Breathing modes and hidden symmetry of trapped atoms in two dimensions

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Atoms confined in a harmonic potential show universal oscillations in two dimensions (2D). We point out the connection of these ''breathing'' modes to the presence of a hidden symmetry. The underlying symmetry SO(2,1), i.e., the two-dimensional Lorentz group, allows pulsating solutions to be constructed for the interacting quantum system and for the corresponding nonlinear Gross-Pitaevskii equation. We point out how this symmetry can be used as a probe for recently proposed experiments of trapped atoms in 2D. [S1050-2947(97)50502-6]

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The problem of Bose-Einstein condensation in an external potential has received a lot of attention after the experimental observation of the condensation in alkali-atom vapors [1]. In real experiments the trapping potential is approximately harmonic with frequency ω_0. This implies specific peculiarities in the behavior of the system. In recent papers [2–4] a few of these properties have been demonstrated. It has been shown [2] that the nonlinear Gross-Pitaevskii (GP) equation (12) for a trapped two-dimensional (2D) system possesses ''breathing'' oscillatory modes with the universal frequency $2\omega_0$, describing a pulsation of the condensate. The same modes show up in [4], where the authors were able to construct explicitly the time evolution of the GP equation in a time-dependent external potential in 2D. In [3] the authors discovered that the energy spectrum of a system of trapped particles interacting with a $1/r^2$ potential is divided into sets of equidistant levels with the separation $2\omega_0$ again. To our knowledge this interesting property has not yet been properly explained and the connection between these different systems has not been established.

In this paper we shall show that in these cases the existence of these $2\omega_0$ oscillations is ensured by a specific symmetry property of the system. Proper use of this symmetry leads to a transformation that permits a set of breathing mode wave functions to be constructed algebraically, not only for the $1/r^2$ problem but also for a local interaction in two dimensions. Actually this symmetry is not only a property of the mean-field theory as found in [2,4] but of the full quantum theory. We will show that the oscillating solutions of [4] are a continuous representation of the underlying group SO(2,1).

To understand the role of symmetry in the problem, it is useful to consider as an instructive example a system of classical particles moving in a harmonic external potential $V_{\rm pot}=\Sigma_i \frac{1}{2} m\omega_0^2 r_i^2$ and interacting with a potential $V(\boldsymbol{r}_i)$ with the scaling property, $V(\lambda \boldsymbol{r}_i)=\lambda^n V(\boldsymbol{r}_i)$. The position of the ith particle is given by $\boldsymbol{r}_i$, its momentum by $\boldsymbol{p}_i$.

Let us consider the quantity $I=\Sigma_i r_i^2$ such that $\partial_t I = 2\Sigma_i \boldsymbol{r}_i\cdot\boldsymbol{p}_i/m$. Following the usual derivation of the virial theorem in classical mechanics we get

$$\partial_t \sum_i \boldsymbol{p}_i\cdot\boldsymbol{r}_i=\sum_i (\partial_t \boldsymbol{r}_i)\boldsymbol{p}_i-\sum_i \boldsymbol{r}_i\cdot\boldsymbol{\nabla}_i(V+V_{\rm pot})$$
$$=2T-nV-2V_{\rm pot}. \quad (1)$$

We now see that for a potential with the scaling exponent $n=-2$ the right-hand side of (1) takes the form $2E-2m\omega_0^2 I$, where E is the total energy of the system. In this case one gets a closed equation for I:

$$\partial_t^2 I=-4\omega_0^2 I+4E/m, \quad (2)$$

with the obvious solution, $I=A\cos(2\omega_0 t+\gamma)+E/(m\omega_0^2)$. Thus the existence of the ''$2\omega_0$'' modes is connected with the $n=-2$ scaling of the interaction potential. In a 3D system the only potential possessing this property is the $1/r^2$ interaction used in [3]. But in the quantum 2D case the Fermi ''pseudopotential'' $\frac{1}{2}g\delta^2(\boldsymbol{r})$, as used in the GP equation [2,4], gives the same scaling.

The equations above can be rewritten by introducing (the notations will become obvious later) $L^+=\Sigma_i \boldsymbol{p}_i\cdot\boldsymbol{r}_i/2 + i(E-m\omega_0^2 I)/(2\omega_0)$. We simply get $\partial_t L^+ = i2\omega_0 L^+$. The phase of L^+ varies linearly with time:

$$\Phi(\boldsymbol{p}_i,\boldsymbol{r}_i)\equiv\frac{1}{2\omega_0}{\rm Im}\ \ln L^+$$
$$=\frac{1}{2\omega_0}\arctan\frac{(E-m\omega_0^2 I)/(2\omega_0)}{\Sigma_i \boldsymbol{p}_i\cdot\boldsymbol{r}_i/2}$$
$$\Phi(\boldsymbol{p}_i,\boldsymbol{r}_i)-\Phi(\boldsymbol{p}_i^0,\boldsymbol{r}_i^0)=t-t_0,$$

where $\boldsymbol{p}_i^0,\boldsymbol{r}_i^0$ are the coordinates at $t=t_0$. Φ can now be used to determine the ''abbreviated action'' $S(E,\boldsymbol{r}_i)$, which is a function of the energy and the coordinates at the end of a path. It is determined by the Hamilton-Jacobi equation $H(\partial_{\boldsymbol{r}_i}S,\boldsymbol{r}_i)=E$ with $t=\partial_E S=\Phi(\partial_{\boldsymbol{r}_i}S,\boldsymbol{r})$. In hyperspherical coordinates in the space of all $\mathbf{r}_i$ with $r=\sqrt{I}$, Φ is only a function of r and $\partial_r S$. Therefore we have $S(E,\boldsymbol{r}_i)=S(E,r)+S_0(\alpha_i)$, where the α_i denote all other coordinates, clearly showing that the coordinate r or I totally separates.

One of the most powerful methods in physics is the use of symmetries and groups. One way is to use the invariance of the Hamiltonian or the action under certain transformations; another is recognizing that the Hamiltonian is a part of some larger algebra. The most famous textbook example is the algebraic solution of the harmonic oscillator using the spectrum generating Heisenberg algebra $[H,a^{\pm}]=\pm\omega_0 a^{\pm}$ (we put $\hbar=1$ throughout the paper).

We will now discuss such a spectrum generating symmetry for the (now quantum-mechanical) problem of interacting particles in a harmonic trap. First we will consider the effect of a scaling transformation for the Hamiltonian

$$H_0=\sum_i -\frac{1}{2m}\Delta_i+\sum_{i<j} V(\boldsymbol{r}_i-\boldsymbol{r}_j)$$

without an external potential:

$$\boldsymbol{r}\rightarrow\lambda\boldsymbol{r},\Psi(\boldsymbol{r})\rightarrow\lambda^{d/2}\Psi(\lambda\boldsymbol{r}),$$

$$H_0\rightarrow\frac{1}{\lambda^2}\sum_i -\frac{1}{2m}\Delta_i+\sum_{i<j} V(\lambda(\boldsymbol{r}_i-\boldsymbol{r}_j)). \tag{3}$$

H_0 is scale invariant if $V(\lambda\boldsymbol{r})=V(\boldsymbol{r})/\lambda^2$. This is the case for an interaction of the form $V(\boldsymbol{r})=g/r^2$ in any dimension, but as mentioned above also for

$$V(\boldsymbol{r}-\boldsymbol{r}')=\frac{1}{2}g\,\delta^2(\boldsymbol{r}-\boldsymbol{r}') \tag{4}$$

in two dimensions. The first case is known as the Calogero-Sutherland model [5] and is widely investigated. All our results apply for both cases, as we will only use symmetry properties connected to scale transformations, but we concentrate on the problem of bosons with a local interaction and only briefly comment on the relation to known results for the Calogero-Sutherland model. Actually the local interaction is the most important case, at least for neutral atoms at low energies, where the range of interaction is small compared to all other scales. The statistics of the particles does not play a role.

It is important to note that a δ-function interaction is not well defined in two dimensions due to logarithmic ultraviolet divergences that are cut off by the finite range a_0 of the interaction. This length a_0 obviously breaks the scale invariance of H_0 and will therefore modify our results; nevertheless, this effect will be small as long as a_0 is smaller than any other scale in the system. This is most clearly seen in the classical wave limit (i.e., the GP equation), which we will discuss later, where such a problem is absent.

Adding an external potential $H=H_0+H_{\rm pot}$, $H_{\rm pot}=\Sigma_i\frac{1}{2}m\omega_0^2 r_i^2$ obviously breaks scale invariance, as $H_{\rm pot}\rightarrow\lambda^2 H_{\rm pot}$ under a scale transformation. However, due to a special property of the harmonic oscillator, a powerful spectrum generating symmetry still exists.

The important step is to recognize that the commutator of the harmonic potential with the Hamiltonian or the time derivative of $\Sigma_i r_i^2$ is proportional to the generator of scale transformations:

$$[H_{\rm pot},H]=\left[\sum_j \frac{1}{2}m\omega_0^2 r_j^2,\sum_i -\frac{1}{2m}\boldsymbol{\nabla}_i^2\right]$$

$$=\sum_i \frac{1}{2}\omega_0^2(\boldsymbol{\nabla}_i\cdot\boldsymbol{r}_i+\boldsymbol{r}_i\cdot\boldsymbol{\nabla}_i)=i\omega_0^2 Q \tag{5}$$

$$Q=\sum_i \frac{1}{2}(\boldsymbol{p}_i\cdot\boldsymbol{r}_i+\boldsymbol{r}_i\cdot\boldsymbol{p}_i).$$

Q is the generator of scale transformations, as it describes the translation of the coordinates $\boldsymbol{r}_i$ by an amount proportional to $\boldsymbol{r}_i$.

We can collect our results (3,5) in the following algebra:

$$[Q,H_0]=2iH_0,\quad [Q,H_{\rm pot}]=-2iH_{\rm pot},$$

$$[H_{\rm pot},H]=i\omega_0^2 Q,$$

or using

$$L_1=\frac{1}{2\omega_0}(H_0-H_{\rm pot}),\quad L_2=\frac{Q}{2},$$

$$L_3=\frac{1}{2\omega_0}(H_0+H_{\rm pot})=\frac{1}{2\omega_0}H, \tag{6}$$

we get the algebra

$$[L_1,L_2]=-iL_3,\quad [L_2,L_3]=iL_1,\quad [L_3,L_1]=iL_2. \tag{7}$$

This is the well-known algebra of SU(1,1) or SO(2,1), the two-dimensional Lorentz group. L_1 and L_2 are the generators of the two ''boosts'' and $L_3=(1/2\omega_0)H$, i.e., the generator of time translations, is the analog of the generator of the rotation. With $L^{\pm}=(1/\sqrt{2})(L_1\pm iL_2)$ this reads

$$[H,L^{\pm}]=\pm 2\omega_0 L^{\pm},\quad [L^+,L^-]=-\frac{1}{2\omega_0}H. \tag{8}$$

Note the minus sign in the last equation, indicating that the group is the Lorentz group SO(2,1) [or SU(1,1)] and not SO(3).

One important consequence of this spectrum-generating symmetry is that the Hilbert space will separate into irreducible representations of the group. If the energy is bounded from below, these are discrete infinite-dimensional representations with no upper bound. Starting from the lowest eigenstate in one of the representations with energy E_0, $H|\Psi_0\rangle=E_0|\Psi_0\rangle$, one can construct higher states with energies $E_0+n2\omega_0$, $n=1,2,\dots$ by applying L^+ [use $HL^+|\Psi_0\rangle=(L^+H+2\omega_0 L^+)|\Psi_0\rangle=(E_0+2\omega_0)L^+|\Psi_0\rangle$]. $|\Psi_0\rangle$ is annihilated by L^-. Obviously an infinite number of excitations with energies $n2\omega_0$ exists, which we will identify with the breathing modes of the system.

Also the time dependence of all the operators, which are part of the algebra, can be given explicitly:

$$L^{\pm}(t)=e^{\mp i2w_0 t}L^{\pm}. \tag{9}$$

Defining the operator for the mean-square displacement of the particles $\hat{I}(t)=\Sigma\boldsymbol{r}_i^2(t)$, one finds

$$\hat{I}(t)=\frac{2}{m\omega_0^2}H_{\rm pot}(t)$$

$$=\frac{1}{m\omega_0^2}[H-2\omega_0\sqrt{2}\,{\rm Re}(L^+e^{-2w_0 t})]. \tag{10}$$

This equation or the corresponding differential equation $\partial_t^2 \hat{I} = -(2\omega_0)^2[\hat{I} - H/(m\omega_0^2)]$, coinciding with the classical Eq. (2), clearly show that the variable $\hat{I}$ is separated from all other variables, in analogy to the center-of-mass motion. This separation is known in the case of the $1/r^2$ interaction [3]. As the frequency ω_0 of the external potential obviously couples only to this coordinate, the full nonlinear response to a change of the potential can also be calculated [3,4].

For the expectation value we directly get

$$I(t) = \langle \hat{I}(t) \rangle = I_0 + A\cos(2\omega_0 t + \gamma), \tag{11}$$

with $I_0 = \langle H \rangle/(m\omega_0^2)$ and $Ae^{-i\gamma} = \sqrt{2}/(m\omega_0)\langle L^+ \rangle_{t=0}$. The same solution was found recently for the GP equation in $D=2$ [2,4], but here we show that it also holds for the full quantum system. It is valid not only for the expectation value but directly for the operators (10) and is due to a simple underlying symmetry. We will discuss the change of this equation under a group transformation later.

The Casimir operator of this group $C = H^2 - (2\omega_0)^2(L_1^2 + L_2^2)$ commutes not only with the Hamiltonian, giving a new conserved quantity, but also with the generators of the group and is the analog of the invariant line element in special relativity.

The classical wave limit of interacting bosons, i.e., the nonlinear GP equation [6], has recently attracted considerable attention, as it accurately describes the Bose-condensate of trapped atoms and allows for reliable analytic, and especially numerical, calculations [7]. It is also an interesting problem on its own for mathematical physics. We will show that the previously discovered analytic solutions [4] actually form a continuous representation of the group SO(2,1) for the GP equation.

In [4] the following transformation of the solution of the GP equation ($m = \hbar = 1$),

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\Delta\Psi + \frac{1}{2}\omega_0^2 r^2 \Psi + g|\Psi|^2\Psi - \mu\Psi, \tag{12}$$

was considered (in our notation):

$$\Psi_2(\boldsymbol{r},t) = \exp[i(b + cr^2)]\frac{1}{\sqrt{a}}\Psi_1(\boldsymbol{u},\tau), \tag{13}$$

$$\boldsymbol{u} = \frac{\boldsymbol{r}}{\sqrt{a(t)}}, \quad \tau(t) = \int^t \frac{dt'}{a(t')}. \tag{14}$$

If the functions $a(t)$, $b(t)$, and $c(t)$ fulfill ($a_t = \partial_t a$, etc.),

$$a_t = 4ac, \quad b_t = \mu\left(1 - \frac{1}{a}\right),$$

$$a_{tt}a - \frac{a_t^2}{2} + \frac{(2\tilde{\omega}_0)^2}{2}a^2 - \frac{(2\omega_0)^2}{2} = 0, \tag{15}$$

and if Ψ_1 is a solution of (12), then Ψ_2 is also a solution of the GP equation with a possibly time-dependent frequency $\tilde{\omega}_0(t)$. Differentiating (15) gives the linear equation $a_{ttt} + (2\tilde{\omega}_0)^2 a_t = -4\tilde{\omega}_0\tilde{\omega}_{0t}a$, which once again demonstrates the universal nature of these modes; the initial values must fulfill (15).

We will now consider the case $\tilde{\omega}_0 = \omega_0 = \text{const}$, where the differential equations (15) can can be solved directly:

$$a = \sinh\eta\cos(2\omega_0 t + \gamma) + \cosh\eta, \tag{16}$$

$$\begin{aligned}\tau &= \frac{1}{\omega_0}\{\arctan[e^{-\eta}\tan(\omega_0 t)] + \pi n\} \\ &= \frac{1}{2\omega_0}\left[\arccos\left(\frac{\sinh\eta + \cosh\eta\cos 2\omega_0 t}{\cosh\eta + \sinh\eta\cos 2\omega_0 t}\right)\right. \\ &\quad \left.\times\,\mathrm{sgn}(\sin 2\omega_0 t) + 2\pi n\right]\end{aligned} \tag{17}$$

and $c(\eta,t), b(\eta,t)$ accordingly. The integer n has to be chosen to get a continuous solution, $n(t) = [\omega_0 t/\pi + 1/2]$, where $[x]$ denotes the greatest integer less than x. A particularly important example is the case when the initial solution is a static one, e.g., the ground state, $\Psi_1 = \Psi_0(\boldsymbol{r})$. The transformation builds then from such a static solution an oscillating breathing solution. In this case the parameter η of the transformation defines the relation of the energy of the new solution E to the static one E_0 according to $E = \cosh(\eta)E_0$. In the following we set the phase $\gamma = 0$ in (16), as a finite γ can be achieved by a simple translation in time.

Together with the time translations the discussed transformation form a group, indeed a continuous representation of SO(2,1). A general group transformation U_{t_2,η,t_1} can be described by an initial translation in time by t_1, the above-described scaling transformation parametrized by η, and a second final translation backwards in time can be described by t_2. This is the analog to the description of rotations by Euler angles. With $\tau' = \tau(\eta, t - t_1)$ and a', b', c' accordingly we have

$$U_{t_2,\eta,t_1}\Psi = e^{i(b' + c'r^2)}\frac{1}{\sqrt{a'}}\Psi\left(\frac{1}{\sqrt{a'}}\boldsymbol{r}, \tau' + t_2\right). \tag{18}$$

It is now simple to work out the multiplication rules of this group — they are the same as for two-dimensional Lorentz transformations. For example, if one performs two successive "Lorentz boosts" in the same direction one has to add the rapidities:

$$U_{0,\eta_2,0} \circ U_{0,\eta_1,0} = U_{0,\eta_1+\eta_2,0}. \tag{19}$$

We can see this by calculating the corresponding function $a(t)$ for two transformations using (17):

$$\begin{aligned}a &= (\cosh\eta_2 + \sinh\eta_2\cos 2\omega_0 t)\cosh\eta_1 \\ &\quad + \sinh\eta_1\cos[2\omega_0\tau(\eta_2,t)] \\ &= \cosh(\eta_1+\eta_2) + \sinh(\eta_1+\eta_2)\cos 2\omega_0 t.\end{aligned} \tag{20}$$

To make the connection to the previously constructed algebra, one has to calculate the generators of the group, i.e. the infinitesimal transformations $\delta\eta \to 0$.

$$a \approx 1+\delta\eta\cos 2\omega_0 t, \quad \tau \approx t-\delta\eta\frac{1}{2\omega_0}\sin 2\omega_0 t.$$

Under such a transformation the wave function changes by $\delta\Psi = U_{0,\delta\eta,0}\Psi - \Psi$:

$$\delta\Psi \approx -i\,\delta\eta\left[\frac{1}{4}(\boldsymbol{r}\cdot\boldsymbol{p}+\boldsymbol{p}\cdot\boldsymbol{r})\cos 2\omega_0 t - \frac{1}{2\omega_0}\sin 2\omega_0 t\left(i\frac{\partial}{\partial t}-2\frac{1}{2}\omega_0^2 r^2\right)\right]\Psi \tag{21}$$

with $\boldsymbol{p}=-i\boldsymbol{\nabla}$. This has to be compared with $L_1 = \frac{1}{4}(\boldsymbol{r}\cdot\boldsymbol{p}+\boldsymbol{p}\cdot\boldsymbol{r})$ and $L_2=(1/2\omega_0)(H-2H_{\rm pot})$. Identifying H with $i\partial_t$ and noting that in the Heisenberg picture $L_2(t)=\sqrt{2}\ {\rm Im}L^+(t)=L_2\cos 2\omega_0 t - L_1 \sin 2\omega_0 t$, we can identify $U_{0,\eta,0}$ with the transformation generated by L_2. Accordingly L_1 generates the transformation $U_{0,\eta,\pi/(2\omega_0)}$ and H obviously $U_{t,0,0}$. Actually the solutions generated by L_1 and L_2 describe the breathing of the system, i.e. a pulsating motion.

To finish this section we directly evaluate the effect of a group transformation onto the mean-square displacement of the bosons $I(t)=\int r^2|\Psi|d^2r$. From (17) we know that, for any given solution Ψ_0, $I(t)$ is of the form $I(t)=C(\cosh\eta_0+\sinh\eta_0\cos 2\omega_0 t)$ as long as the system does not collapse, i.e., as long as $I(t)>0$. (Such a collapse is of course possible only for an attractive interaction, i.e., for $g<0$.) With $\Psi'=U_{0,\eta,0}\Psi$ and using (20) we get

$$I'(t)=\int r^2|\Psi'(\boldsymbol{r},t)|d^2r=a(t)I(\tau(t)) = C(\cosh(\eta_0+\eta)+\sinh(\eta_0+\eta)\cos 2\omega_0 t). \tag{22}$$

In particular, any solution without a breathing motion ($\eta_0=0$) will be transformed into a breathing one and on the other side any breathing motion can be transformed to 0 by choosing $\eta=-\eta_0$.

As mentioned below Eq. (11) all these properties are a direct consequence of the underlying group structure and $\omega_0^2 C\cosh(\eta_0+\eta)$ can be identified with the energy of the system, while the amplitude of the oscillations $C\sinh(\eta_0+\eta)$ is $2/\omega_0\sqrt{\langle L_1\rangle^2+\langle L_2\rangle}$. The quantity $\widetilde{C}=\langle H\rangle^2-(2\omega_0)^2(\langle L_1\rangle^2+\langle L_2\rangle^2)$ is conserved under a group transformation ($\cosh^2-\sinh^2=1$) and plays the role of the Casimir invariant. Generally, if $\widetilde{C}$ is positive, one can always find a group transformation, so that $\langle L_1'\rangle^2=\langle L_2'\rangle^2=0$ and therefore $I(t)'=\sqrt{\widetilde{C}}/\omega_0^2=$const. For $\widetilde{C}<0$ this is not possible and $I(t)$ describes the collapse of the system, as has been discussed in [2].

To conclude, we have shown that the existence of breathing oscillations of 2D atoms in a harmonic trap is ensured by a hidden symmetry of the system. If the atoms interact to a good approximation by a Fermi pseudopotential, i.e., by a local interaction, we expect well-defined modes with a frequency of exactly $2\omega_0$. We identified the corresponding solutions of [4] with a continuous representation of the underlying ''Lorentz'' group SO(2,1).

The experimental conditions for the realization of a 2D system in a magnetic trap look quite promising in the case when most of the atoms are in the condensate. The system can be considered as 2D if $\omega_z \gg \omega_0$ is fulfilled, but our calculations are also valid for a system without condensation, e.g., for fermions. In this case the condition for two dimensionality also demands that the effective temperature be less than the level separation in the z direction: $T\ll\hbar\omega_z$. On the contrary, one can also use an asymmetric trap [9], which is prolonged in the z direction, and excite oscillations in the condensate with no z dependence.

Important new possibilities are opened by a recent proposal [8] to confine atoms in a 2D optical dipole trap. In such an experiment the appearance of sharp $2\omega_0$ frequencies would be the first demonstration of the two-dimensional nature of the system. Experimentally it is also easy to excite the $2\omega_0$ modes by a change of the external potential as described in [4]. Precision measurements of this response permit one to check the assumptions of the model. Especially, the validity of a local interaction, which was generally accepted so far, could be investigated. It will be interesting to study deviations from such a description both experimentally and theoretically.

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