Resummation of anisotropic quartic oscillator: Crossover from anisotropic to isotropic large-order behavior

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We present an approximative calculation of the ground-state energy for the anisotropic oscillator with a potential

$$V(x,y)=\frac{\omega^2}{2}(x^2+y^2)+\frac{g}{4}[x^4+2(1-\delta)x^2y^2+y^4].$$

Using an instanton solution for the isotropic limit $\delta=0$, we obtain the imaginary part of the ground-state energy for small negative g as a series expansion in the anisotropy parameter δ. From this, the large-order behavior of the g expansions accompanying each power of δ are obtained by means of a dispersion relation in g. The g expansions are summed by a Borel transformation, yielding an approximation to the ground-state energy for the region near the isotropic limit. This approximation is found to be excellent in a rather wide region of δ around $\delta=0$. Special attention is devoted to the immediate vicinity of the isotropy point. Using a simple model integral we show that the large-order behavior of a δ-dependent series expansion in g undergoes a crossover from an isotropic to an anisotropic regime as the order k of the expansion coefficients passes the value $k_{\rm cross}\sim 1/|\delta|$. [S1050-2947(97)08801-X]

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I. INTRODUCTION

Phase transitions in anisotropic systems with cubic symmetry have attracted much interest in the literature [1–5]. Especially well studied are corresponding models in quantum mechanics. To gain an analytic insight into the latter, Banks, Bender, and Wu (BBW) [6] investigated a Hamiltonian with a potential

$$V(x,y)=\frac{\omega^2}{2}(x^2+y^2)+\frac{g}{4}[x^4+2(1-\delta)x^2y^2+y^4]. \quad (1)$$

Using multidimensional WKB techniques they derived the large-order behavior of the perturbation series for the ground-state energy

$$E=\sum_k E_k(\delta)g^k \quad (2)$$

as a function of the anisotropy parameter δ.

Janke [7] obtained the same results with more efficiency from a path integral for the imaginary part of the energy E. The imaginary part contains information on the tunneling decay rate of the ground state for $g<0$, and determines directly the large-order behavior of the perturbation coefficients $E_k(\delta)$ via a dispersion relation in the complex coupling constant plane. Both BBW and Janke find a different large-order behavior of the isotropic system $\delta=0$ and the anisotropic system $\delta\neq 0$. They do not discuss, however, the interesting question of how the latter goes over into the former as δ tends to zero.

It is the purpose of this paper to fill this gap. For an optimal understanding of the expected behavior we shall not attack directly the path integral involving the potential (1), but first only the corresponding simple integral. For this integral, a perturbation expansion of the form (2) yields exact δ-dependent perturbation coefficients. The coefficients $E_k(\delta)$ are shown to have a large-order behavior which undergoes a crossover between the earlier derived isotropic and anisotropic behaviors when the order k passes the crossover value $k_{\rm cross}\sim 1/|\delta|$. A small anisotropy $\delta\approx 0$ is hence only reflected in very high orders of perturbation theory.

The expansion terms of a model integral with the potential (1) count the number of Feynman diagrams in a perturbation expansion of quantum mechanical and field-theoretic path integrals. Since this number grows factorially, the bare model integral is sufficient to derive nontrivial information on the large-order behavior of the eventual object of interest, quantum field theory. It turns out that for resumming the g series, asymptotic large-order estimates for the δ-dependent coefficients can be used only in the anisotropic regime $k|\delta|\gg 1$. In the isotropic regime $k|\delta|\ll 1$, on the other hand, it is impossible to truncate the large-order expansion of the perturbation coefficients after a finite number of terms. Thus the neighborhood of the isotropic system $\delta=0$ needs an extra investigation. In the context of quantum field theory, this was recently presented in [8].

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The imaginary parts of physical quantities at small negative g can be calculated with the help of classical solutions called instantons. In systems sufficiently close to the isotropic point it is not necessary to know the exact instanton solutions for all δ. The knowledge of the solution at the symmetry point $\delta=0$ is perfectly sufficient, and the imaginary parts can be expanded around it in powers of δ.

After having understood the model integral, we shall perform the same analysis for an anisotropic quantum mechanical system, which represents a one-dimensional ϕ^4-field theory with cubic anisotropy.

The paper is organized as follows. In Sec. II we review a simple resummation procedure [9] by which the divergent power-series expansion of a function $Z(g)=\Sigma_k Z_k g^k$ is converted into an almost convergent series $\Sigma_p a_p I_p(g)$. Here $I_p(g)$ are certain confluent hypergeometric functions which possess power-series expansions in g with the same leading large-order behavior as the system under study. In Sec. III we shall analyze the above-mentioned crossover in the large-order behavior for the simple model integral. In particular, we shall justify the resummation procedure of Sec. II and the methods in [8] to be perfect tools in approximating the integral for the region near the isotropic limit $\delta\to 0$. In Sec. IV, finally, we present a similar calculation for the ground-state energy of the anharmonic potential with cubic anisotropy.

In addition to this more standard resummation procedure we analyze the model also within the variational perturbation theory developed in [10–13]. Variational perturbation theory yields uniformly and exponentially fast converging expansion for quantum mechanical systems with quartic potentials [14]. The uniform convergence was first proven for the partition function of the anharmonic integral, and later for the quantum mechanical anharmonic oscillator with coupling strength g in several papers [15]. Recently, the proof was sharpened and carried from the partition function to the energies [16].

The input for the quantum mechanical model is provided by the exact Rayleigh-Schrödinger perturbation coefficients of the ground-state energy which we derive from an extension of recursion relations due to Bender and Wu (BW) [17].

II. RESUMMATION

We begin by describing a practical algorithm [9] for a Borel resummation of a divergent perturbation series

$$Z(g)=\sum_k Z_k g^k. \tag{3}$$

The method will be most efficient under the following conditions.

(1) From low-order perturbation theory we know the expansion coefficients Z_k up to a certain finite order N.

(2) From semiclassical methods we are in the possession of the high-order information in the form

$$Z_k \overset{k\to\infty}{\to} \gamma(-1)^k k!\, k^\beta \sigma^k\left(1+\frac{\gamma_1}{k}+\frac{\gamma_2}{k^2}+\cdots\right). \tag{4}$$

(3) By some scaling arguments we are able to assure a power behavior in the strong-coupling limit

$$Z(g) \overset{g\to\infty}{\to} \kappa g^\alpha. \tag{5}$$

The idea of the algorithm is the following: It is possible to construct an infinite, complete set of Borel summable functions $I_p(g)$ which satisfy the high-order and strong-coupling conditions (4) and (5). These functions can be used as a new basis in which to reexpand $Z(g)$:

$$Z(g)=\sum_{p=0}^{\infty} a_p I_p(g). \tag{6}$$

The series (6) should be such that the knowledge of the first $(N+1)$ coefficients in the power-series expansion (3) is sufficient to determine directly the first $(N+1)$ coefficients a_p, yielding an approximation

$$Z(g)\approx Z^{(N)}(g)\equiv\sum_{p=0}^{N} a_p I_p(g). \tag{7}$$

This would then be a new representation of the function $Z(g)$ with the same power series up to g^N, which makes additional use of large-order and strong-coupling informations (4) and (5). In the limit of large N, the series (7) is expected to be applicable for much larger values of g than the original series (3).

The functions $I_p(g)$ being Borel summable have a Borel representation

$$I_p(g)=\int_0^\infty dt e^{-t} t^{b_0} B_p^{b_0}(gt), \tag{8}$$

parametrized by some b_0 and integer p. What are the conditions on $B_p^{b_0}(gt)$, such that $I_p(g)$ satisfies Eqs. (4) and (5) for all p? The answer is most easily found with the help of the hypergeometric functions

$${}_2F_1(a,b;c;-\sigma g t)=\sum_{k=0}^{\infty}\frac{(a)_k (b)_k}{(c)_k}\frac{(-\sigma g t)^k}{k!}, \tag{9}$$

with appropriate parameters a, b, and c. The Pochhammer symbol $(a)_k$ stands for $(a)_k=\Gamma(a+k)/\Gamma(a)$. These functions have the following virtues: First, they are standard special functions of mathematical physics whose properties are well known. Second, they have a cut running from $t=-1/|\sigma g|$ to minus infinity which is necessary to generate the large-order behavior (4). Third, they have enough free parameters to fit all input data. The first property permits an immediate calculation of the Borel integral (8), which is simply a Laplace transformation of $t^{b_0}{}_2F_1(a,b;c;-\sigma g t)$,

$$\int_0^\infty dt e^{-t} t^{b_0}{}_2F_1(a,b;c;-\sigma g t)$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} E(a,b,b_0+1:c:1/\sigma g). \tag{10}$$

The resulting $E(a,b,b_0+1:c:1/\sigma g)$ is MacRobert's E function. Using its asymptotic expansion (see Ref. [18], p. 203) it is easy to verify that our ansatz reproduces the large-order behavior (4). Indeed, for large k the power series

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}E(a,b,b_0+1:c:1/\sigma g)\equiv\sum_{k=0}^{\infty}e_k g^k \qquad (11)$$

has coefficients which grow like

$$e_k \overset{k\to\infty}{\to} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}(-1)^k k!\,k^{a+b-c+b_0-1}\sigma^k. \qquad (12)$$

Moreover, this property is unchanged if the original hypergeometric function is multiplied by a power $(\sigma g t)^p$. A possible set of Borel functions is therefore

$$B_p(gt)=(\sigma g t)^p {}_2F_1(a,b;c;-\sigma g t). \qquad (13)$$

Looking at Eq. (12) we see that the functions (13) are not completely fixed by a given large-order behavior. The parameter β in Eq. (4) merely imposes the following relation upon the parameters a, b, c, and b_0:

$$a+b-c+b_0-1=\beta, \qquad (14)$$

and there are many different ways to satisfy this. A specific choice will be suggested by practical considerations. One is that the I_p's should possess a simple integral representation in order to avoid complicated numerical work. In addition, we would like to work with parameters a, b, and c, for which the hypergeometric function ${}_2F_1$ reduces to simple algebraic functions. This happens only for special sets of the parameters. A simple possibility is, for instance (see Ref. [19], p. 556),

$${}_2F_1\left(a,a+\frac{1}{2};2a+1;-z\right)=4^a(1+\sqrt{1+z})^{-2a}, \qquad (15)$$

which arises by choosing the parameters a, b, and c to be related by

$$a+b-c=-\frac{1}{2};\quad c-2b=0. \qquad (16)$$

With this, the relation (14) can be satisfied for an arbitrary value of the parameter a by choosing

$$b_0=\beta+\frac{3}{2}, \qquad (17)$$

and we are left with only one free parameter. This parameter may be used to accommodate the strong-coupling behavior of $Z(g)$ if it is known. Equation (7) yields the condition $I_p(g)\to\text{const}\times g^\alpha$ on the functions $I_p(g)$. From Eq. (8) we see that such a power behavior emerges if all Borel functions B_p satisfy $B_p(z)\to\text{const}\times z^\alpha$ and thus ${}_2F_1(a,b;c;-z)\to\text{const}\times z^{-p+\alpha}$ [see Eq. (13)]. The explicit representation (15) shows that the parameter a has to be taken as

$$a=p-\alpha. \qquad (18)$$

We thus obtain the approximation $Z^{(N)}=\Sigma_{p=0}^{N}a_p I_p$, with the expansion functions

$$\begin{aligned} I_p(g)&=\int_0^\infty dt\frac{e^{-t}t^{b_0}}{\Gamma(b_0+1)}\frac{(\sigma g t)^p}{4^p}\\ &\quad\times{}_2F_1\left(p-\alpha,p-\alpha+\frac{1}{2};2(p-\alpha)+1;-\sigma g t\right)\\ &=\int_0^\infty dt\frac{e^{-t}t^{b_0}}{\Gamma(b_0+1)}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\sigma g t}\right)^{2\alpha}\\ &\quad\times\frac{(\sigma g t)^p}{(1+\sqrt{1+\sigma g t})^{2p}}, \end{aligned} \qquad (19)$$

where the Borel parameter b_0 is fixed by Eq. (17). The normalization constant $1/4^p\Gamma(b_0+1)$ in front of the expansion functions was introduced for convenience.

Let us now derive equations for the expansion coefficients a_p in terms of the perturbation coefficients Z_k. All one has to do is take the asymptotic expansions

$$I_p(g)=\sum_{k=0}^{\infty}I_k^p g^k, \qquad (20)$$

insert these into Eq. (7), collect terms of equal power g^k, and compare these with the perturbation series (3). This gives the $(N+1)$ algebraic equations

$$Z_k^{(N)}\equiv\sum_{p=0}^{N}I_k^p a_p=Z_k;\quad k=0,1,\ldots,N. \qquad (21)$$

By assumption, the series on the left hand side contains only the coefficients a_p with $p\leqslant N$. Thus the a_p's can be computed, in principle, by inverting the $(N+1)\times(N+1)$ matrix $(I)_{kp}=I_k^p$. Even though this may be done recursively for any given case, it is preferable to find an explicit algebraic solution for a_p in terms of Z_k. This is possible using the following trick. We rewrite the asymptotic expansion of $Z^{(N)}$ in Borel form,

$$\begin{aligned} Z^{(N)}(g)&\equiv\sum_{p=0}^{N}a_p I_p(g)=\sum_{k=0}^{\infty}Z_k^{(N)}g^k\\ &=\int_0^\infty dt e^{-t}t^{b_0}\sum_{k=0}^{\infty}\frac{Z_k^{(N)}(gt)^k}{\Gamma(k+b_0+1)}, \end{aligned} \qquad (22)$$

insert the expression (19) for $I_p(g)$, and compare directly both integrands,

$$\begin{aligned} &\sum_{p=0}^{N}a_p\frac{(\frac{1}{2}+\frac{1}{2}\sqrt{1+\sigma g t})^{2\alpha}}{\Gamma(b_0+1)}\left[\frac{\sigma g t}{(1+\sqrt{1+\sigma g t})^2}\right]^p\\ &\quad=\sum_{k=0}^{\infty}\frac{Z_k^{(N)}(gt)^k}{\Gamma(k+b_0+1)}. \end{aligned} \qquad (23)$$

Introducing the new variable

$$w\equiv\frac{\sigma g t}{(1+\sqrt{1+\sigma g t})^2}=\frac{\sqrt{1+\sigma g t}-1}{\sqrt{1+\sigma g t}+1}, \qquad (24)$$

we obtain from Eq. (23) the relation valid for all α:

$$\sum_{p=0}^{N} a_p w^p = \sum_{k=0}^{\infty} \frac{Z_k^{(N)}}{(b_0+1)_k} \left(\frac{4}{\sigma}\right)^k \frac{w^k}{(1-w)^{2(k-\alpha)}}. \quad (25)$$

In order to compare equal powers in w, we expand on the right hand side

$$(1-w)^{-2(k-\alpha)} = \sum_{l=0}^{\infty} \binom{-2(k-\alpha)}{l} (-w)^l, \quad (26)$$

which gives after a shift of the summation index from l to $p=k+l$ the first $(N+1)$ coefficients a_p in terms of the perturbation coefficients Z_k

$$a_p = \sum_{k=0}^{p} \frac{Z_k}{(b_0+1)_k} \left(\frac{4}{\sigma}\right)^k \binom{-2(k-\alpha)}{p-k} (-1)^{p-k} \quad (27)$$

(recall that $Z_k^{(N)} = Z_k$ for $k=0,1,\ldots,N$). Finally, rewriting the binomial coefficients by means of the identity

$$\binom{x}{p} = (-1)^p \binom{p-x-1}{p}, \quad (28)$$

we obtain the more convenient expression

$$a_p = \sum_{k=0}^{p} \frac{Z_k}{(b_0+1)_k} \left(\frac{4}{\sigma}\right)^k \binom{p+k-1-2\alpha}{p-k}. \quad (29)$$

Thus, we have solved the original matrix inversion problem (21) by translating it to a simple problem in function theory, namely, that of inverting the function $w(\sigma g t)$ in Eq. (24). For the purpose of calculating the integrals $I_p(g)$ numerically, we may use the variable w itself as a variable of integration, and rewrite the integral representation for $I_p(g)$ in the form:

$$I_p(g) = \left(\frac{4}{\sigma g}\right)^{b_0+1} \int_0^1 dw \frac{(1+w)w^{b_0+p}}{\Gamma(b_0+1)(1-w)^{2b_0+2\alpha+3}} \times \exp\left[-\frac{4w}{(1-w)^2 \sigma g}\right]. \quad (30)$$

Together with the explicit formula (29) for the coefficients a_p we thus have solved the resummation problem, and it is now straightforward to calculate the approximation (7).

III. MODEL INTEGRAL

In order to set up an approximation method for an anisotropic model in the neighborhood of the isotropic point $\delta=0$, we study first, as mentioned in the Introduction, a simple toy model whose partition function is defined by a two-dimensional integral:

$$Z \equiv \frac{1}{2\pi} \int \int_{-\infty}^{+\infty} dx dy \exp\left\{-\frac{1}{2}(x^2+y^2) - \frac{g}{4}[x^4 + 2(1-\delta)x^2y^2 + y^4]\right\}. \quad (31)$$

This can be interpreted as a partition function of a ϕ^4 theory in zero space-time dimensions with cubic anisotropy. The integral (31) is well defined for all $g>0$ and $-\infty<\delta<2$. Introducing polar coordinates $x=r\cos\varphi$ and $y=r\sin\varphi$, we obtain the more convenient form of this integral:

$$Z = \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} d\rho d\varphi \exp[-\rho - G(g,\delta,\varphi)\rho^2], \quad (32)$$

with $\rho = r^2/2$ and

$$G(g,\delta,\varphi) = g\left[1 - \frac{\delta}{2}\sin^2(2\varphi)\right]. \quad (33)$$

After an integration over the angle φ, we find the integral

$$Z = \int_0^{\infty} d\rho \exp\left[-\rho - g\left(1-\frac{\delta}{4}\right)\rho^2\right] I_0\left(\frac{\delta}{4} g \rho^2\right), \quad (34)$$

where $I_0(x)$ is a modified Bessel function $I_\nu(x)$ for $\nu=0$. Equation (34) is useful for a numerical calculation of $Z(g,\delta)$. It will serve as a testing ground for our approximations.

Thanks to the special space-time dimensionality of the model, the perturbation expansion of $Z(g,\delta)$ can be obtained explicitly, and we can calculate the large-order behavior without doing a saddle point approximation, which is unavoidable in quantum mechanics and field theory.

At first glance it seems useful to expand $Z(g,\delta)$ in the form

$$Z(g,\delta) = \sum_{k=0}^{\infty} Z_k(\delta) g^k, \quad (35)$$

where the perturbation coefficients depend on the anisotropy parameter δ. The coefficients $Z_k(\delta)$ may be found by expanding the integrand of Eq. (32) in powers of g and performing the integral term by term, yielding

$$Z_k(\delta) = \frac{(-1)^k}{k!} \Gamma(2k+1) \left(1-\frac{\delta}{2}\right)^{k/2} P_k\left(\frac{4-\delta}{2\sqrt{4-2\delta}}\right), \quad (36)$$

where $P_k(x)$ are Legendre polynomials.

In the isotropic limit $\delta=0$, this reduces to

$$Z_k(0) = \frac{(-1)^k}{k!}\Gamma(2k+1) = \frac{(-1)^k}{\sqrt{\pi}} 4^k \Gamma\left(k+\frac{1}{2}\right) = \frac{(-1)^k}{\sqrt{\pi}} 4^k k! k^{-1/2}[1+O(1/k)], \quad (37)$$

showing that in the rotationally symmetric case, the large-order parameter β takes the value $\beta=-1/2$. To obtain (37), use was made of the duplication formula for gamma functions,

$$\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z) \Gamma\left(z+\frac{1}{2}\right), \quad (38)$$

and the expansion

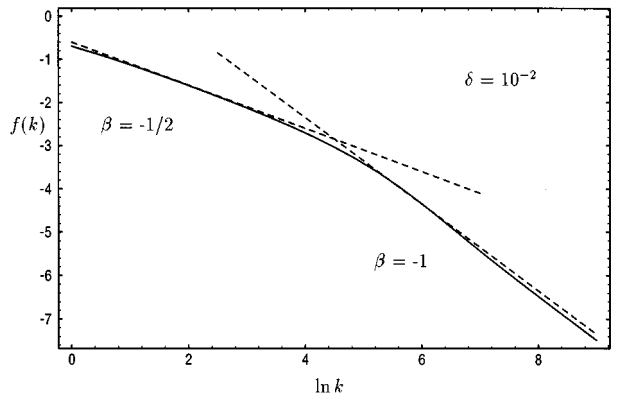

FIG. 1. Crossover of large-order behavior of the expansion coefficients Z_k in Eq. (35) from the isotropic regime ($\beta=-1/2$) to the anisotropic regime ($\beta=-1$). Plotted is the function $f(k)=\ln[Z_k/(-4)^k k!]$ for the anisotropy $\delta=10^{-2}$. In this case the crossover value is given by $k_{\rm cross}\sim 1/|\delta|=10^2$ ($\ln k_{\rm cross}\approx 4.6$).

$$\Gamma(k+\varepsilon+1)=k!k^{\varepsilon}[1+O(1/k)], \tag{39}$$

yielding the large-k behavior of $\Gamma(2k+1)$,

$$\Gamma(2k+1)=\pi^{-1/2}4^k(k!)^2k^{-1/2}[1+O(1/k)]. \tag{40}$$

In Fig. 1 we have plotted the order dependence of the coefficients (36) for the anisotropy parameter $\delta=10^{-2}$. What we can see is a crossover of the large-order behavior from an isotropic to an anisotropic regime in the vicinity of a special crossover value $k_{\rm cross}\sim 1/|\delta|=10^2$. In the anisotropic regime $k|\delta|\gg 1$, the large-order parameter β has the value $\beta=-1$. For $k|\delta|\ll 1$, however, we can read off the large-order behavior of the isotropic case, i.e., $\beta=-1/2$ (see also Fig. 2).

In order to understand the different large-order behavior for $\delta\neq 0$ we note that an approximation of the Legendre polynomials $P_k(x)$ for large k including contributions of the order $O(1/k)$ can be derived from Hobson's formula (see Ref. [20], p. 305):

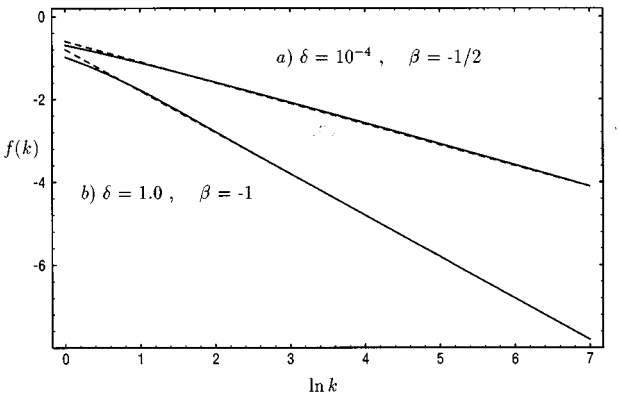

FIG. 2. Example for the two different large-order regimes of Z_k in Eq. (35), where $f(k)$ is the same function as in Fig. 1. (a): Isotropic regime ($\delta = 10^{-4}$, $k_{\rm cross}=10^4$, $\beta=-1/2$). (b): Anisotropic regime ($\delta = 1$, $k_{\rm cross}=1$, $\beta=-1$).

$$P_k(x)=(2\pi k)^{-1/2}(x^2-1)^{-1/4}(x+\sqrt{x^2-1})^{k+1/2}\times\left[1+\frac{1-\sqrt{x^2-1}(x+\sqrt{x^2-1})}{8k\sqrt{x^2-1}(x+\sqrt{x^2-1})}+O(1/k^2)\right]. \tag{41}$$

Substituting

$$x=\frac{4-\delta}{2\sqrt{4-2\delta}}, \tag{42}$$

we obtain for $0<\delta<2$

$$P_k\left(\frac{4-\delta}{2\sqrt{4-2\delta}}\right)=\sqrt{\frac{2}{\pi}}k^{-1/2}\delta^{-1/2}\left(\frac{1}{1-\delta/2}\right)^{k/2}\times\left[1+\frac{1}{k\delta}\left(\frac{4-3\delta}{8}\right)+O\left(\frac{1}{k^2\delta^2}\right)\right]. \tag{43}$$

The combination of Eqs. (40) and (43) yields the large-order behavior of the perturbation coefficients $Z_k(\delta)$ for $0<\delta<2$:

$$Z_k(\delta)=\frac{2^{1/2}}{\pi}(-1)^k4^kk!k^{-1}\delta^{-1/2}\left\{1+\frac{1}{k\delta}\left[\frac{1}{2}+O(\delta)\right]+O\left(\frac{1}{k^2\delta^2}\right)\right\}. \tag{44}$$

A similar calculation can be done for $\delta<0$ with the result:

$$Z_k(\delta)=\frac{(2-\delta)^{1/2}}{\pi}(-1)^k(4-2\delta)^kk!k^{-1}(-\delta)^{-1/2}\times\left\{1-\frac{1}{k\delta}\left[\frac{1}{2}+O(\delta)\right]+O\left(\frac{1}{k^2\delta^2}\right)\right\}, \tag{45}$$

in perfect agreement with the general symmetry relation

$$Z(g,\delta)=Z(\overline{g},\overline{\delta});\quad \overline{g}=g(2-\delta)/2;\quad \overline{\delta}=-\frac{2\delta}{2-\delta}, \tag{46}$$

which maps $\delta<0$ onto $0<\overline{\delta}<2$. This relation follows from a simple change of variables,

$$\overline{x}=(x+y)/\sqrt{2},\quad \overline{y}=(x-y)/\sqrt{2}, \tag{47}$$

in Eq. (31).

It is easy to verify that the large-order expansions (44) and (45) agree with the results derived by means of the steepest descent method by Janke [7]. For resumming the series (35), these large-order expansions can be used only for $k|\delta|\gg 1$. In the regime $k|\delta|\ll 1$, on the other hand, it is impossible to truncate the series in Eqs. (44) and (45) after a finite order of $1/k|\delta|\gg 1$. Thus, the isotropic regime cannot be described by resumming the perturbation series (35) using the asymptotic results (44) and (45). Being interested in the

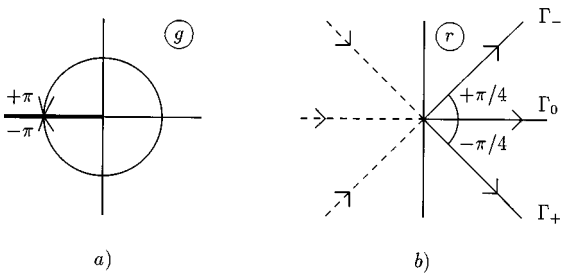

FIG. 3. Analytic continuation $g\to|g|\exp(\pm i\pi)$: (a) Rotation by angles $\pm\pi$ in the cut complex g plane. (b) Two rotated paths of integration in the r plane ($r>0$).

region close to the isotropic limit, we therefore use an expansion different from Eq. (35), and rewrite $Z(g,\delta)$ as

$$Z(g,\delta)=\sum_{k=0}^{\infty}\sum_{n=0}^{k} Z_{kn}g^k\delta^n. \tag{48}$$

Then, for the reason given in [8], reliable results should be obtained by resumming the g series accompanying each power δ^n. The explicit form of the coefficients Z_{kn} is given by

$$Z_{kn}=\pi^{-1/2}(-1)^{k+n}\frac{\Gamma(n+\frac{1}{2})\Gamma(2k+1)}{2^n\Gamma(n+1)^2\Gamma(k-n+1)}, \tag{49}$$

and $Z_{kn}=0$ for $k<n$. For these coefficients, the expansion of the gamma functions yields the behavior for large $k\gg n$:

$$Z_{kn}=(-1)^n\frac{\Gamma(n+\frac{1}{2})}{2^n\Gamma(n+1)^2}(-1)^k\frac{4^k}{\pi}k!k^{n-1/2}[1+O(1/k)]. \tag{50}$$

In the following we shall calculate the coefficients (50) by means of the steepest descent method using the saddle points of the limit $\delta\to0$. This will serve as a preparation for the analogous method in quantum mechanics and quantum field theory.

As a function of the anisotropy parameter $\delta<2$ and the complex coupling g, the integral (31) is only defined in the half-plane $\mathrm{Re}g\geqslant0$. For $\mathrm{Re}g<0$, the integral can be calculated by an analytic continuation from the right into the left half-plane, keeping the integrand in Eq. (31) real. This continuation can be achieved by a joint rotation in the complex g plane and of the integration contour in the $\vec{r}=(x,y)$ plane. The convergence of the integral is maintained by the substitution $g\to g\exp(i\theta)$ and $\vec{r}\to\vec{r}\exp(-i\theta/4)$, where θ is the rotation angle in the complex g plane. Let us assume that the function $Z(g,\delta)$ is analytic in the g plane, with a cut along the negative g axis, and a discontinuity for $g<0$. Then the rotation in the complex g plane by an angle $\theta=\pm\pi$ yields on the upper (lower) lip of the cut:

$$Z_{\pm}(-|g|,\delta)=Z(|g|e^{\pm i\pi},\delta). \tag{51}$$

The corresponding rotated integration contours ($\Gamma_{\mp}$) are drawn in Fig. 3. The discontinuity across the cut is given by

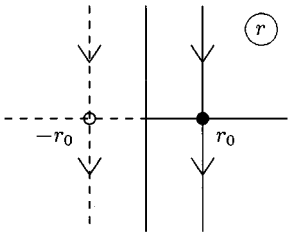

FIG. 4. Deformation of the contours of integration to make them pass through the saddle point r_0.

$$\mathrm{disc}Z=\int_0^{2\pi}\frac{d\varphi}{2\pi}\int_\Gamma r dr\,\exp\left[-\frac{r^2}{2}-G(g,\delta,\varphi)\frac{r^4}{4}\right], \tag{52}$$

where the combined contour $\Gamma=\Gamma_+-\Gamma_-$ runs for $r>0$ entirely through the right half-plane.

In a perturbatively expansion in powers of δ, the discontinuity can be computed from an expansion around the saddle point

$$r_0=\sqrt{\frac{1}{|g|}} \tag{53}$$

of the isotropic case $\delta=0$. Since $r>0$, only the positive square root contributes, the negative one is automatically taken into account by the integration over the angle φ. Now, the contour of integration Γ in the right half-plane can be deformed to run vertically across the saddle point (see Fig. 4), i.e., we can integrate along a straight line:

$$r=\sqrt{\frac{1}{|g|}}-i\xi. \tag{54}$$

The exponent in Eq. (52) plays the role of an action, and the deviations ξ may be considered as radial fluctuations around the extremal solution. The angle φ is analogous to a collective coordinate along the motion of the instanton in the isotropic limit. Expanding the action up to the second order in ξ around the extremum of the isotropic action, we obtain

$$\mathrm{disc}Z=-\frac{i}{2\pi}\left(\frac{1}{|g|}\right)^{1/2}\int_{-\infty}^{+\infty}\int_0^{2\pi}d\xi d\varphi\,\exp\left[-\frac{1}{4|g|}-\frac{\delta}{8|g|}\sin^2(2\varphi)-\xi^2+O\left(\frac{\delta}{\sqrt{|g|}}\right)\right]. \tag{55}$$

Integrating out the radial fluctuations ξ and the azimuthal collective coordinate φ, and using the equation

$$\mathrm{Im}Z=\frac{1}{2i}\mathrm{disc}Z, \tag{56}$$

we obtain the following imaginary part for Z:

$$\mathrm{Im}Z=-\sum_{n=0}^{\infty}(-1)^n\delta^n\frac{\Gamma(n+\frac{1}{2})}{2^n\Gamma(n+1)^2}\left(\frac{1}{4|g|}\right)^{n+1/2}\times\exp\left(-\frac{1}{4|g|}\right)[1+O(g)]. \tag{57}$$

Each power δ^n has its own n-dependent imaginary part. Given such an expansion, the large-order estimates for the coefficients Z_{kn} (with $k \gg n$) follows from a dispersion relation in g [see, for example, Eq. (6) in Ref. [7]],

$$Z_{kn} = \frac{1}{\pi} \int_{-\infty}^{0} dg \frac{\mathrm{Im} Z^{(n)}(g+i0)}{g^{k+1}}, \tag{58}$$

where $Z^{(n)}(g)$ is the coefficient of δ^n. In general, if a real analytic function $F(g)$ has on top of the cut along $g \in (-\infty,0)$ an imaginary part

$$\mathrm{Im} F(g+i0) = -\pi\gamma \left(\frac{1}{\sigma|g|} \right)^{\beta+1} \exp\left(-\frac{1}{\sigma|g|} \right) [1+O(g)], \tag{59}$$

then a dispersion relation of the form (58) leads to the asymptotic behavior

$$F_k = \gamma(-1)^k \sigma^k k^\beta k! [1+O(1/k)]. \tag{60}$$

With $\sigma=4$ and $\beta = n-1/2$, we obtain again the result (50).

Thus, the steepest descent method using the isotropic saddle point is a perfect tool for calculating the large-order behavior of the expansion coefficients Z_{kn} in the expansion (48). An important advantage of this method with respect to the exact calculation (49) is the fact that it can be generalized to quantum mechanics and field theory where exact calculations would be impossible.

Before applying the resummation algorithm of the previous section we have to study the strong-coupling behavior, i.e., the limit of large g. This can simply be done by rescaling the integral (32)

$$Z = \int_0^\infty dy \int_0^{2\pi} \frac{d\varphi}{2\pi} G(g,\delta,\varphi)^{-1/2} \exp\left(-\frac{y}{\sqrt{G(g,\delta,\varphi)}} - y^2 \right) \tag{61}$$

with G from Eq. (33) and $y = \rho\sqrt{G}$. Taking the limit of large g (i.e., large G) and integrating out the angle φ, we find

$$Z(g,\delta)) \overset{g\to\infty}{\to} \kappa(\delta) g^{-1/2}, \tag{62}$$

with

$$\kappa(\delta) = \frac{\pi^{1/2}}{2} \sum_{n=0}^{\infty} \frac{(2n)!^2}{(n!)^4 2^{5n}} \delta^n. \tag{63}$$

Now, a resummation of the g series in Eq. (48) yields a generalization of Eq. (7):

$$Z^{(N)}(g,\delta) \equiv \sum_{n=0}^{N} \left(\sum_{p=n}^{N} a_{pn} I_{pn}(g) \right) \delta^n, \tag{64}$$

with the complete set of Borel summable functions

$$I_{pn}(g) = \left(\frac{4}{\sigma g} \right)^{b_0(n)+1} \int_0^1 dw \times \frac{(1+w) w^{b_0(n)+p}}{\Gamma[b_0(n)+1](1-w)^{2b_0(n)+2\alpha+3}} \times \exp\left[-\frac{4w}{(1-w)^2 \sigma g} \right], \tag{65}$$

and the coefficients

$$a_{pn} = \sum_{k=n}^{p} \frac{Z_{kn}}{(b_0(n)+1)_k} \left(\frac{4}{\sigma} \right)^k \binom{p+k-1-2\alpha}{p-k}, \tag{66}$$

where the perturbation coefficients Z_{kn} are given by Eq. (49). The parameters $b_0(n)$, σ, and α follow from the large-order behavior (50) and the strong-coupling expansion (62), respectively:

$$b_0(n) = n+1,$$

$$\sigma = 4,$$

$$\alpha = -\frac{1}{2}. \tag{67}$$

From Eq. (66) it is possible to derive the following closed formula for the coefficients a_{pn}:

$$a_{pn} = \frac{2^n}{\pi} \frac{\Gamma(n+\frac{1}{2})^2 \Gamma(n+2)}{\Gamma(2n+2)} \times (-1)^p \frac{(-p)_n (p-n)!}{p!n!} \binom{2\alpha+1}{p-n}. \tag{68}$$

Inserting the exact strong-coupling parameter $\alpha = -1/2$, we obtain

$$a_{pn} = \begin{cases} \dfrac{1}{8^n} \dfrac{(n+1)}{(2n+1)} \dfrac{(2n)!}{(n!)^2} & \text{for } p=n, \\ 0 & \text{else.} \end{cases} \tag{69}$$

Thus we have the general result that for the integral model (31) the approximants $\Sigma_{p=n}^{N} a_{pn} I_{pn}(g)$ in Eq. (64) posses no terms with $p>n$, where n is the power of δ. In such a way the n-dependent functions of g associated with each δ^n are recovered exactly.

The approximation $Z^{(N)}(g,\delta)$ may then be compared with the numerically calculated integral (34). In Figs. 5 and 6 we show the result for various N and coupling constants $g/4$.

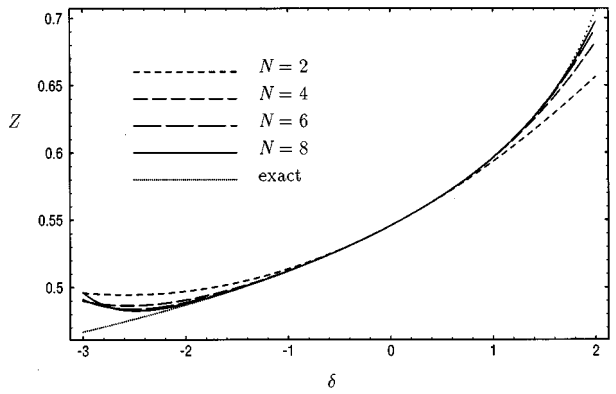

FIG. 5. Partition function Z of the simple integral model as a function of the anisotropy parameter δ with the coupling constant $g/4=0.25$. Comparison is made between the exact result obtained by numerical integration of Eq. (34) and the resummed perturbation series $Z^{(N)}$ [see Eq. (64)] for various orders N.

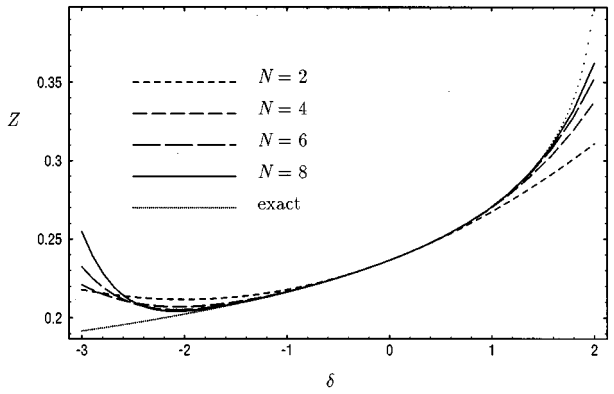

FIG. 6. The same functions as in Fig. 5, but with $g/4=2.5$.

IV. QUANTUM MECHANICS

Having understood the model integral, we now turn to the ϕ^4 theory in one space-time dimension with a cubic anisotropy, which is equivalent to the quantum mechanics of an anisotropic anharmonic oscillator.

A. Recursions relations for the ground-state perturbation coefficients

Consider an anharmonic oscillator with cubic anisotropy and a Hamiltonian

$$H=-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{\omega^2}{2}(x^2+y^2)+\frac{g}{4}[x^4+2(1-\delta)x^2y^2+y^4]. \tag{70}$$

Introducing reduced variables by a rescaling

$$x\to\sqrt{\frac{1}{\omega}}x,\quad y\to\sqrt{\frac{1}{\omega}}y,$$

$$g\to\omega^3 g,\quad E^{(n)}\to\omega E^{(n)}, \tag{71}$$

renders a dimensionless time-independent Schrödinger equation

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{1}{2}r^2+\frac{g}{4}(r^4-2\delta x^2y^2)\right]\Psi_n(x,y)=E^{(n)}\Psi_n(x,y), \tag{72}$$

with the boundary condition

$$|\Psi_n(x,y)|\to 0,\quad \text{for } r=\sqrt{x^2+y^2}\to\infty. \tag{73}$$

The boundary condition selects only the discrete energy eigenvalues $E^{(n)}$. We now consider the ground-state energy $E^{(0)}=E$, whose perturbation expansion has the form

$$E=\sum_{m=0}^{\infty}\sum_{l=m}^{\infty}E_{lm}\left(\frac{g}{4}\right)^l(2\delta)^m, \tag{74}$$

where $E_{00}=1$ is the unperturbed ground-state energy. In the following, we refer to Eq. (74) as the Rayleigh-Schrödinger series, and to E_{lm} as a Rayleigh-Schrödinger coefficients.

In general, the ground-state energy is available from the sum of all connected Feynman diagrams with no externel legs. For an efficient computation of the Rayleigh-Schrödinger coefficients at large orders we shall derive recursions for the E_{lm} extending a method due to Bender and Wu [17]. In this way we obtain a difference equation for the Rayleigh-Schrödinger coefficients.

Separating out the unperturbed ground-state wave function, $\Psi_0(x,y)=\exp[-(x^2+y^2)/2]$, we expand

$$\Psi(x,y)=\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\left(-\frac{g}{4}\right)^k(2\delta)^n\times\exp[-(x^2+y^2)/2]\Phi_{kn}(x,y), \tag{75}$$

where $\Phi_{kn}(x,y)$ are polynomials in x,y with $\Phi_{00}\equiv 1$. Inserting the perturbation expansions (74), (75) into the differential equation (72), and collecting powers of g and δ, we find

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_{kn}+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\Phi_{kn}-r^4\Phi_{k-1,n}+x^2y^2\Phi_{k-1,n-1}=\sum_{l=1}^{k}(-1)^lE_{l0}\Phi_{k-l,n}+\sum_{m=1}^{n}\sum_{l=m}^{k}(-1)^lE_{lm}\Phi_{k-l,n-m}. \tag{76}$$

Finally, the ansatz

$$\Phi_{kn}(x,y)=\sum_{i,j=0}^{2k-n} A_{ij}^{kn} x^{2i} y^{2j} \tag{77}$$

with

$$A_{ij}^{kn}=0 \quad \text{for } i,j>2k-n; \quad i,j,k,n<0; \quad k<n \tag{78}$$

gives the difference equation

$$\begin{aligned} 2(i+j)A_{ij}^{kn} &= (2i+1)(i+1)A_{i+1,j}^{kn}+(2j+1)(j+1)A_{i,j+1}^{kn} \\ &\quad +A_{i-2,j}^{k-1,n}+A_{i,j-2}^{k-1,n}+2A_{i-1,j-1}^{k-1,n}-A_{i-1,j-1}^{k-1,n-1} \\ &\quad -\sum_{l=1}^{k}(A_{10}^{l0}+A_{01}^{l0})A_{ij}^{k-l,n} \\ &\quad -\sum_{m=1}^{n}\sum_{l=m}^{k}(A_{10}^{lm}+A_{01}^{lm})A_{ij}^{k-l,n-m}. \end{aligned} \tag{79}$$

The A_{ij}^{kn} yield the desired Rayleigh-Schrödinger coefficients E_{kn} via the simple formula

$$E_{kn}=-(-1)^k(A_{10}^{kn}+A_{01}^{kn}). \tag{80}$$

These can be determined recursively via Eq. (79). The recursion must be initiated with

$$A_{00}^{kn}=\delta_{k0}\delta_{n0}, \tag{81}$$

and solved for increasing $k=0,1,2,\ldots$; $n=0,1,2,\ldots,k$ and, at each k and n, for decreasing $i=2k-n,\ldots,0$; $j=2k-n,\ldots,0$ (omitting $i=j=0$). The procedure is most easily performed with the help of an algebraic computer program.[1] The list of the first Rayleigh-Schrödinger coefficients up to $k=12$ ($n=0,\ldots,k$) is given in Table I.

B. Large-order coefficients

Working with Langer's formulation [21] (which is related to Lipatov's [22] by a dispersion relation) and making use of known results for the isotropic anharmonic oscillator, we now derive the large-order behavior of perturbation expansion for the ground-state energy.

The method is based on the path-integral representation of the quantum partition function

$$Z=\int \mathcal{D}x\mathcal{D}y\ \exp[-A(x,y)] \overset{\beta\to\infty}{\to} \exp(-\beta E), \tag{82}$$

where

$$A=\int_{-\beta/2}^{+\beta/2} d\tau\left\{\frac{1}{2}(\dot{x}^2+\dot{y}^2)+\frac{1}{2}(x^2+y^2) +\frac{g}{4}[x^4+2(1-\delta)x^2y^2+y^4]\right\} \tag{83}$$

is the Euclidean action corresponding to the Hamiltonian (70). For $g>0$, the system is stable and Z is real. On the other hand, if the coupling constant g is negative the system becomes unstable and Z develops an exponentially small imaginary part related to the decay-rate Γ of the ground-state resonance. The imaginary part of the ground-state energy may be obtained by taking the large β limit in Eq. (82),

$$\mathrm{Im}E=\frac{1}{2}\Gamma=-\frac{1}{\beta}\frac{\mathrm{Im}Z}{\mathrm{Re}Z}, \quad \beta\to\infty. \tag{84}$$

In the above equation the fact was used that $\mathrm{Im}Z \propto \exp(-\beta)\exp[-1/(\sigma|g|)]$ is much smaller than $\mathrm{Re}Z=\exp\{-\beta[1+O(g)]\}$. For small $g<0$, this imaginary part can be computed perturbatively in the anisotropy parameter δ by an expansion around the isotropic instanton solution $r_c(\tau)$:

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}(r_c+\xi)+\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}\eta,$$

$$r_c=\sqrt{\frac{2}{|g|}}\frac{1}{\cosh(\tau-\tau_0)}. \tag{85}$$

For brevity, we shall set $\tau_0=0$ in the sequel. In Eq. (85) we have separated out the rotation angle φ of the isotropic instanton in the (x,y) plane and denoted the radial and azimuthal degrees of freedom by ξ and η. Inserting the expansion (85) into the action (83), we obtain the expression

$$A=\frac{4}{3|g|}+\frac{\delta}{|g|}\frac{2\sin^2(2\varphi)}{3}+\frac{1}{2}\int d\tau\left[\xi\left(-\frac{d^2}{d\tau^2}+1 -\frac{6}{\cosh^2\tau}\right)\xi+\eta\left(-\frac{d^2}{d\tau^2}+1-\frac{2}{\cosh^2\tau}\right)\eta\right]+O\left(\frac{\delta}{\sqrt{|g|}}\right), \tag{86}$$

where we have split the action into terms responsible for the leading contributions in an expansion of the form (57), and a remainder $O(\delta/\sqrt{|g|})$. Then the δ dependence of the azimuthal quadratic fluctuations belongs to the omitted terms. Expanding Eq. (82) in δ and integrating out the quadratic fluctuations we obtain

[1]The program can be obtained from the World Wide Web page at the following address: http://www.physik.fu-berlin.de/~kleinert/kleiner_re245/preprint.html

TABLE I. Coefficients E_{kn} in the perturbation series (74) for the ground-state energy up to $k=12$ ($n=0,\ldots,k$).

k	n	E_{kn}	k	n	E_{kn}
0	0	1	9	1	−4840190554455/512
1	0	2	9	2	785448510795415/124416
1	1	−1/4	9	3	−172109470699495/62208
2	0	−9	9	4	5484677894663731/6635520
2	1	9/4	9	5	−2714832036203789/15925248
2	2	−3/16	9	6	1910496739715441/79626240
3	0	89	9	7	−468820318449871/212336640
3	1	−267/8	9	8	1223678377567247/10192158720
3	2	177/32	9	9	−29878788733243/10192158720
3	3	−11/32	10	0	−34427971992123/128
4	0	−5013/4	10	1	172139859960615/512
4	1	5013/8	10	2	−757445337006448801/2985984
4	2	−9943/64	10	3	95243865818145949/746496
4	3	2465/128	10	4	−2665713995581312 1627/597196800
4	4	−973/1024	10	5	661928984385561 8939/597196800
5	0	88251/4	10	6	−1868810838686785 2767/9555148800
5	1	−441255/32	10	7	28739651906357 9707/1194393600
5	2	874757/192	10	8	−6016110774357344761/305764761600
5	3	−216751/256	10	9	588950135128273907/611529523200
5	4	171049/2048	10	10	−52319976745196951/2446118092800
5	5	−20987/6144	11	0	1196938085820951/128
6	0	−3662169/8	11	1	−13166318944030461/1024
6	1	10986507/32	11	2	4849536413117402 49799/44789760
6	2	−327063703/2304	11	3	−1224987392783925 49037/19906560
6	3	81133049/2304	11	4	2731647374892748 32749/110592000
6	4	−64093757/12288	11	5	−8576021167229490768493/11943936000
6	5	31487347/73728	11	6	4857488765802597 09683/3185049600
6	6	−4401593/294912	11	7	−1109806816323066 8731/471859200
7	0	86716929/8	11	8	7861086018093480 99491/305764761600
7	1	−607018503/64	11	9	−3857115751084320 09551/2038431744000
7	2	32603176343/6912	11	10	763166529190515 0913/90596966400
7	3	−40534191905/27648	11	11	−1259395219006727 1863/73383542784000
7	4	32089547489/110592	12	0	−179761724871375777/512
7	5	−15794879119/442368	12	1	539285174614127331/1024
7	6	4423646695/1769472	12	2	−6584877044071318 31592119/1343692800
7	7	−135064261/1769472	12	3	835370295662057 5386361/268738560
8	0	−18380724429/64	12	4	−1870114571495468628478319/13271040000
8	1	18380724429/64	12	5	420826707520785 0881725247/89579520000
8	2	−6932983533833/41472	12	6	−1673706430871417 3333404777/1433272320000
8	3	216189163547/3456	12	7	383485326168139 53055927/17694720000
8	4	−6865860756773/442368	12	8	−2723466451149412 0875149389/91729428480000
8	5	847050762955/331776	12	9	133955544635750 1564974269/45864714240000
8	6	−951145969207/3538944	12	10	−5312983172484495 1147579/27179089920000
8	7	116411434099/7077888	12	11	702917278261458 74647867/880602513408000
8	8	−151575359341/339738624	12	12	−5292021388168607 6606297/35224100536320000
9	0	537798950495/64			

$$Z=f_\xi f_\eta\int_0^{2\pi}d\varphi\sum_{n=0}^{\infty}\frac{(-\delta/4)^n}{n!}[2\sin^2(2\varphi)]^n\left(\frac{4}{3|g|}\right)^n \times\exp\left(-\frac{4}{3|g|}\right)[1+O(g)]. \tag{87}$$

$$\int_0^{2\pi}d\varphi[2\sin^2(2\varphi)]^n=8^n2\frac{\Gamma^2\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)} =8^n2B\left(n+\frac{1}{2},n+\frac{1}{2}\right), \tag{88}$$

The angle integral can be done with the result

where $B(z,w)$ is the beta function. The contribution f_ξ and f_η from the quadratic radial and azimuthal fluctuations coincide with those appearing in the isotropic oscillator problem, and are therefore known. With the isotropic classical action $A_{0c}=4/(3|g|)$, the well-known results are

$$f_\xi = -\frac{i}{2}\sqrt{\frac{A_{0c}}{2\pi}}\,\beta\left|\frac{\det'(-d^2/d\tau^2+1-6/\cosh^2\tau)}{\det(-d^2/d\tau^2+1)}\right|^{-1/2} Z_{\rm osc}$$
$$= -\frac{i}{2}\sqrt{\frac{A_{0c}}{2\pi}}\beta\sqrt{12}\exp(-\beta/2) \tag{89a}$$

and

$$f_\eta = \sqrt{\frac{3A_{0c}}{2\pi}}\left[\frac{\det'(-d^2/d\tau^2+1-2/\cosh^2\tau)}{\det(-d^2/d\tau^2+1)}\right]^{-1/2} Z_{\rm osc}$$
$$= \sqrt{\frac{3A_{0c}}{2\pi}}\,2\exp(-\beta/2), \tag{89b}$$

where we have used the partition function of the harmonic oscillator,

$$Z_{\rm osc}\equiv\det(-d^2/d\tau^2+1)^{-1/2}=\frac{1}{2\sinh(\beta/2)}\overset{\beta\to\infty}{\to}\exp(-\beta/2), \tag{90}$$

to normalize the determinants. In the upper determinants, the zero eigenvalues are excluded. This fact is recorded by the prime.

The radial fluctuations in the variable ξ contain a negative eigenmode, this being responsible for the factor $-i/2$ and the absolute value sign, and a zero eigenmode associated with the translation invariance which is spontaneously broken by the special choice $\tau_0=0$. The separation of this zero eigenmode in the framework of collective coordinates yields the factor $\beta\sqrt{A_{0c}/(2\pi)}$ (see Chap. 17 in Ref. [12]). Collecting the contributions of the negative and all positive eigenmodes one obtains the remaining factor $\sqrt{12}$ in (89a).

In contrast to the radial case the azimuthal fluctuations η do not contain a negative mode. The azimuthal fluctuation operator has one zero eigenvalue due to the rotational invariance in the limit $\delta\to 0$. The associated eigenmode is extracted from the integration measure via the change of variables (85). The Jacobian of this coordinate transformation can be deduced from the isotropic system. It contributes the factor $\sqrt{3A_{0c}/(2\pi)}$. The remaining factor 2 results from all other modes with positive eigenvalues in (89b).

Collecting all contributions to the imaginary part of the ground-state energy (84), a cancellation of all β-dependent factors leads to

$$\mathrm{Im}E\overset{g\to 0^-}{\to}\frac{2|f_\xi|f_\eta}{\beta\exp(-\beta)}\sum_{n=0}^{\infty}\frac{(-2\delta)^n}{n!}B\left(n+\frac{1}{2},n+\frac{1}{2}\right)\left(\frac{4}{3|g|}\right)^n\exp\left(-\frac{4}{3|g|}\right)=\frac{6}{\pi}\sum_{n=0}^{\infty}\frac{(-2\delta)^n}{n!}B\left(n+\frac{1}{2},n+\frac{1}{2}\right)\left(\frac{4}{3|g|}\right)^{n+1}$$
$$\times\exp\left(-\frac{4}{3|g|}\right). \tag{91}$$

Finally, by means of the dispersion relation (58) we find the corresponding large-order behavior of the coefficients in the expansion (74):

$$E_{kn}\overset{k\to\infty}{\to}-\frac{6}{\pi^2}\frac{(-2)^n}{n!}B\left(n+\frac{1}{2},n+\frac{1}{2}\right)(-1)^k\left(\frac{3}{4}\right)^k k!\,k^n. \tag{92}$$

C. Resummation

After having derived the large-order behavior of E_{kn} and the low-order perturbation coefficients via the Bender and Wu-like recursions (79) and (80), we are in the position to resum the g series accompanying each power δ^n in the expansion (74).

The remaining strong-coupling expansion follows from Symanzik scaling [23]:

$$E(g,\delta)=\sum_{m=0}^{\infty}\kappa_m(\delta)g^{(1-2m)/3}, \tag{93}$$

i.e., the power behavior in the strong-coupling limit is given by

$$E(g,\delta)\overset{g\to\infty}{\to}\kappa_0(\delta)g^{1/3}, \tag{94}$$

where the exponent 1/3 coincides with that appearing in the one-dimensional oscillator problem. Similar to the integral model, the δ dependence enters only in the prefactor κ_0.

Combining Eq. (17) and the formulas (64), (65), and (66), the resummation proceeds with the parameters

$$b_0(n)=n+\frac{3}{2},$$
$$\sigma=\frac{3}{4},$$
$$\alpha=\frac{1}{3}. \tag{95}$$

In Figs. 7 and 8 we have plotted the δ dependence of the resummed ground-state energy E for two different values of

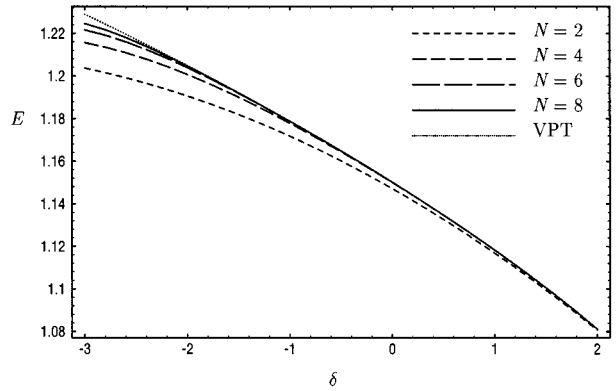

FIG. 7. Ground-state energy E of the anisotropic anharmonic oscillator with $g/4=0.1$ as a function of the anisotropy parameter δ. Shown are the resummed perturbation series for various orders of approximation N and the approximation $W_5(\Omega_5)$ [see the text after Eq. (103)] from the variational perturbation theory (VPT). Differences between $W_5(\Omega_5)$ and the exact ground-state energy are existent only on a finer energy scale.

the coupling constant $g/4$ and various orders N. As N increases, the curves approach the extremely accurate dotted curve obtained numerically from the variational perturbation theory described in the next subsection.

C. Variational perturbation theory

It is useful to compare the above results with those of another recently developed resummation procedure known as variational perturbation theory (for an introduction see Ref. [12], Chap. 5). Consider the Rayleigh-Schrödinger expansion of the ground-state energy:

$$E(g,\delta)=\omega\sum_{l=0}^{\infty}\sum_{m=0}^{l}E_{lm}(2\delta)^m\left(\frac{g/4}{\omega^3}\right)^l, \tag{96}$$

where the Rayleigh-Schrödinger coefficients E_{lm} are obtained from the recursion relation (79) via Eq. (80).

A variational parameter Ω is introduced as follows: First, the potential is separated into an arbitrary harmonic term and a remainder:

$$\frac{\omega^2}{2}(x^2+y^2)=\frac{\Omega^2}{2}(x^2+y^2)+\frac{\omega^2-\Omega^2}{2}(x^2+y^2). \tag{97}$$

In contrast to ordinary perturbation theory, an interacting potential $V_{\rm int}$ is defined by

$$V(x,y)=\frac{\Omega^2}{2}(x^2+y^2)+V_{\rm int}(x,y), \tag{98}$$

such that

$$V_{\rm int}(x,y)=\frac{g}{4}(\rho r^2+r^4-2\delta x^2y^2);\quad \rho=\frac{2}{g}(\omega^2-\Omega^2). \tag{99}$$

Perturbing around the trial oscillator of frequency Ω, an expansion is now found in powers of g at fixed ρ and δ:

$$E_N(g,\delta,\rho)=\Omega\sum_{l=0}^{N}\varepsilon_l(\rho,\delta)\left(\frac{g/4}{\Omega^3}\right)^l. \tag{100}$$

The calculation of the new coefficients $\varepsilon_l(\rho,\delta)$ up to a specific order N does not require much additional work, since they are easily obtained from the ordinary perturbation series (96). We simply replace ω by the identical expression

$$\omega=\sqrt{\Omega^2+\omega^2-\Omega^2}=\sqrt{\Omega^2+g\rho/2}, \tag{101}$$

reexpand $E(g,\delta)$ in powers of g, and truncate the series after an order $l>N$. This yields the reexpansion coefficients

$$\varepsilon_l(\rho,\delta)=\sum_{j=0}^{l}\sum_{n=0}^{j}E_{jn}(2\delta)^n\binom{(1-3j)/2}{l-j}(2\rho\Omega)^{l-j}. \tag{102}$$

The truncated power-series

$$W_N(g,\delta,\Omega):=E_N[g,\delta,2(\omega^2-\Omega^2)/g] \tag{103}$$

is certainly independent of Ω for N going to infinity. However, at any finite order it does depend on Ω. The optimal value of Ω is found by calculating all extrema and the turn-

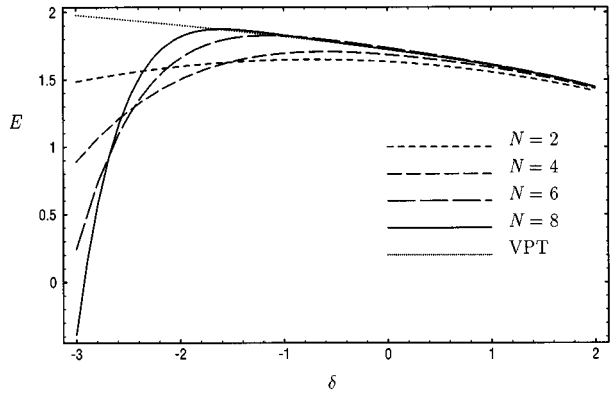

FIG. 8. The same functions as in Fig. 7, but with $g/4=1.0$.

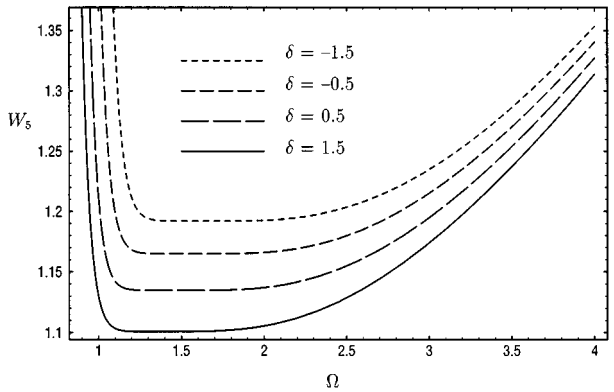

FIG. 9. The function W_5 at constant coupling strength $g/4=0.1$ for various anisotropy parameters δ.

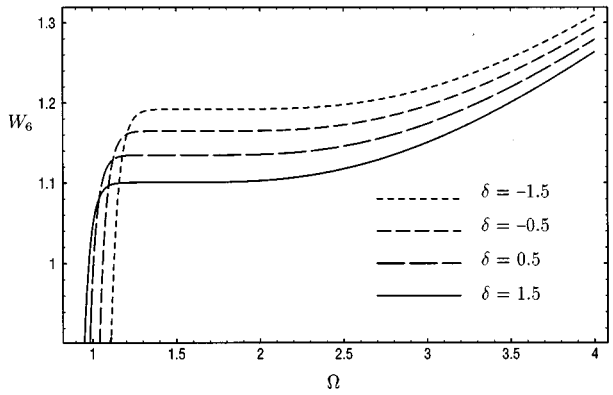

FIG. 10. The function W_6 for $g/4=0.1$ and various δ.

ing points. The smallest among these order-dependent points is used as an optimal trial value and is denoted by $\Omega_N(g,\delta)$. The associated energy $W_N[g,\delta,\Omega_N(g,\delta)]$ constitutes the desired approximation to the ground-state energy. In Figs. 9 and 10 we have plotted the Ω dependence of W_5 and W_6 for various anisotropy parameters δ at the coupling constant $g/4=0.1$. The shape of the curves depends little on δ. Only in the case of odd N does a minimum exist. For even N, there is no extremum and the optimal Ω value lies at a turning point.

For an isotropic gx^4 model, the precision of the variational perturbation method has been illustrated by a comparison with accurate numerical energies [14]. At increasing N the approach of W_N to the exact energy is quite rapid and its mechanism is well understood [16].

In Table II we display the ground-state energies for odd N, which we have obtained for the anisotropic model at various δ and $g/4$. The speed of convergence to fixed energy values is comparable to that for a simple gx^4 interaction. So we may safely assume that these numbers coincide with the exact ground-state energy values at least up to the first four digits.

TABLE II. Convergence of the ground-state energy in the variational perturbation expansion for various anisotropy parameters δ.

			$g/4=0.1$		
$N \setminus \delta$	-2.5	-1.5	-0.5	0.5	1.5
1	1.222923	1.19626	1.167751	1.137	1.103438
3	1.217193	1.192062	1.164807	1.134739	1.100658
5	1.217109	1.192032	1.164801	1.134734	1.100607
7	1.217107	1.192033	1.164803	1.134735	1.100604
9	1.217107	1.192034	1.16481	1.134736	1.100604
11	1.217107	1.192035	1.16481	1.134739	1.100604
			$g/4=1.0$		
$N \setminus \delta$	-2.5	-1.5	-0.5	0.5	1.5
1	1.969986	1.88556	1.791636	1.684863	1.559412
3	1.941934	1.863112	1.773978	1.669261	1.536823
5	1.941196	1.862803	1.773867	1.669156	1.535609
7	1.941172	1.862806	1.773888	1.669172	1.535454
9	1.941172	1.862815	1.773909	1.669188	1.535425
11	1.94118	1.862823	1.773924	1.669199	1.535418

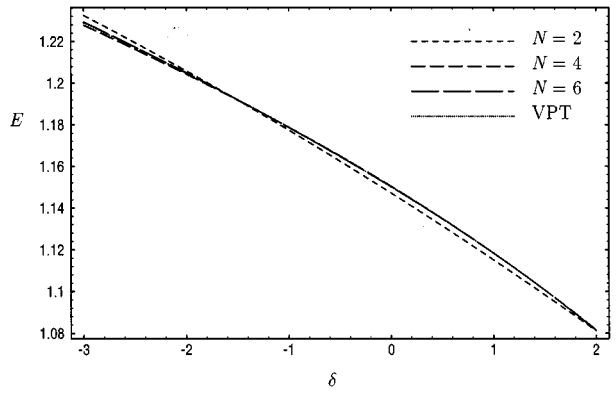

FIG. 11. The same functions as in Fig. 7 ($g/4=0.1$), but with the large-order parameter $\sigma=3$ (explained in the text).

V. SUMMARY

With the help of a simple model integral containing a quadratic and two quartic terms of different symmetry, we have investigated in detail the large-order behavior of the δ-dependent g series in a function $f(g,\delta)=\Sigma_k f_k(\delta)g^k$ for the region near the isotropic limit $\delta\to 0$. We have shown that the large-order behavior of $f_k(\delta)$ undergoes a crossover from an effectively isotropic to the anisotropic regime near the order of perturbation theory $k_{\rm cross}\approx 1/|\delta|$.

In quantum mechanics, the extreme large-order behavior of perturbation theory for the anisotropic regime $k|\delta|\gg 1$ is identical with earlier results of BBW [6] and Janke [7]. In displaying the crossover behavior we have gone beyond these earlier works.

In particular, our resummation algorithm is shown to work very well in the vicinity of $\delta=0$ and for $\delta>0$, the latter being relevant to the question of a stable cubic fixed point in field theory. With increasing coupling constant $g/4$, the error of the result for the ground-state energy becomes larger. However, for $N=6$ (this is the largest available order for the β functions in quantum field theory, see Ref. [5]) and in the wide region $\delta\in(-0.5,2)$ and $g/4\in(0,1)$, the error remains smaller than 0.8%. The increasing error for large negative values of δ can intuitively be understood by comparing the first two terms in the action (86): For

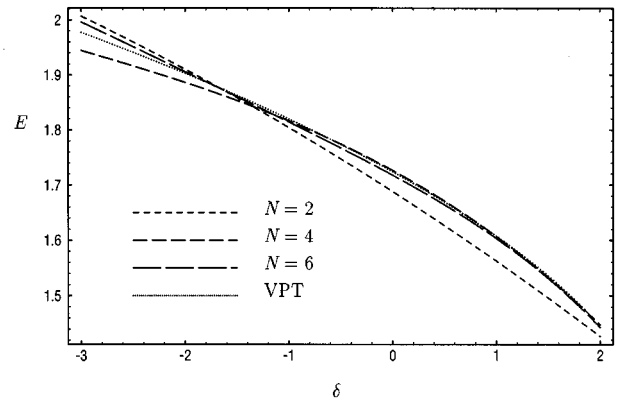

FIG. 12. The same functions as in Fig. 8 ($g/4=1.0$), but with the large-order parameter $\sigma=3$.

$\delta<0$, the ''tunneling paths'' of extremal action are obviously straight lines along the two diagonals in the (x,y) plane $(\varphi=\pi/4)$. Along these diagonals, the basic factor $\exp[-1/(c|g|)]$ related to the decay rate disappears for $\delta\to-2$, and the ensuing expansion of Eq. (82) in powers δ^n becomes meaningless. An improved fit for $\delta<0$ can be obtained by choosing larger values of the large-order parameter σ. In Figs. 11 and 12 we display the result for $\sigma=3$ and $N=6$, where for $g/4=0.1$ the accurate and the resummed curve coincide.

To obtain the correct description of the neighborhood of the isotropic system $\delta=0$, we have used the method developed in the context of an anisotropic quantum field theory in [8]: By replacing the series $\Sigma_k f_k(\delta)g^k$ by $\Sigma_n\Sigma_k f_{kn}g^k\delta^n$ and resumming the g series accompanying each power δ^n, we obtain very good results for the model integral and the ground-state energy of the anisotropic anharmonic oscillator. In this way our results justify the earlier field-theoretic analysis, and should be useful for understanding similar problems in other systems.

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