(t)\rangle=\left[\cos\left(\frac{\Omega}{\hbar}(t-t_0)\right)-i\left(\frac{X}{\Omega}\sigma_1+\frac{Z}{\Omega}\sigma_3\right)\times\sin\left(\frac{\Omega}{\hbar}(t-t_0)\right)\right]|\psi(t_0)\rangle \tag{6}$$

where $\Omega=\sqrt{X^2+Z^2}$. By taking $H_0=Z\sigma_3$ and $V=X\sigma_1$ the above perturbation scheme gives

$$\begin{aligned}|\psi(t)\rangle={}&\exp\left(-i\frac{X}{\hbar}\sigma_1[t+\phi(t_0)]\right)\\&\times\Bigg\{I-\epsilon\left[\exp\left(2i\frac{X}{\hbar}\sigma_1[t+\phi(t_0)]\right)-1\right]\sigma_1\sigma_3\\&+\epsilon^2\left[\exp\left(2i\frac{X}{\hbar}\sigma_1[t+\phi(t_0)]\right)-1\right]\\&-\epsilon^2\frac{i}{\hbar}2X\sigma_1(t-t_0)+O(\epsilon^3)\Bigg\}|\psi(t_0)\rangle,\end{aligned} \tag{7}$$

with $\phi(t_0)=-t_0$ and $\epsilon=Z/2X$. In the above series it should be noticed that there is a term where the substitution $-t_0\rightarrow\phi(t_0)$ is not done at all; this is the secular term. The parameter $\phi(t_0)$ enters just in the regular terms. This rule permits us to get meaningful computations at any order by the method we are going to describe. The above result can also be obtained by the interaction picture, interchanging the role of H_0 and V, so that V is now a small perturbation; then, we have the same problem with both approaches. The problem is that we have a secular term that goes to infinity as $t_0\rightarrow\infty$ at fixed t so that no meaning can be attached to higher-order corrections unless we are able to get rid of it. The divergent part t_0 is the analogous of the logarithm of the cutoff in quantum field theory. By this analogy, we can consider applying the renormalization group as already devised in Ref. [1].

The method of the renormalization group makes the following formal steps. First, let us consider two constants Z_1 and Z_2 so defined:

$$\begin{aligned}|\psi(t_0)\rangle&=Z_1(t_0,\tau)|\psi(\tau)\rangle_R,\\\phi(t_0)&=\phi_R(\tau)+Z_2(t_0,\tau),\end{aligned} \tag{8}$$

with τ an arbitrary time introduced to eliminate the divergent part t_0. Next, we expand in series of ϵ both the constants as

$$Z_1(t_0,\tau)=1+\epsilon a_1(t_0,\tau)+\epsilon^2a_2(t_0,\tau)+O(\epsilon^3),$$

$$Z_2(t_0,\tau)=\epsilon b_1(t_0,\tau)+\epsilon^2 b_2(t_0,\tau)+O(\epsilon^3), \tag{9}$$

with the coefficients a_n and b_n to be determined to eliminate the divergences at t_0 at the various orders. There is a freedom in the choice of those renormalization constants [9] but we make the minimal one by putting

$$a_1=0, \quad a_2=-\frac{i}{\hbar}2X\sigma_1(t_0-\tau),$$

$$b_1=0, \quad b_2=0. \tag{10}$$

Substituting the expressions obtained in this way for $\phi(t_0)$ and $|\psi(t_0)\rangle$ into Eq. (7), we get the ''renormalized'' expression

$$|\psi(t)\rangle=\exp\left(-i\frac{X}{\hbar}\sigma_1[t+\phi_R(\tau)]\right)$$
$$\times\left\{I-\epsilon\left[\exp\left(2i\frac{X}{\hbar}\sigma_1[t+\phi_R(\tau)]\right)-1\right]\sigma_1\sigma_3\right.$$
$$+\epsilon^2\left[\exp\left(2i\frac{X}{\hbar}\sigma_1[t+\phi_R(\tau)]\right)-1\right]$$
$$\left.-\epsilon^2\frac{i}{\hbar}2X\sigma_1(t-\tau)+O(\epsilon^3)\right\}|\psi(\tau)\rangle_R. \tag{11}$$

But $|\psi(t)\rangle$ should not depend on τ; the same happens in quantum field theory where observables should not depend on the renormalization scale. So it must be that

$$\frac{d|\psi(t)\rangle}{d\tau}=0. \tag{12}$$

As we have a dependence on τ in both ϕ_R and $|\psi\rangle_R$, we get nontrivial equations to compute them. The last step consists in taking $\tau=t$ in Eqs. (11) and (12) as τ can be chosen arbitrarily. So the following renormalization group equations are obtained:

$$\frac{d|\psi(t)\rangle_R}{dt}+O(\epsilon^3)=0,$$

$$\frac{d\phi_R(t)}{dt}-2\epsilon^2+O(\epsilon^3)=0, \tag{13}$$

where the terms going like $\epsilon d\phi_R/dt$ and $\epsilon^2 d\phi_R/dt$ are neglected, being of order $O(\epsilon^3)$ and $O(\epsilon^4)$, respectively. This can be seen from the corresponding renomalization group equation for ϕ_R in Eqs. (13). Then, the global solution up to second order is

$$|\psi(t)\rangle= \exp\left[-\frac{i}{\hbar}\left(X+\frac{Z^2}{2X}\right)\sigma_1(t-t_0)\right]$$
$$\times\left(I-\epsilon\left\{\exp\left[2\frac{i}{\hbar}\left(X+\frac{Z^2}{2X}\right)\sigma_1(t-t_0)\right]-1\right\}\sigma_1\sigma_3\right.$$
$$+\epsilon^2\left\{\exp\left[2\frac{i}{\hbar}\left(X+\frac{Z^2}{2X}\right)\sigma_1(t-t_0)\right]-1\right\}$$
$$\left.+O(\epsilon^3)\right)\times|\psi(t_0)\rangle, \tag{14}$$

to be compared with the exact solution, Eq. (6), in the same limit $X\gg Z$. We note at this point that Eqs. (13) can be obtained by computing the envelope of Eq. (7), with $|\psi(t)\rangle$ considered as a function of both t and t_0, at the tangency point $t_0=t$. In order to accomplish this task, we have to compute [10]

$$\left.\frac{d|\psi(t,t_0)\rangle}{dt_0}\right|_{t_0=t}=0; \tag{15}$$

then, taking $|\psi(t,t_0)\rangle$, as given by Eq. (7), at $t_0=t$, Eqs. (13) and (14) are easily recovered. Both methods give the same results, but while the former makes clear that we are using a renormalization group method, the latter is surely simpler to use.

All we need to apply these methods is a naive perturbation series that is generally straightforward to compute. As a result we obtain improved asymptotic solutions without secular terms.

III. VELOCITY-DEPENDENT SPONTANEOUS EMISSION BY A TWO-LEVEL ATOM

Let us consider a two-level atom in a Fabry-Pérot cavity having the Rabi frequency $g\gg\kappa,\gamma$, κ being the constant describing the cavity losses and γ the rate of spontaneous emission. The equations for the probability amplitudes $c_g(t)$ for the ground state and $c_e(t)$ for the excited state are [6]

$$\frac{dc_e(t)}{dt}=-ig\cos(\Omega_d t)e^{-i\Delta t}c_g(t),$$

$$\frac{dc_g(t)}{dt}=-ig\cos(\Omega_d t)e^{i\Delta t}c_e(t)-\frac{\kappa}{2}c_g(t), \tag{16}$$

$\Omega_d=\omega v/c$ being the Doppler shift of the frequency of the field in the cavity if the atom motion along the axis of the cavity is given by $z=vt$, and $\Delta=\omega_0-\omega$ is the detuning. Taking the ideal lossless limit, $\kappa=0$ (otherwise, a non-Hermitian Hamiltonian should be considered). We realize very easily that Eqs. (16) can be derived from the effective Hamiltonian

$$H=\hbar\Delta\sigma_z+\hbar g(\sigma_++\sigma_-)\cos(\Omega_d t), \tag{17}$$

where σ_z, σ_+, and σ_- are the pseudospin operators for a two-level atom [5]. That Hamiltonian can be used to describe the strong-coupling regime. So we are in the situation where we find the first-order correction to the solution in Ref. [6]

when the detuning Δ is not zero. We will find that a secular term appears and we need to resort to the renormalization group method.

By considering $V=\hbar g(\sigma_+ + \sigma_-)\cos(\Omega_d t)$ a strong perturbation for the two-level atom and taking it starting at the time t_0 to avoid ordering problems, the perturbation scheme described in Sec. II gives, until first order,

$$|\psi(t)\rangle = \exp\left(-i\frac{g}{\Omega_d}(\sigma_+ + \sigma_-)\sin\Omega_d[t+\phi(t_0)]\right)\left[I - i\Delta\sum_{n\neq 0} J_n\left(\frac{2g}{\Omega_d}\right)\left(\frac{\sin n\Omega_d[t+\phi(t_0)]}{n\Omega_d} - i(\sigma_+ + \sigma_-)\frac{\cos n\Omega_d[t+\phi(t_0)]-1}{n\Omega_d}\right)\sigma_z - i\Delta J_0\left(\frac{2g}{\Omega_d}\right)(t-t_0)\sigma_z + O\left(\frac{\Delta^2}{\Omega_d^2}\right)\right]|\alpha(t_0)\rangle \tag{18}$$

where, as before, $\phi(t_0) = -t_0$ is a renormalizable parameter and use has been made of the relation [11]

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{+\infty} J_n(z)e^{in\theta}, \tag{19}$$

with $J_n(z)$ the Bessel function of integer order n. Equation (18) holds just for small times as a secular term appears. Our goal is to get a global solution and get rid of the secularity. We can apply both the methods in Refs. [1,10] to obtain the renormalization group equations. For example, considering $|\psi(t)\rangle$ as a function of both t and t_0 and applying Eq. (15) one obtains

$$\frac{d|\alpha(t)\rangle}{dt} + i\Delta J_0\left(\frac{2g}{\Omega_d}\right)\sigma_z|\alpha(t)\rangle + O\left(\frac{\Delta^2}{\Omega_d^2}\right) = 0, \quad \frac{d\phi(t)}{dt} + O\left(\frac{\Delta^2}{\Omega_d^2}\right) = 0. \tag{20}$$

Solving the above renormalization group equations and putting $t_0 = t$ into Eq. (18) gives the global solution

$$|\psi(t)\rangle = \exp\left(-i\frac{g}{\Omega_d}(\sigma_+ + \sigma_-)\sin\Omega_d(t-t_0)\right)\left[I - i\Delta\sum_{n\neq 0} J_n\left(\frac{2g}{\Omega_d}\right)\left(\frac{\sin n\Omega_d(t-t_0)}{n\Omega_d} - i(\sigma_+ + \sigma_-)\frac{\cos n\Omega_d(t-t_0)-1}{n\Omega_d}\right)\sigma_z + O\left(\frac{\Delta^2}{\Omega_d^2}\right)\right]\exp\left[-i\Delta J_0\left(\frac{2g}{\Omega_d}\right)\sigma_z(t-t_0)\right]|\alpha(t_0)\rangle. \tag{21}$$

In the limit $\Delta \to 0$ one gets the solution of Ref. [6] for the strong-coupling regime in a lossless cavity at zero detuning. It is not difficult to see a frequency shift of the two levels of the atom, proportional to the detuning Δ multiplied by the factor $J_0(2g/\Omega_d)$. Then, we can conclude that a Fabry-Pérot cavity produces a shift in the levels of the atom critically dependent on the ratio between the coupling constant g, that is, the ''bare'' Rabi frequency, and the Doppler shift of the field frequency of the cavity Ω_d.

IV. TWO-LEVEL ATOM IN A STANDING WAVE: THE RAMAN-NATH REGIME

We consider a beam of two-level atoms interacting with a classical standing wave in a cavity, in a largely nonresonant regime in order to neglect spontaneous emission. The following Hamiltonian holds for this model [5,7]:

$$H = \hbar\omega_0\sigma_z + \frac{p^2}{2m} + \hbar g(\sigma_- e^{i\omega t} + \sigma_+ e^{-i\omega t})\cos(kx), \tag{22}$$

which, transforming in the interaction picture by taking $H_0 = \hbar\omega_0\sigma_z$, becomes

$$H = \frac{k^2}{2m}p_\xi^2 + \hbar g(\sigma_- e^{-i\Delta t} + \sigma_+ e^{i\Delta t})\cos(\xi), \tag{23}$$

where, as above, $\Delta = \omega_0 - \omega$ is the detuning and we have set $\xi = kx$ and $p_\xi = -i\hbar\partial/\partial\xi$. By the perturbation scheme in Sec. II, we have to solve, for the time evolution operator, the equation

$$i\frac{\partial U(t,t_0)}{\partial t} = g\cos(\xi)(\sigma_- e^{i\Delta t} + \sigma_+ e^{-i\Delta t})U(t,t_0). \tag{24}$$

The solution of this equation is not straightforward, but we can rewrite it as an integral equation and try to see what happens in the limit of a very large detuning, that is, when $\Delta \gg g$. We have

$$U(t,t_0) = I - ig\cos(\xi)\int_{t_0}^{t} dt'(\sigma_- e^{i\Delta t'} + \sigma_+ e^{-i\Delta t'})U(t',t_0), \tag{25}$$

an integral equation to be solved iteratively. We get the small perturbation series in the parameter g/Δ,

$$U(t,t_0)=I+g\cos(\xi)\left(\sigma_-\frac{e^{-i\Delta t}-e^{-i\Delta t_0}}{\Delta}-\sigma_+\frac{e^{i\Delta t}-e^{i\Delta t_0}}{\Delta}\right)-i\frac{2g^2\cos^2(\xi)}{\Delta}\sigma_z(t-t_0)+O\left(\frac{g^2}{\Delta^2}\right), \tag{26}$$

where a secular term is evident. We can eliminate it by the renormalization group methods by rewriting Eq. (26) as

$$U(t,t_0)=e^{i\phi(t_0)}\left[I+g\cos(\xi)\left(\sigma_-\frac{e^{-i\Delta t}-e^{i\Delta\theta(t_0)}}{\Delta}-\sigma_+\frac{e^{i\Delta t}-e^{-i\Delta\theta(t_0)}}{\Delta}\right)-i\frac{2g^2\cos^2(\xi)}{\Delta}\sigma_z(t-t_0)+O\left(\frac{g^2}{\Delta^2}\right)\right], \tag{27}$$

$\theta(t_0)=-t_0$ and $\phi(t_0)$ being renormalizable parameters. We take the derivative of Eq. (27) with respect to t_0 and put $t_0=t$ so that the renormalization group equations are

$$\frac{d\phi(t)}{dt}+\frac{2g^2\cos^2(\xi)}{\Delta}\sigma_z+O\left(\frac{g^3}{\Delta^3}\right)=0,$$

$$\frac{d\theta(t)}{dt}+O\left(\frac{g^2}{\Delta^2}\right)=0. \tag{28}$$

Then, solving those equations and putting $t_0=t$ into Eq. (27), we obtain the known result at leading order [5,7]:

$$U(t,t_0)\approx\exp\left(-i\frac{2g^2\cos^2(\xi)}{\Delta}\sigma_z(t-t_0)\right). \tag{29}$$

So, as expected, in the small-recoil and large-detuning limit, we find again the solution that holds in the Raman-Nath regime. The higher-order correction, in the same approximation of Eq. (29), gives

$$|\psi(\xi,t)\rangle=U(t,t_0)\left[I+i\frac{\hbar k^2}{2m}(t-t_0)\frac{\partial^2}{\partial\xi^2}+O\left(\frac{g}{\Delta}\right)\right]|\xi,t_0\rangle, \tag{30}$$

where again a secular term appears. We see that this term can be made harmless if we take $(\hbar k^2/2m)(t-t_0)\ll 1$, that is, the condition for the Raman-Nath regime. Otherwise, we can apply the renormalization group methods as above and obtain the renormalization group equation

$$i\hbar\frac{\partial|\xi,t\rangle}{\partial t}=-\frac{\hbar^2k^2}{2m}\frac{\partial^2|\xi,t\rangle}{\partial\xi^2}, \tag{31}$$

i.e., the Schrödinger equation for a free particle. Then, if the beam has an initial Gaussian shape, a spread occurs as the particles would be free, independently of the interaction time.

V. CONCLUSIONS

The improvement of the method of strong perturbations [8] by use of renormalization group methods [1,10] has permitted us to study two interesting problems of quantum optics. Particularly, higher-order corrections to the known results at leading order [6,7] can now be obtained without difficulty. We have computed the shift in a two-level atom in a Fabry-Pérot cavity and the effect of the spreading in a two-level atom interacting with a classical standing wave without resorting to the Raman-Nath regime condition obtained from the secular terms of the higher-order corrections in the perturbation series. This is just a first step toward an improved study of the application of perturbation methods to problems of quantum optics.

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