Nonunitary connection between explicitly time-dependent and nonlinear approaches for the description of dissipative quantum systems

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Caldirola [Nuovo Cimento **18**, 393 (1941)] and Kanai's [Prog. Theor. Phys. **3**, 440 (1948)] time-dependent approach for dissipative systems can be traced back to the conventional system-plus-reservoir approach; however, it apparently violates the uncertainty principle. This discrepancy can be avoided by considering a consistent transition between the Caldirola-Kanai formalism and a logarithmic nonlinear Schrödinger equation that requires not only a transformation of the operators, but also a nonunitary transformation of the wave function. This procedure also shows the equivalence of three at first sight incompatible approaches to describe dissipative quantum systems. [S1050-2947(97)03102-8]

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I. INTRODUCTION

Although irreversibility and dissipation are the rule in the perceivable macroscopic world, these phenomena do not explicitly enter the microscopic equations of motion on either the classical or the quantum-mechanical level. The question of how to modify the classical Lagrangian and Hamiltonian formalisms and the corresponding quantization methods in order to include these aspects has not yet been answered in a unique way that would be generally accepted. Different approaches exist in the literature that have their peculiar advantages and shortcomings and most of them seem to be incompatible with each other. In this paper three types of these apparently incompatible approaches that are frequently applied in the literature shall be discussed and it will be shown that they are actually intimately connected. These approaches are (i) the system-plus-reservoir model, (ii) attempts to use explicitly time-dependent Lagrangians and Hamiltonians, and (iii) nonlinear Schrödinger equations (NLSEs). The existence of a unique transformation between the latter two approaches will be explicitly shown and it will become clear how presumed shortcomings can be dispensed with if the problem is properly treated.

II. PROBLEMS WITH THE CALDIROLA-KANAI APPROACH

The system-plus-reservoir ansatz is based on conventional conservative quantum mechanics. It couples the system of interest with a large number of environmental degrees of freedom. Subsequent elimination of these degrees of freedom reduces the entire conservative system to a ''relevant'' dissipative system. Well-known models of this kind have been proposed by Caldeira and Leggett [1] and by Ford *et al.* [2]. However, the consideration of a large number of degrees of freedom, each of which is governed by a partial differential equation, [3] might, in certain cases, e.g., for the study of the quantum limit of chaotic dissipative systems, exhaust or even go beyond the power of the largest and fastest computers.

Since the main interest is usually focused only on a few degrees of freedom, more phenomenologically motivated approaches have been proposed that do not take the environmental degrees of freedom explicitly into account, but try to describe the system of interest with the help of some kind of effective Hamiltonian (see, e.g., [4] and references cited therein). These Hamiltonians can be either explicitly time dependent or they may contain additional nonlinear friction terms, thus leading to NLSEs.

The historically first and most frequently used time-dependent Hamiltonian was proposed by Caldirola [5] and Kanai [6]; a similar approach, using an expanding coordinate system and a Hamiltonian that is a time-dependent invariant, has been discussed by the present author [7]. Both models can be connected on the classical level via a canonical transformation. The application of (linear) time-dependent Hamiltonians would permit the direct use of standard schemes of quantization. Although the classical Hamiltonian function of Caldirola and Kanai,

$$H_{\mathrm{CK}}=\frac{1}{2m}e^{-\gamma t}\tilde{p}_{\mathrm{CK}}^{2}+e^{\gamma t}V(x), \tag{1}$$

with friction constant γ, yields the proper equation of motion for a system under the influence of a linear velocity- (or momentum-) dependent frictional force

$$m\ddot{x}+m\gamma\dot{x}+\frac{\partial}{\partial x}V(x)=0, \tag{2}$$

a problem arises after canonical quantization. Since the canonical momentum

$$\tilde{p}_{\mathrm{CK}}=m\dot{x}e^{\gamma t}=pe^{\gamma t}, \tag{3}$$

which corresponds to the operator $\tilde{p}_{\mathrm{op}}=(\hbar/i)(\partial/\partial x)$, differs from the physical kinetic momentum $p=m\dot{x}$, the commutator of physical momentum and position, and hence the uncertainty product, decay exponentially. This apparent violation of the uncertainty principle gave rise to serious criticisms [8–10] as well as attempts to avoid [11,12] or

explain [13] this peculiarity of the Caldirola-Kanai (CK) model. (For a review, see also [14].)

Studying the literature carefully, one is under the impression that the explanations given for the violation of the uncertainty principle are not totally convincing and that the critics might be correct. However, a different aspect was recently added, when Sun and Yu [15,16] were able to show that it is possible to obtain the CK Hamiltonian *operator*, starting with the conventional system-plus-reservoir approach. This leads to the paradoxical situation where a Hamiltonian operator, which can be traced back to conventional quantum mechanics, apparently causes violation of one of the most fundamental principles of this theory, the uncertainty principle. After a brief discussion of the third type of approach, the NLSEs, it will be shown how this paradox can be eliminated.

III. NONLINEAR APPROACHES

A way to circumvent the problem of finding a classical Hamiltonian for dissipative systems that can be canonically quantized is to add a friction term directly to the Hamiltonian operator,

$$H_{\mathrm{NL}}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)+W=H_L+W. \tag{4}$$

There are several attempts of this kind in the literature (see, e.g., [4,17–19]). Most of them use as a guideline to finding the proper form of W, that Ehrenfest's theorem in the form

$$\frac{d}{dt}\langle p\rangle+\gamma\langle p\rangle+\left\langle\frac{\partial}{\partial x}V\right\rangle=0 \tag{5}$$

has to be fulfilled, where $\langle\ \rangle=\int\Psi^*\cdots\Psi\,dx$ denotes mean values calculated with the help of the wave function $\Psi(x,t)$ and its complex conjugate Ψ^*. From the Heisenberg equations of motion it follows then that the additional term W is defined via

$$\left\langle\frac{\partial}{\partial x}W\right\rangle=\gamma\langle p\rangle. \tag{6}$$

This definition, however, is by far not unique as it contains only the mean value of the derivative of W. In addition, different definitions of momenta occur in quantum mechanics, all of which have the same mean value.

In Schrödinger's first communication on wave mechanics [20] he starts with the Hamilton-Jacobi (HJ) equation

$$\frac{\partial}{\partial t}S_C+H\left(x,\frac{\partial}{\partial x}S_C,t\right)=0, \tag{7}$$

with the action function S_C and the momentum $p_C=(\partial/\partial x)S_C$. He introduces the wave function $\Psi(x,t)$ through $S_C=(\hbar/i)\ln\Psi$ and arrives, via a variational ansatz, at the Hamiltonian operator H_L. With his definition of S_C, the momentum reads

$$p_C=\frac{\hbar}{i}\frac{\partial}{\partial x}\ln\Psi=\frac{\hbar}{i}\frac{(\partial/\partial x)\,\Psi}{\Psi}, \tag{8}$$

which is a *complex* quantity, if Ψ is complex, i.e., $S_C=S_R+S_I$. Obviously, the mean value of p_C is identical to that of the quantum-mechanical momentum operator $p_{\mathrm{op}}=(\hbar/i)(\partial/\partial x),\langle p_C\rangle=\langle(\hbar/i)(\partial/\partial x)\rangle$. The density $\varrho=\Psi^*\Psi$ obeys the continuity equation (CE)

$$\frac{\partial}{\partial t}\varrho+\frac{1}{m}\frac{\partial}{\partial x}(\varrho p_R)=0, \tag{9}$$

where only the *real* part of p_C, namely, $p_R=(\hbar/2mi)(\partial/\partial x)\ln(\Psi/\Psi^*)$, occurs. A comparison shows, however, that p_R also has the same mean value as p_C and p_{op} since the mean value of the imaginary part $p_I=(\hbar/2mi)(\partial/\partial x)\ln\varrho$ always vanishes.

In the most-often-discussed and ''rederived'' nonlinear approach, Kostin [17] clearly uses only the real part p_R, thus arriving at the real friction term

$$W_K=\gamma\frac{\hbar}{2i}\left(\ln\frac{\Psi}{\Psi^*}-\left\langle\ln\frac{\Psi}{\Psi^*}\right\rangle\right), \tag{10}$$

where the mean value occurring guarantees that $\langle W_K\rangle=0$ and thus $\langle E\rangle=\langle H_L\rangle$ is still valid. As stated in the literature [4], this term leads to results that are difficult to explain in physical terms, as it also admits the stationary states of the undamped harmonic oscillator (HO) as solutions and yields the unshifted frequency instead of the reduced classical one. In addition, the density $\varrho(x,t)$ fulfills the *reversible* CE, although the system displays *irreversible dynamics*.

IV. CONNECTION BETWEEN THE LOGARITHMIC NLSE AND THE CALDIROLA-KANAI APPROACH

It has been shown [21,22] that all these problems can be solved or avoided by an approach starting with a modified irreversible density equation and using a separation method, which was introduced by Madelung [23] and Mrowka [24], to rederive the conventional linear Schrödinger equation (SE), without applying Hamilton's form of classical mechanics, but only Newton's form. The modification consists of an additional diffusion term to the CE, thus arriving at the Fokker-Planck-type equation (FPE)

$$\frac{\partial}{\partial t}\varrho+\frac{1}{m}\frac{\partial}{\partial x}(\varrho p_R)-D\frac{\partial^2}{\partial x^2}\varrho=0. \tag{11}$$

The inclusion of a diffusion current density was later supported by group-theoretical arguments [25]. In order to achieve separation into two equations for the complex wave amplitudes Ψ and Ψ^*, the additional condition

$$-D\frac{(\partial^2/\partial x^2)\,\varrho}{\varrho}=\gamma(\ln\varrho-\langle\ln\varrho\rangle) \tag{12}$$

has to be fulfilled, where the mean value on the right-hand side guarantees normalizability. After separation, the SE contains an additional logarithmic nonlinear term (the logarithmic NLSE)

$$i\hbar\frac{\partial}{\partial t}\Psi=\left(H_L+\gamma\frac{\hbar}{i}(\ln\Psi-\langle\ln\Psi\rangle)\right)\Psi. \tag{13}$$

When Ehrenfest's theorem is considered, the physical meaning of the nonlinear (NL) term becomes obvious, as the logarithmic term gives rise to the additional friction term $\gamma\langle p\rangle$ in the classical equation of motion. Here, however, the NL term in the NLSE is not derived from the classical friction term, but, to the contrary, the correct classical friction term results from the additional irreversible diffusion term in connection with the separation condition (12). For further details see [21,22].

The connection between our logarithmic NLSE and the time-dependent CK approach shall now be elucidated. For this purpose, we reverse the procedure of Schrödinger's original quantization method, divide the NLSE by Ψ, and use the definition of action S_C, yielding

$$\left(\frac{\partial}{\partial t}+\gamma\right)S_C+H=-\gamma\langle S_C\rangle. \tag{14}$$

This is, of course, as little rigorous as Schrödinger's first attempt was; however, it follows his way of thinking of how to connect the classical HJ theory with a wave (mechanical) equation.

Equation (14) now shows some similarity to an approach used by Razavy [26] and also discussed by Wagner [27]. However, in agreement with Kostin, these authors also use only the real part S_R of the complex action function S_C. S_R enters a (real) HJ-type equation (with an additional so-called quantum potential) if Madelung's hydrodynamical form of quantum mechanics is used. The imaginary part is ignored, as it would yield an additional term to the CE and might thus violate the normalizability of the wave functions. The above-mentioned FPE shows that this is not necessarily the case as it still allows normalization.

The term $-\gamma\langle S_C\rangle$ is necessary mainly for normalization purposes and shall be neglected for the moment. Multiplying the remaining left-hand side of Eq. (14) by $e^{\gamma t}$ and using the definitions

$$\widetilde{S}_C=e^{\gamma t}S_C, \quad \widetilde{H}=e^{\gamma t}H \tag{15}$$

finally yields the HJ equation

$$\frac{\partial}{\partial t}\widetilde{S}_C+\widetilde{H}=0. \tag{16}$$

From the definitions of the action functions, it follows that the wave function $\widetilde{\Psi}$ in the transformed system is connected with the wave function Ψ in the physical system via

$$\ln\widetilde{\Psi}=e^{\gamma t}\ln\Psi. \tag{17}$$

Consequently, the complex momenta in the two systems are connected via

$$\widetilde{p}_C=\frac{\hbar}{i}\frac{\partial}{\partial x}\ln\widetilde{\Psi}=e^{\gamma t}\frac{\hbar}{i}\frac{\partial}{\partial x}\ln\Psi=e^{\gamma t}p_C, \tag{18}$$

which is equivalent to the connection between canonical and kinetic momentum in the CK theory. The *noncanonical* connection between the classical variables (x,p) and $(x,\widetilde{p}_{\mathrm{CK}})$ corresponds to the *nonunitary* connection between Ψ and $\widetilde{\Psi}$. Although Ψ and $\widetilde{\Psi}$ both depend explicitly only on x and t, the connection between the momentum and $\ln\Psi$ has the consequence that $\widetilde{\Psi}(x,t)$ in the transformed system is *different* from $\Psi(x,t)$ in the physical system.

Expressing $\widetilde{H}$ in terms of the canonical momentum $\widetilde{p}_C$,

$$\widetilde{H}(x,\widetilde{p}_C,t)=e^{-\gamma t}\frac{\widetilde{p}_C^{\,2}}{2m}+e^{\gamma t}V(x), \tag{19}$$

and following Schrödinger's quantization procedure, but now using the canonical momentum $\widetilde{p}_C$, finally yields

$$i\hbar\frac{\partial}{\partial t}\widetilde{\Psi}(x,t)=\widetilde{H}_{\mathrm{op}}\widetilde{\Psi}(x,t)=\left[e^{-\gamma t}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)+e^{\gamma t}V\right]\widetilde{\Psi}$$
$$=H_{\mathrm{CK,op}}\widetilde{\Psi}, \tag{20}$$

where the Hamiltonian *operator* $\widetilde{H}_{\mathrm{op}}$ is identical to the one of CK and $\widetilde{p}_{\mathrm{op}}=(\hbar/i)(\partial/\partial x)$.

Here it is remarked that $\widetilde{p}_C^{\,2}$ in Eq. (19) would actually lead to a term proportional to $[(\partial/\partial x)\widetilde{\Psi}/\widetilde{\Psi}]^2$ that differs from $(\partial^2/\partial x^2)\widetilde{\Psi}$ by a term proportional to $(\partial/\partial x)\widetilde{p}_C=(\partial/\partial x)[(\partial/\partial x)\widetilde{\Psi}/\widetilde{\Psi}]$. Following Schrödinger's variational ansatz for the time-independent case, the contribution from the latter term must disappear, which leads to restrictions in the behavior of the variation of the wave function at infinity. For the time-dependent case, when considering problems with Gaussian wave-packet solutions (and these are the ones that will be discussed in detail in this paper), $(\partial/\partial x)\widetilde{p}_C$ is either zero or a mere time-dependent function that can be included in the normalization factor of the wave packet. (Further details are under investigation and are planned to be discussed in forthcoming papers.)

N.B. The major difference between Eq. (20) and the CK theory is that in the latter, the operator $H_{\mathrm{CK,op}}$ acts on the wave function $\Psi(x,t)$ in physical space, instead of acting on $\widetilde{\Psi}(x,t)$ in the transformed system. However, for consistency reasons it is necessary, if the momentum is transformed according to $p_C\rightarrow\widetilde{p}_C$, that the wave function also has to be transformed accordingly, $\Psi\rightarrow\widetilde{\Psi}$, since $p_C=(\hbar/i)(\partial/\partial x)\ln\Psi\rightarrow\widetilde{p}_C=(\hbar/i)(\partial/\partial x)\ln\widetilde{\Psi}$, where Ψ and $\widetilde{\Psi}$ are connected via Eq. (17).

Since the connection between Ψ and $\widetilde{\Psi}$ is not unitary, it follows that if the solution $\widetilde{\Psi}$ of Eq. (20) is normalized, the solution Ψ after transformation, i.e., the solution of the logarithmic NLSE without the term $\gamma\langle S_C\rangle$, is not normalized. So, for normalization purposes, $\gamma\langle S_C\rangle$ has to be subtracted from the right-hand side of the equation for Ψ, thus leading to the NLSE (13). In Fig. 1 the connection between the different levels of description, classical (left) as well as quantum mechanical, is shown schematically.

Considering effective Hamiltonians for dissipative systems, one has to distinguish between two levels of description: (i) the physical level with physical coordinate x and kinetic momentum p [and wave function $\Psi(x,t)$] and (ii) the canonical level with canonical coordinate $\widetilde{x}$ (which in our example is identical to the physical coordinate x) and canonical momentum $\widetilde{p}$.

On the physical level, these dissipative systems cannot be described by the canonical formalism (and probably also not

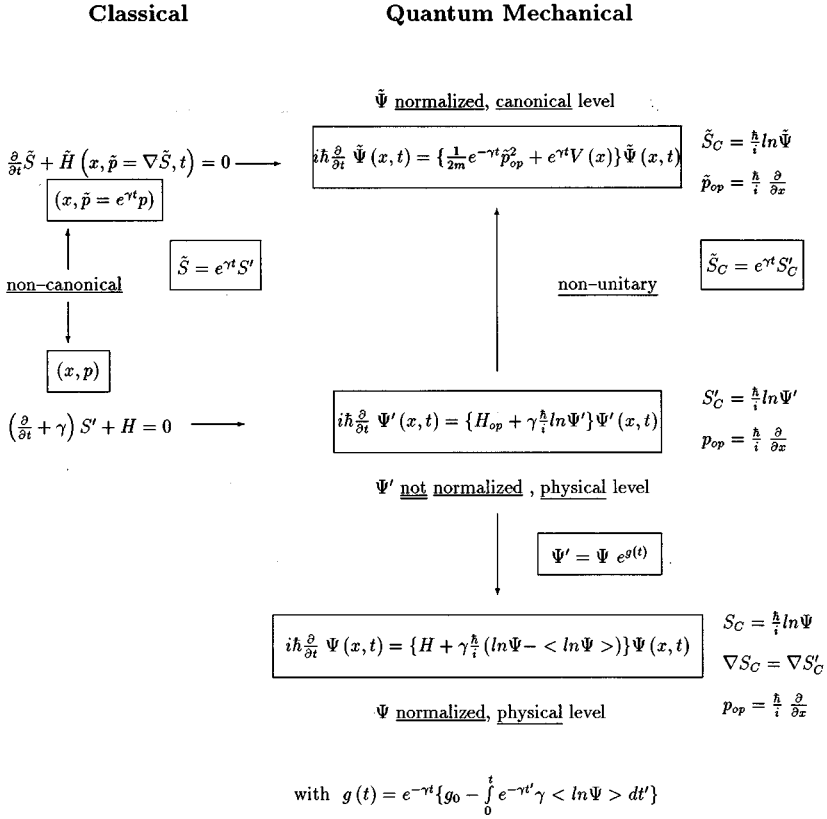

FIG. 1. Connections between the canonical and the physical level in classical as well as in quantum mechanics.

by a linear effective Hamiltonian operator). However, via a noncanonical transformation (connected with a nonunitary transformation of the wave function), a level can be reached where the description within the canonical formalism can be possible.

V. UNCERTAINTY PRODUCTS IN CANONICAL AND PHYSICAL DESCRIPTIONS

As an example it shall be shown that the apparent violation of the uncertainty principle vanishes if operators *and* wave functions are transformed consistently. For this purpose we consider exact analytic Gaussian wave packet (WP) solutions of the damped HO. On the canonical level, i.e., where the CK Hamiltonian is applied, the WP $\tilde{\Psi}_{\mathrm{CK}}(x,t)$ is proportional to

$$\tilde{\Psi}_{\mathrm{CK}}(x,t) \propto \exp\left(i\tilde{y}(x - \langle x\rangle(t))^2 + \frac{i}{\hbar}\langle\tilde{p}\rangle x \right), \tag{21}$$

where $\tilde{y}(t) = \tilde{y}_R(t) + i\tilde{y}_I(t)$ is complex, $\langle x\rangle$ is the mean value of position and $\langle\tilde{p}\rangle = \langle\tilde{p}_C\rangle = \langle\tilde{p}_{\mathrm{op}}\rangle = me^{\gamma t}(d/dt)\langle x\rangle$ is identical to the classical *canonical* momentum $\tilde{p}_{\mathrm{CK}}$ of Caldirola and Kanai [see Eq. (3)]. The physical momentum is therefore given by $\langle p\rangle = m(d/dt)\langle x\rangle$ and, since inserting $\tilde{\Psi}_{\mathrm{CK}}(x,t)$ in Eq. (20) shows that $\langle x\rangle(t)$ fulfills the classical equation of motion (2) including friction, the mean value of the physical momentum, due to dissipation, decreases in the same way as the corresponding classical quantity.

The mean-square deviation of position is inversely proportional to the imaginary part of $\tilde{y}(t)$,

$$(\overline{\Delta x^2})_{\mathrm{CK}} = \langle x^2\rangle_{\mathrm{CK}} - \langle x\rangle_{\mathrm{CK}}^2 = \frac{1}{4\tilde{y}_I}, \tag{22}$$

where the subscript CK indicates that the mean values are calculated using $\tilde{\Psi}_{\mathrm{CK}}$, i.e., $\langle\ \rangle_{\mathrm{CK}} = \int\tilde{\Psi}_{\mathrm{CK}}^*\cdots\tilde{\Psi}_{\mathrm{CK}}dx$. Similarly, the fluctuation of the *canonical* momentum is given by

$$(\overline{\Delta\tilde{p}_{\mathrm{op}}^2})_{\mathrm{CK}} = \langle\tilde{p}_{\mathrm{op}}^2\rangle_{\mathrm{CK}} - \langle\tilde{p}_{\mathrm{op}}\rangle_{\mathrm{CK}}^2 = \hbar^2\left(\frac{\tilde{y}_R^2 + \tilde{y}_I^2}{\tilde{y}_I}\right), \tag{23}$$

hence the uncertainty product

$$U_{\mathrm{CK}} = (\overline{\Delta x^2})_{\mathrm{CK}}(\overline{\Delta\tilde{p}_{\mathrm{op}}^2})_{\mathrm{CK}} = \frac{\hbar^2}{4}\left[1 + \left(\frac{\tilde{y}_R}{\tilde{y}_I}\right)^2\right] \geqslant \frac{\hbar^2}{4} \tag{24}$$

obviously does not violate Heisenberg's principle.

The violation seems to occur if the canonical momentum operator $\tilde{p}_{\mathrm{op}} = (\hbar/i)(\partial/\partial x)$ is replaced by the kinetic momentum, according to $p_{\mathrm{op}} = e^{-\gamma t}\tilde{p}_{\mathrm{op}}$, and the momentum fluctuation is calculated using the *same* WPs $\tilde{\Psi}_{\mathrm{CK}}$ as used for $\tilde{p}_{\mathrm{op}}$, i.e., $(\overline{\Delta p_{\mathrm{op}}^2})_{\mathrm{CK}} = e^{-2\gamma t}(\overline{\Delta\tilde{p}_{\mathrm{op}}^2})_{\mathrm{CK}}$. However, the operator p_{op} is not defined on the space to which $\tilde{\Psi}_{\mathrm{CK}}$ belongs, but on the physical space. Therefore, to be consistent, if the operators are transformed, the WPs have to be transformed accordingly.

The WPs that apply to the physical level are, as shown above, the solutions of the logarithmic NLSE (13). These WPs can be written in a form equivalent to Eq. (21),

$$\Psi_{\mathrm{NL}}(x,t) \propto \exp\left(iy(x - \langle x\rangle(t))^2 + \frac{i}{\hbar}\langle p\rangle x \right), \tag{25}$$

but $\tilde{y}$ has to be replaced by $y(t)=e^{-\gamma t}\tilde{y}(t)$ and $\langle p\rangle = \langle p_c\rangle\langle p_{\rm op}\rangle = m(d/dt)\langle x\rangle$ is identical to the classical *kinetic* momentum p, since $\langle x\rangle$ still fulfills the classical equation of motion (2).

The fluctuations of position and *kinetic* momentum, calculated with $\Psi_{\rm NL}(x,t)$, are now given by

$$(\overline{\Delta x^2})_{\rm NL}=\frac{1}{4y_I}=e^{\gamma t}\frac{1}{4\tilde{y}_I}=e^{\gamma t}(\overline{\Delta x^2})_{\rm CK}, \tag{26}$$

$$(\overline{\Delta p_{\rm op}{}^2})_{\rm NL}=\hbar^2\left(\frac{y_R^2+y_I^2}{y_I}\right)=e^{-\gamma t}\left(\frac{\tilde{y}_R^2+\tilde{y}_I^2}{\tilde{y}_I}\right)$$
$$=e^{-\gamma t}(\overline{\Delta\tilde{p}_{\rm op}^2})_{\rm CK}. \tag{27}$$

So the uncertainty product does not violate Heisenberg's principle

$$U_{\rm NL}=\frac{\hbar^2}{4}\left[1+\left(\frac{y_R}{y_I}\right)^2\right]=U_{\rm CK}\geqslant\frac{\hbar^2}{4}, \tag{28}$$

but is identical to the one on the canonical level, as only the ratio of y_R and y_I occurs.

VI. DISCUSSION AND CONCLUSIONS

It shall be mentioned here briefly (details will be discussed elsewhere) that the discussed transition from a nonlinear theory on conventional spaces over conventional fields to a linear theory on generalized spaces over generalized fields is a specific example for the so-called isolinearization (here because of irreversibility in particular genolinearization), introduced by Santilli in the framework of hadronic mechanics (see, e.g., [28]).

In a recent paper by Gisin and Rigo [29] it is claimed that ''any deterministic nonlinear Schrödinger equation allows one to send signals in a finite time over arbitrarily large distances'' and hence should be considered non-physical. Considering our logarithmic NLSE (13) in this context, there are two kinds of nonlinearity to be distinguished. One is due to the occurrence of the mean value $\langle\,\rangle=\int\Psi^*\cdots\Psi dx$ in the equation, but this is mainly for normalization purposes; the other, a true nonlinearity, is the appearance of the logarithm of the wave function. This term, however, can be linearized as shown above if wave functions and operators are transformed simultaneously in a nonunitary way. This linearizability might also have an effect on the problem of superluminal communication. There might further be close connections with nonlinear gauge theories discussed by Goldin [30] in connection with a family of NLSEs and its physical relevance. These questions are still under investigation and a final answer cannot yet be given.

At least the following can already be stated about the physical relevance of the logarithmic NLSE (13). In [29] Gisin and Rigo discuss what they call an ''interesting'' NLSE [see Eqs. (4)–(7) therein], which can be associated with a master equation of Lindblad form for the time evolution of density operators. However, this interesting NLSE is also deterministic, whereas the master equation is stochastic in the sense that it turns pure states into mixed states. These authors then show that, by adding randomness to the NLSE, the master equation can be recovered and the NLSE becomes ''relevant'' in the sense that the contradiction with relativity no longer occurs. For further details on the quantum state diffusion equation obtained in this way or similar stochastic quantum state evolution equations see [29–35] and references cited therein. Since the logarithmic NLSE (13) can also be written in a form similar to that of the interesting NLSE of Gisin and Rigo [see, e.g., [36], especially for Gaussian WPs in a form similar to their Eq. (7)], analogous addition of noise to our NLSE would avoid the signaling problem and turn it into an equation that is physically relevant according to Gisin and Rigo's definition.

In conclusion, it can be stated that the many-body problem of a system dissipating energy due to contact with an environment can, at least in certain cases, be reduced to a description with the help of an effective Hamiltonian for the system of interest. This Hamiltonian can be explicitly time dependent, like the one of CK, allowing canonical quantization to yield a *linear* effective Hamiltonian operator. In this case, the *canonical variables* are connected to the *physical* ones, classically, via a *noncanonical transformation*. Consequently, the wave functions that belong to the domain of definition of the canonical operators are connected to the wave functions of the physical operators via a *nonunitary transformation*. An equivalent description can be achieved by directly using the conventional operators corresponding to the *physical variables* and the respective wave functions. This advantage has as a consequence the disadvantage of the corresponding Hamiltonian operator being *nonlinear*. In any case, both descriptions do not yield unphysical results if operators *and* wave functions are both properly considered on the same level, either canonical or physical. A more detailed discussion is planned for forthcoming papers.

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