Wigner's problem for a precessing magnetic dipole

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For a magnetic dipole precessing in a magnetic field, it is shown that the quantum dynamical vector field does not determine the commutation relations for the observables. Several nonlinearly related observables with identical quantum dynamics are constructed explicitly. [S1050-2947(97)07607-5]

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I. INTRODUCTION

Wigner posed the problem of whether quantum dynamical equations determine completely the commutation relations for the observables [1]. He found an ambiguity for the quantum harmonic oscillator. This question was also addressed by Okubo [2], and he stated that the ambiguity in the commutation relations for the same quantum vector field may be understood as an ambiguity in the canonical quantization canonical procedure due to essentially different Lagrangians giving the same classical equations of motion [3–5]. The Wigner procedure has been discussed in the context of parastatistics [6], and in connection with using parastatistics and supersymmetry [7] (also see the discusion in [8,9]). Self-adjointness of operators in relation with Wigner's problem for the harmonic oscillator has been considered by Watanabe [10], and he reduces the problem to considering a singular oscillator problem [11,12].

The problem of the commutation relations of different observables which are compatible with the same quantum dynamics was studied in detail by Man'ko *et al.* [13], where a class of nonlinearly related quadrature components, obeying the same dynamics of harmonic vibrations, was constructed. It was also shown that the same dynamics exists for the quadratures which may or may not satisfy the uncertainty relations, since they have different commutation relations. In fact, in another context, not related to the Wigner problem but to that of constructing coherent states, some oscillator dynamics for observables obeying different commutation relations has been exhibited [14], and general deformations for the harmonic oscillator algebra were proposed [15–18].

Discussion of Wigner's problem usually has been done for the harmonic oscillator. It is interesting to consider systems different from the harmonic oscillator where Wigner's problem emerges. This may suggest experiments which could clarify the physical meaning of this ambiguity in the commutation relations.

The aim of this work is to demonstrate that analogously to the Wigner's problem solved for the harmonic oscillator, the dynamics of precessing the magnetic dipole is compatible with a broad class of different commutation relations which obey the same dynamical equations. Indeed we will show that this is also compatible with q-commutation relations [19–21]. We establish a nonlinear relation (a deformation) which connects the different components of the angular momentum of the system.

In the second section we write the quantum system that we consider, the precessing magnetic dipole, and show deformations, both on the observables and in the commutation relations, which are compatible with the same dynamics of the standard problem. In the third section it is shown that this phenomenon has a classical counterpart. In Sec. IV we review the deformed, both in observables and commutation relations, two-dimensional harmonic oscillator, and in Sec. V we establish the connection between this problem and the precessing dipole. In the last section some conclusions are presented.

II. ROTATING MAGNETIC DIPOLE

Let us consider a magnetic dipole precessing in a magnetic field. This system is described by the Hamiltonian

$$H=-\mu(\vec{J}\cdot\vec{B}). \tag{1}$$

Here $\vec{J}$ is the angular-momentum operator, and μ is the magnetic moment. Using the standard Heisenberg picture we obtain the equations of motion for the angular-momentum components

$$\dot{\vec{J}}=\mu(\vec{J}\times\vec{B}). \tag{2}$$

Let us choose a reference frame in which the magnetic field is along the Oz axis: $\vec{B}=B\hat{e}_z$. We introduce the ladder operators

$$J_\pm=\frac{1}{\sqrt{2}}(J_x\pm iJ_y),$$
$$J_0=J_z, \tag{3}$$

which satisfy the commutation relations

$$[J_+,J_-]=2J_0,\quad [J_0,J_\pm]=\pm J_\pm . \tag{4}$$

We can rewrite the equations of motion (2) as

$$\dot{J}_\pm=\mp i\mu BJ_\pm$$
$$\dot{J}_0=0, \tag{5}$$

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whose solutions are

$$J_{\pm}(t)=\exp(\mp i\mu Bt)J_{\pm}(0),$$
$$J_0(t)=J_0(0). \tag{6}$$

One notices that the equations of motion for J_+ and J_- are identical with the equations of motion for creation and annihilation operators for the harmonic oscillator considered in [13] [see also Eq. (32) for the harmonic-oscillator case]. Therefore we can use our previous knowledge on the one-dimensional harmonic oscillator discussed in [13].

The equations of motion (2), have as integrals of the motion the zero-angular-momentum component J_0 and the square of the angular momentum $\vec{J}^2$. Due to this one can deform the angular momentum in the form

$$\tilde{J}_k=J_kF(\vec{J}^2,J_0) \tag{7}$$

and one can check that the new deformed components satisfy the equation

$$\dot{\tilde{\vec{J}}}=\mu(\tilde{\vec{J}}\times\vec{B}). \tag{8}$$

To get $\vec{J}^2$ and J_0, in terms of the deformed operators $\tilde{\vec{J}}^2$ and $\tilde{J}_0$, we need to invert the relations

$$\tilde{\vec{J}}^2=f(\vec{J}^2,J_0)\{[f(\vec{J}^2,J_0-1)+f(\vec{J}^2,J_0+1)]\vec{J}^2$$
$$+[f(\vec{J}^2,J_0)-f(\vec{J}^2,J_0-1)-f(\vec{J}^2,J_0+1)]J_0^2$$
$$+[f(\vec{J}^2,J_0-1)-f(\vec{J}^2,J_0+1)]J_0\},$$
$$\tilde{J}_0=f(\vec{J}^2,J_0)J_0. \tag{9}$$

We see that equations of motion (2), for the usual angular momentum, and equations of motion (8), for the deformed angular momentum, have the same form. The relation between the angular momentum and the deformed angular momentum (7) is provided by a noncanonical nonlinear transformation, which preserves the linear character of the dynamics but changes the commutation relations. One can check that the commutation relations of the deformed components of the angular momentum are

$$[\tilde{J}_+,\tilde{J}_-]=\left\{1-\frac{f(\vec{J}^2,J_0+1)}{f(\vec{J}^2,J_0-1)}\right\}\tilde{J}_+\tilde{J}_-+2f(\vec{J}^2,J_0+1)\tilde{J}_0,$$
$$[\tilde{J}_0,\tilde{J}_\pm]=\left\{1-\frac{f(\vec{J}^2,J_0\mp1)}{f(\vec{J}^2,J_0)}\right\}\tilde{J}_0\tilde{J}_\pm\pm f(\vec{J}^2,J_0\mp1)\tilde{J}_\pm. \tag{10}$$

The established result shows that one and the same dynamics, i.e., the same vector field (2), with the same solutions for the magnetic dipole precessing in a magnetic field, is compatible with two different sets of commutation relations for the angular-momentum components given in standard form (4), and the commutation relations of the deformed angular momentum given by Eq. (10).

We can introduce deformed commutation relations defined by (e.g., see [19,20])

$$[A,B]\rightarrow[A,B]_{Q\Lambda}=A\Lambda(\vec{J}^2,J_0)B-QB\Lambda(\vec{J}^2,J_0)A. \tag{11}$$

The deformed commutation relations for the deformed angular-momentum components are

$$[\tilde{J}_+,\tilde{J}_-]_{Q\Lambda}=\left\{\Lambda(\vec{J}^2,J_0-1)-Q\Lambda(\vec{J}^2,J_0+1)\right.$$
$$\left.\times\frac{f(\vec{J}^2,J_0+1)}{f(\vec{J}^2,J_0-1)}\right\}\tilde{J}_+\tilde{J}_-+2Qf(\vec{J}^2,J_0+1)$$
$$\times\Lambda(\vec{J}^2,J_0+1)\tilde{J}_0, \tag{12}$$

$$[\tilde{J}_0,\tilde{J}_\pm]_{Q\Lambda}=\left\{\Lambda(\vec{J}^2,J_0)-Q\Lambda(\vec{J}^2,J_0\mp1)\right.$$
$$\left.\times\frac{f(\vec{J}^2,J_0\mp1)}{f(\vec{J}^2,J_0)}\right\}\tilde{J}_0\tilde{J}_\pm\pm Qf(\vec{J}^2,J_0\mp1)$$
$$\times\Lambda(\vec{J}^2,J_0\mp1)\tilde{J}_\pm. \tag{13}$$

To recover the original Hamiltonian description of our dynamical system we require

$$f(\vec{J}^2,J_0)\Lambda(\vec{J}^2,J_0)=1. \tag{14}$$

III. CLASSICAL DIPOLE PRECESSING

Let us demonstrate in this section that the phenomenon of preserving dynamics by nonlinear noncanonical transformation exists also in the classical counterpart of our system. In fact, let $\vec{l}=\vec{r}\times\vec{p}$ be the classical angular momentum with the Poisson brackets

$$\{l_i,l_j\}=\varepsilon_{ijk}l_k. \tag{15}$$

Then the classical Hamiltonian

$$H=-\mu(\vec{B}\cdot\vec{l}) \tag{16}$$

gives the classical equations of motion, i.e., the classical vector field,

$$\dot{\vec{l}}=\mu(\vec{l}\times\vec{B}). \tag{17}$$

Let us introduce c-number classical coordinates

$$l_\pm=\frac{1}{\sqrt{2}}(l_1\pm il_2), \tag{18}$$

which obey the equations of motion

$$\dot{l}_\pm=\mp i\mu Bl_\pm,$$
$$\dot{l}_0=0. \tag{19}$$

Again one sees that the equations of motion for complex variables $l_\pm$ have the same form as the equations of motion for the complex amplitude of the classical harmonic oscillator. This system admits

$$l=\sqrt{\vec{l}^2},l_0 \tag{20}$$

as constants of motion. Then, in analogy with the quantum case of the preceding section, let us consider deformed classical variables related to the previous ones by a nonlinear noncanonical transformation

$$\vec{\tilde{l}}=\vec{l}f_{cl}(l,l_0). \tag{21}$$

One can check that $\vec{\tilde{l}}$ obeys the same dynamics (19); i.e.,

$$\dot{\tilde{l}}_\pm=\mp i\mu B\tilde{l}_\pm,$$

$$\dot{\tilde{l}}_0=0. \tag{22}$$

If the function $f_{cl}(l,l_0)$ is regular so that the map is invertible, the equations of motion in the new variables are equivalent to the old ones.

We see that equations of motion (19), for the usual classical angular momentum, and equations of motion (22) for the deformed classical angular momentum are identical in form. The relation between the angular momentum and deformed angular momentum (21) is a noncanonical nonlinear transformation, which preserve the linear dynamics but changes the Poisson brackets. The Poisson brackets for the deformed angular momentum are

$$\{\tilde{l}_\pm,\tilde{l}_0\}=\mp\Big(f_{cl}(l(\tilde{l}_a),l_0(\tilde{l}_a)) + l_0(\tilde{l}_a)\frac{\partial}{\partial l_0}\ln|f(l(\tilde{l}_a),l_0(\tilde{l}_a))|\Big)\tilde{l}_\pm$$

$$\{\tilde{l}_+,\tilde{l}_-\}=f_{cl}(l(\tilde{l}_a),l_0(\tilde{l}_a))\tilde{l}_0 -2\frac{\partial}{\partial l_0}\ln|f(l(\tilde{l}_a),l_0(\tilde{l}_a))|\tilde{l}_+\tilde{l}_-. \tag{23}$$

The Hamiltonian producing the same dynamics with the above Poisson brackets for the deformed angular momentum has the form

$$H=-\mu f_{cl}(l(\tilde{l}_a),l_0(\tilde{l}_a))(\vec{B}\times\vec{\tilde{l}}). \tag{24}$$

Thus we have constructed an example of a classical counterpart of the quantum Wigner's problem on precessing dynamics of a *generalized angular momentum*. We point out that we call physical angular momentum the one given by $\vec{l}=\vec{r}\times\vec{p}$. The *deformed angular momentum*, in spite that has the same dynamics, is physically different. The angular momentum physically meaningful may be established only through a measuring procedure.

IV. DEFORMED TWO-DIMENSIONAL HARMONIC OSCILLATOR

Let us consider a two-dimensional harmonic oscillator

$$H=\sum_{i=1,2}\omega_j n_j, \tag{25}$$

where the number operators n_j are defined through the creation and annihilation operators a_j and $a_j^\dagger$ in the usual way

$$[a_j,a_j^\dagger]=\delta_{ij},\quad [a_j,a_j]=0,$$

$$n_j=a_j^\dagger a_j. \tag{26}$$

The equations of motion are

$$\dot{a}_j=-i\omega_j a_j=i[H,a_j], \tag{27}$$

which implies

$$a_j=\exp(-i\omega_j t)a_j(0). \tag{28}$$

Let us define the deformed operators

$$\tilde{a}_j=a_jf(n_1,n_2)\equiv a_jf(n_k)=f(n_k+\delta_{jk})a_j. \tag{29}$$

The generators of the $U(2)$ algebra in terms of annihilation and creation operators are transformed as

$$\tilde{a}_j^\dagger\tilde{a}_k=f(n_l)f(n_l+\delta_{kl}-\delta_{jl})a_j^\dagger a_k. \tag{30}$$

Thus, the number operators are related through the relation

$$\tilde{n}_j=f^2(n_l)n_j. \tag{31}$$

The equations of motion in the new deformed variables are given by

$$\dot{\tilde{a}}_j=-i\omega_j\tilde{a}_j,$$

$$\dot{\tilde{a}}_j^\dagger=i\omega_j\tilde{a}_j^\dagger, \tag{32}$$

i.e., they are still linear equations.

We now define the new deformed commutation relations, in analogy to Eq. (11), by

$$[A,B]\to[A,B]_{Q\Lambda}=A\Lambda(n_k)B-QB\Lambda(n_k)A. \tag{33}$$

The basic commutation relations for our deformed operators under the deformed commutation relations shown above are

$$[\tilde{a}_i,\tilde{a}_j]_{Q\Lambda}=\left\{\Lambda(n_k+\delta_{ik})-Q\Lambda(n_k+\delta_{jk})\frac{f(n_k+\delta_{jk})}{f(n_k+\delta_{ik})}\right\}\tilde{a}_i\tilde{a}_j, \tag{34}$$

$$[\tilde{a}_i,\tilde{a}_j^\dagger]_{Q\Lambda}=\Big\{\Lambda(n_k+\delta_{ik})-Q\Lambda(n_k-\delta_{jk}) \times\frac{f(n_k)f(n_k+\delta_{ik}-\delta_{jk})}{f^2(n_k+\delta_{ik})}\Big\}\tilde{a}_i\tilde{a}_j^\dagger + Q\Lambda(n_k-\delta_{jk})f^2(n_k)\delta_{ij}, \tag{35}$$

$$[\tilde{a}_i^\dagger,\tilde{a}_j]_{Q\Lambda}=-\left\{Q\Lambda(n_k+\delta_{jk})-\Lambda(n_k-\delta_{ik})\times\frac{f(n_k)f(n_k-\delta_{ik}+\delta_{jk})}{f^2(n_k+\delta_{jk})}\right\}\tilde{a}_j\tilde{a}_i^\dagger-\Lambda(n_k-\delta_{ik})f^2(n_k)\delta_{ij}, \tag{36}$$

$$[\tilde{a}_i^\dagger,\tilde{a}_j^\dagger]_{Q\Lambda}=\left\{\Lambda(n_k-\delta_{ik})-Q\Lambda(n_k-\delta_{jk})\frac{f(n_k-\delta_{jk})}{f(n_k-\delta_{ik})}\right\}\tilde{a}_i^\dagger\tilde{a}_j^\dagger \tag{37}$$

$$[\tilde{n}_i,\tilde{a}_j]_{Q'\Lambda}=\left\{\Lambda(n_k)-Q'\Lambda(n_k+\delta_{jk})\frac{f^2(n_k+\delta_{jk})}{f^2(n_k)}\right\}\tilde{n}_i\tilde{a}_j-Q'\Lambda(n_k+\delta_{jk})f^2(n_k+\delta_{jk})\delta_{ij}\tilde{a}_j \tag{38}$$

$$[\tilde{n}_i,\tilde{a}_j^\dagger]_{Q'\Lambda}=\left\{\Lambda(n_k)-Q'\Lambda(n_k-\delta_{jk})\frac{f^2(n_k-\delta_{jk})}{f^2(n_k)}\right\}\tilde{n}_i\tilde{a}_j^\dagger+Q'\Lambda(n_k-\delta_{jk})f^2(n_k-\delta_{jk})\delta_{ij}\tilde{a}_j^\dagger. \tag{39}$$

It is not difficult to show that our equations of motion are compatible with these deformed commutation relations. Namely,

$$\frac{d}{dt}[\tilde{a}_j,\tilde{a}_k]_{Q\Lambda}=\left[\frac{d}{dt}\tilde{a}_j,\tilde{a}_k\right]_{Q\Lambda}+\left[\tilde{a}_j,\frac{d}{dt}\tilde{a}_k\right]_{Q\Lambda}, \tag{40}$$

i.e., the dynamics acts as a derivation for the new bracket.

We are able to recover the original Hamiltonian description of our dynamical system (27) by requiring

$$f^2(n_k)\Lambda(n_k)=1. \tag{41}$$

It is also possible to get more well-known forms of quantization in this framework. For example, the one-dimensional q-deformed quantization by Biedenharn [19] and Macfarlane [20] is obtained by adding to Eq. (41) the condition

$$f^2(n+1)=(n+1)\left\{1+[(n+1)Q-Q^{n+1}]\frac{f^2(n-1)}{f^2(n)}\right\}^{-1}, \quad n\geqslant 1. \tag{42}$$

We remark that in order to be able to make these kinds of comparisons, we need different Q's in q commutators (34)–(39), if we deal with q commutators involving the number operators or not.

In generic cases, because our commutation relations are compatible with the dynamical vector field, we can find a Hamiltonian description, i.e., the derivation becomes an inner derivation.

For use in Sec. V, we notice that different labels may be introduced, namely,

$$N=n_1+n_2$$

$$J_3=\tfrac{1}{2}(n_1-n_2), \tag{43}$$

instead of the labels n_1 and n_2; i.e., $f(n_1,n_2)\equiv f(N,J_3)$. With this notation we could write

$$f(n_k\pm\delta_{jk})=f[N\pm 1,J_3+\tfrac{1}{2}(\delta_{j1}-\delta_{j2})].$$

V. JORDAN-SCHWINGER MAPPING

The realization of the angular-momentum algebra using the $SU(2)$ generators is expressed by the Jordan-Schwinger mapping [22,23],

$$J_+=a_1^\dagger a_2,$$

$$J_-=a_2^\dagger a_1,$$

$$J_0=\tfrac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2). \tag{44}$$

Using the dynamics expressed by Eq. (27), we obtain the equations of motion

$$\dot{J}_\pm=\pm i(\omega_1-\omega_2)J_\pm,$$

$$\dot{J}_0=0. \tag{45}$$

Then the dynamical system coincides with the one described in Eq. (6) if the following condition is satisfied:

$$\omega_2-\omega_1=\mu B. \tag{46}$$

Because the Jordan-Schwinger mapping makes the identification $J_0=\frac{1}{2}(n_1-n_2)$ and $\vec{J}^2=\frac{1}{2}N(\frac{1}{2}N+1)$, $N=n_1+n_2$, we see that condition (14) is similar to condition (41), but not identical, because the angular-momentum components transform in an anisotropic way through the Jordan-Schwinger mapping; i.e.,

$$\tilde{J}_\alpha=f(N,J_0)J_\alpha f(N,J_0), \quad \alpha=\pm,0. \tag{47}$$

VI. CONCLUSIONS

In this work we show that analogously to Wigner's problem solved for the harmonic oscillator [13], the dynamics of a magnetic dipole moving under the influence of a constant magnetic field is compatible with a broad class of different commutation relations, including standard q-deformed ones, which obey the same dynamical equations. We establish a nonlinear relation (a deformation) which connects the different components of the angular momentum of the system, both quantum and classically. Finally, through the Jordan-Schwinger map, we connect the problem considered to the already known results about the harmonic oscillator [13]. We hope that examples more manageable for the experimental test may be useful to understand the experimental meaning of ambiguities in the Hamiltonian descriptions.

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