ARTICLES

Geometric phases for generalized squeezed coherent states

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A simple technique is used to obtain a general formula for the Berry phase (and the corresponding Hannay angle) for an arbitrary Hamiltonian with an equally spaced spectrum and appropriate ladder operators connecting the eigenstates. The formalism is first applied to a general deformation of the oscillator involving both squeezing and displacement. Earlier results are shown to emerge as special cases. The analysis is then extended to multiphoton squeezed coherent states and the corresponding anholonomies deduced.
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I. INTRODUCTION

Generalized coherent states of various kinds have been discussed in recent years in the literature (see, e.g., [1–7]). These states play an important role in multiphoton processes in quantum optics, and also have applications in quantum measurement theory. A unified description of multiphoton coherent states has been given recently [8]. Many of these states can be identified with eigenstates of Hamiltonians that are essentially number operators in appropriate Fock spaces, so that the corresponding levels are equally spaced. While some of these states have classical properties, others like the ''cat states'' [9,10] (eigenstates of a^m, where a is the annihilation operator) are nonclassical. In view of the significance of these states in optics, an investigation of their quantum (and wherever possible, classical) properties is of interest.

An important aspect in this regard is the geometric phase or anholonomy associated with the evolution of these states in certain circumstances. Originally derived by Berry [11] for Hamiltonians with a nondegenerate spectrum under a cyclic, adiabatic variation of parameters, the formalism has been extended to Hamiltonians with a degenerate spectrum [12] as well as nonadiabatic [13] and noncyclic [14] variation of parameters. Even more general settings for the geometric phase have been pointed out, involving a group-theoretic approach [15] and a quantum-kinematic approach [16].

In terms of physical applications, the geometric phase and its generalizations have attracted a lot of interest in a wide variety of fields (e.g., see [17,18]), especially in quantum and coherent optics. Examples include studies on the effect of the geometric phase on the coherent excitation and photoionization of atoms driven by an intense laser field [19], on the photon statistics of the output field in a degenerate parametric amplifier [20], and on coherent pulse propagation [21]. Recently, it has been shown [22] that the geometric phase arising in the propagation of a single-mode electromagnetic field through a nonlinear medium is sensitive to the photon statistics of the initial field. A measurement of the geometric phase would thus be a way to obtain information on the photon distribution of the field. Other practical applications in optics include the construction of achromatic phase shifters [23] using the geometric phase for white light phase-stepping interferometry in surface-profile studies [24,25]. It is therefore clear that coherent optics is eminently suited for a practical realization of the geometric phase in various cases, thus providing a valuable probe to study nonclassical states of radiation [26].

With this motivation, it is therefore of interest to determine the geometric phase for different kinds of coherent states. In particular, there is a wide class of Hamiltonians whose eigenstates are generalized coherent states, and it is for this class that we calculate the Berry phase. We shall be concerned with the Berry phase in its original setting: the cyclic, adiabatic variation of parameters in a Hamiltonian with a discrete, nondegenerate spectrum. Berry's seminal work [11] established a well-known formula for the geometric phase γ_n of the nth level, as the line integral of a certain vector field over a closed contour in parameter space. Earlier works on Berry phases in the context of squeezed coherent states [27] make direct use of this formula. In this paper, we adopt a more general approach. We show that for a Hamiltonian system with equally spaced levels, γ_n is a linear function of n. Hence all the information on the Berry phases of the various eigenstates is contained in the corresponding phases γ_0 and γ_1 of the ground state and the first excited state, respectively. In turn, this implies that in the semiclassical limit, the anholonomy (the Hannay angle) is simply the difference between γ_0 and γ_1. This relationship simplifies enormously the computation of the semiclassical anholonomy, besides clarifying exactly why the latter vanishes in some cases, although the corresponding Berry phase does not.

The plan of the paper is as follows. In Sec. II we derive the linear relationship mentioned above between γ_n, γ_0, and γ_1. In Sec. III we use this to obtain an explicit expression for γ_n for the generalized harmonic oscillator coherent states, and show how earlier results follow as special cases. Finally, in Sec. IV we extend the discussion to sets of multiphoton coherent states built up from the squeezed vacuum ground state. Section V contains some concluding remarks.

II. GENERAL FORMULA FOR γ_n FOR EQUALLY SPACED LEVELS

We begin with the simple observation that the geometric phase is specific to the actual system under consideration, in the following sense: in a given Hamiltonian, a clear identification must first be made of the actual dynamical (or ''fast'') variables **r** versus the adiabatic, externally varied ''slow'' variables **R**. In this sense, the Hamiltonian $H_1=\hbar\omega(a^\dagger a+\frac{1}{2})$ (where **r** comprises a and $a^\dagger$, **R** is represented by the single parameter ω, and $[a,a^\dagger]=1$) is not identical, *a priori*, to the Hamiltonian $H_2=p^2/(2m)+\frac{1}{2}m\omega^2q^2$ (where **r** comprises q and p, **R** stands for m and ω, and $[q,p]=i\hbar$). Of course, H_2 may be *rewritten* in the form $H_2=\hbar\omega[a^\dagger(\mathbf{R})a(\mathbf{R})+\frac{1}{2}]$ by defining the *parameter-dependent* operators $a(\mathbf{R})=(m\omega/2\hbar)^{1/2}q+i(2m\omega\hbar)^{-1/2}p$ and its Hermitian conjugate $a^\dagger(\mathbf{R})$. Their commutator turns out to be $[a(\mathbf{R}),a^\dagger(\mathbf{R})]=1$ for *all* **R**, and it is this invariance of the operator algebra that makes it convenient to *analyze* the Hamiltonian H_2 using its *representation* in terms of $a(\mathbf{R})$ and $a^\dagger(\mathbf{R})$. Our approach is essentially based on this property adapted to more general cases, as we shall see. We mention in passing that the distinction drawn above between different Hamiltonians (exemplified here by H_1 and H_2) is what is essentially responsible for the fact [28] that a canonical transformation can convert (wholly or partly) a geometric phase into a dynamical phase, or vice versa.

Consider a Hamiltonian $H(\mathbf{R})$ with equally spaced, nondegenerate eigenvalues, where **R** denotes the set of ''external'' parameters to be varied adiabatically in some physical range. A form for $H(\mathbf{R})$ that describes all the systems of interest to us is given (up to constants) by the Hermitian operator

$$H(\mathbf{R})=G^\dagger(\mathbf{R})X(\mathbf{R})G(\mathbf{R}), \tag{1}$$

where $X(\mathbf{R})$ is a positive-definite, Hermitian operator, together with the equal-time commutation relation

$$[X(\mathbf{R})G(\mathbf{R}),G^\dagger(\mathbf{R})]=1 \tag{2}$$

on a suitable Hilbert space of states, for every **R**. For the standard oscillator, $G=a$ while X is a multiple of the unit operator. In more general instances, as in the case of Hamiltonians whose eigenstates are certain coherent states [8], X may be a nontrivial function of $a^\dagger a$. Equation (2) leads to $[XG,H]=XG$ and $[G^\dagger,H]=-G^\dagger$ for every **R**. It is then readily deduced that the spectrum of $H(\mathbf{R})$ is the set of nonnegative integers, i.e., there exists normalized eigenstates $|n,\mathbf{R}\rangle$ such that

$$H(\mathbf{R})\,|n,\mathbf{R}\rangle=n\,|n,\mathbf{R}\rangle \quad (n=0,1,2,\ldots). \tag{3}$$

Further, since XG and $G^\dagger$ act as lowering and raising operators, respectively, we have

$$X(\mathbf{R})G(\mathbf{R})\,|0,\mathbf{R}\rangle=0, \tag{4}$$

$$X(\mathbf{R})G(\mathbf{R})\,|n,\mathbf{R}\rangle=c_n\,|n-1,\mathbf{R}\rangle, \tag{5}$$

and

$$G^\dagger(\mathbf{R})\,|n,\mathbf{R}\rangle=d_n\,|n+1,\mathbf{R}\rangle, \tag{6}$$

where the time-independent constants c_n and d_n can be determined if we know also the commutators $[G,G^\dagger]$ and $[G,X]$. We note in passing that Eq. (2) implies that $[G,G^\dagger X]=1$ so that we could also have chosen G as the lowering operator and $G^\dagger X$ as the corresponding raising operator. However, we shall use the choice made earlier, as it is more convenient for the calculations to be presented in Sec. IV.

Let the parameter **R** be varied adiabatically and cyclically with a time period T. Denoting by $|n,\mathbf{R}\rangle$ the nth eigenstate at $t=0$ and by $|n,\mathbf{R}\rangle_T$ the state to which it evolves at time T, we have

$$|n,\mathbf{R}\rangle_T=\exp\left[i\gamma_n-\frac{i}{\hbar}\int_0^T E_n[\mathbf{R}(t)]dt\right]|n,\mathbf{R}\rangle, \tag{7}$$

where $E_n(\mathbf{R})$ is the corresponding eigenvalue of the Hamiltonian $H(\mathbf{R})$. Using Eq. (3), we have in the present instance

$$|n,\mathbf{R}\rangle_T=\exp\left[i\gamma_n-\frac{inT}{\hbar}\right]|n,\mathbf{R}\rangle, \tag{8}$$

keeping in mind that $\mathbf{R}(0)=\mathbf{R}(T)$. The Berry phase γ_n is given by [11]

$$\gamma_n=i\oint\langle n,\mathbf{R}|(\nabla_{\mathbf{R}}\,|n,\mathbf{R}\rangle)\cdot d\mathbf{R}, \tag{9}$$

where the integral runs over a closed contour in parameter space. Analogous to Eq. (8), we have also

$$|n-1,\mathbf{R}\rangle_T=\exp\left[i\gamma_{n-1}-\frac{i(n-1)T}{\hbar}\right]|n-1,\mathbf{R}\rangle. \tag{10}$$

But $X(\mathbf{R})G(\mathbf{R})$ is the lowering operator for *each* value of **R**, so that at time T we must have

$$[X(\mathbf{R})G(\mathbf{R})]_T\,|n,\mathbf{R}\rangle_T=c_n\,|n-1,\mathbf{R}\rangle_T, \tag{11}$$

where we have denoted by $[X(\mathbf{R})G(\mathbf{R})]_T$ the annihilation operator at time T. However, we now recall that c_n does not depend on t, as it is determined by the equal-time commutators $[G,G^\dagger]$ and $[G,X]$. Substituting from Eqs. (8) and (10) for the kets in Eq. (11), we find that $(XG)_T$ must be given by

$$[X(\mathbf{R})G(\mathbf{R})]_T=X(\mathbf{R})G(\mathbf{R})\exp\left[i\left(\gamma_{n-1}-\gamma_n+\frac{T}{\hbar}\right)\right]. \tag{12}$$

As this operator relation holds good for every n, it follows immediately that $(\gamma_n-\gamma_{n-1})$ must be independent of n. In other words,

$$\gamma_n=\gamma_0+n(\gamma_1-\gamma_0), \tag{13}$$

which is also consistent with the requirement that $\{[X(\mathbf{R})G(\mathbf{R})]_T\}^n$ acting on $|n,\mathbf{R}\rangle_T$ yield the state $|0,\mathbf{R}\rangle_T$. We note that the formula obtained for γ_n is only contingent on the existence of (i) an equally spaced spectrum, and (ii) raising and lowering operators connecting the eigenstates. The corresponding classical anholonomy is the Hannay angle (the shift in the angle variable), for which the familiar semiclassical connection gives the formula [29] $\Delta\theta=-\partial\gamma_n/\partial n$. From Eq. (13), we have therefore

$$\Delta\theta=\gamma_0-\gamma_1. \tag{14}$$

III. ANHOLONOMIES FOR SQUEEZED COHERENT STATES

The following results are well known [29,30]: the Berry phase $\gamma_n=0$ for the linear harmonic oscillator with Hamiltonian $p^2/(2m)+\frac{1}{2}m\omega^2q^2$ or $\hbar\omega(a^\dagger a+\frac{1}{2})$, under the variation of the parameters m and ω, but the generalized oscillator with a cross term $(pq+qp)$ may have $\gamma_n\neq 0$. Classically, the quadratic Hamiltonian $Ap^2+2Bpq+Cq^2$ has a nonvanishing Hannay angle if and only if there is also a *rotation* of the axes of the ellipse in the (q,p) plane under the adiabatic, cyclic variation of A,B, and C: mere translation of its center and scaling of its axes lead to $\Delta\theta=0$. Turning to coherent states [27], for the displaced oscillator with Hamiltonian $\hbar\omega[(a^\dagger-\alpha^*)(a-\alpha)+1/2]$, whose ground state is a coherent state, one finds $\gamma_n\neq 0$, but $\Delta\theta=0$, under the adiabatic variation of the complex parameter α. However, if one considers *squeezed* coherent states, $\Delta\theta\neq 0$ under the adiabatic variation of the squeezing parameter β.

We now consider the general deformation of the oscillator Hamiltonian that includes the foregoing as special cases. Let

$$H'=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2=\hbar\omega(a^\dagger a+\tfrac{1}{2}) \tag{15}$$

(where the dependence of a and $a^\dagger$ on the parameters m and ω is implicit). Transforming H' with the squeezing operator [31]

$$S(\beta,\beta^*)=\exp\left(\frac{\beta a^{\dagger 2}-\beta^* a^2}{2}\right)(\beta\in\mathcal{C}) \tag{16}$$

and the displacement operator

$$D(\alpha,\alpha^*)=\exp(\alpha a^\dagger-\alpha^* a),\quad (\alpha\in\mathcal{C}) \tag{17}$$

we have the Hamiltonian

$$H=D(\alpha,\alpha^*)S(\beta,\beta^*)H' S^\dagger(\beta,\beta^*)D^\dagger(\alpha,\alpha^*). \tag{18}$$

Let $\{|n\rangle\}(n=0,1,2,\ldots)$ denote the eigenstates of H', and $\mathcal{H}_0$ the Fock space spanned by these states. We note that both $S(\beta,\beta^*)$ and $D(\alpha,\alpha^*)$ are unitary operators in $\mathcal{H}_0$. Comparing Eq. (18) with Eq. (1) we identify the operators

$$G(\mathbf{R})=DSaS^\dagger D^\dagger,\quad X(\mathbf{R})=1, \tag{19}$$

where

$$\mathbf{R}=\{m,\omega,\alpha=\alpha_1+i\alpha_2,\beta=\beta_1+i\beta_2\}. \tag{20}$$

Moreover

$$[G(\mathbf{R}),G^\dagger(\mathbf{R})]=1 \tag{21}$$

in this case. Also

$$H(\mathbf{R})|n,\mathbf{R}\rangle=\hbar\omega(n+\tfrac{1}{2})|n,\mathbf{R}\rangle, \tag{22}$$

where

$$|n,\mathbf{R}\rangle=D(\alpha,\alpha^*)S(\beta,\beta^*)|n\rangle,\quad n=0,1,2,\ldots. \tag{23}$$

Therefore, under an adiabatic, cyclic variation of the six real parameters comprising $\mathbf{R}$, the Berry phase and Hannay angle are given by Eqs. (13) and (14), respectively. Hence we have merely to compute explicitly the quantities

$$\gamma_0=i\oint\langle 0,\mathbf{R}|\nabla_{\mathbf{R}}|0,\mathbf{R}\rangle\cdot d\mathbf{R} \tag{24}$$

and

$$\gamma_1=i\oint\langle 1,\mathbf{R}|\nabla_{\mathbf{R}}|1,\mathbf{R}\rangle\cdot d\mathbf{R}. \tag{25}$$

(The gradient is understood to act on the ket to its right.) Now $\langle 1,\mathbf{R}|\nabla_{\mathbf{R}}|1,\mathbf{R}\rangle$ can be simplified by noting that $|1,\mathbf{R}\rangle=G^\dagger(\mathbf{R})|0,\mathbf{R}\rangle,\langle 1,\mathbf{R}|=\langle 0,\mathbf{R}|G(\mathbf{R})$. Moreover, using the fact that $[G(\mathbf{R}),G^\dagger(\mathbf{R})]=1$ in this case, and that $G(\mathbf{R})|0,\mathbf{R}\rangle=0$, we find

$$\langle 1,\mathbf{R}|\nabla_{\mathbf{R}}|1,\mathbf{R}\rangle=\langle 0,\mathbf{R}|\nabla_{\mathbf{R}}|0,\mathbf{R}\rangle+\langle 0,\mathbf{R}|[G(\mathbf{R}),(\nabla_{\mathbf{R}}G^\dagger(\mathbf{R}))]|0,\mathbf{R}\rangle. \tag{26}$$

Therefore

$$\gamma_1=\gamma_0+i\oint\langle 0,\mathbf{R}|[G(\mathbf{R}),(\nabla_{\mathbf{R}}G^\dagger(\mathbf{R}))]|0,\mathbf{R}\rangle\cdot d\mathbf{R}. \tag{27}$$

The emergence of the first term (γ_0) on the right-hand side is entirely a consequence of the commutation relation $[G,G^\dagger]=1$. (In Sec. IV, we shall see what happens when $X\neq 1$, $[G,G^\dagger]\neq 1$.) To evaluate the commutator in Eq. (27), it is helpful to use the fact that Eq. (19) can be reduced to the explicit expression [16]

$$G(\mathbf{R})=(a-\alpha)\cosh|\beta|-(a^\dagger-\alpha^*)\frac{\beta}{|\beta|}\sinh|\beta|. \tag{28}$$

Carrying out the calculations involved (the salient features are given in the Appendix), we arrive finally at the following results. It turns out that variations in m and ω are both included in that of the single parameter

$$\lambda=\ln(m\omega). \tag{29}$$

Moreover, there occurs a natural separation of the contributions of the squeezing and displacement parameters to the Berry phase γ_n acquired by $|n,\mathbf{R}\rangle$. We find

$$\gamma_n=\gamma_n^{(D)}+\gamma_n^{(S)}, \tag{30}$$

with

$$\gamma_n^{(D)}=\oint(\alpha_2 d\alpha_1-\alpha_1 d\alpha_2-\alpha_1\alpha_2 d\lambda) \tag{31}$$

and

$$\gamma_n^{(S)} = \left(n+\frac{1}{2}\right) \oint \left(\frac{\sinh|\beta|}{|\beta|}\right)^2 (\beta_2 d\beta_1 - \beta_1 d\beta_2) - \frac{\beta_2}{|\beta|}\sinh|\beta|\cosh|\beta| d\lambda, \tag{32}$$

where $\oint$ stands for the integral over the closed contour traversed in the space of the six parameters **R**. We are now ready to read off a number of special cases.

(i) *α = const, β = const, m and ω varied*: It is evident that $\gamma_n = 0$, and hence $\Delta\theta = 0$, in this case. Varying m and ω does not produce a geometric phase, as the variation appears as a perfect differential, $d(\ln m\omega)$. The original oscillator corresponds to the trivial case $\alpha = 0, \beta = 0$.

(ii) *β = const*: In this case (which includes $\beta = 0$, or no squeezing) we have a nonvanishing Berry phase that is just $\gamma_n = \gamma_n^{(D)}$, but this is n independent, so that the Hannay angle $\Delta\theta = 0$. This remains so, of course, even if m and ω are also kept constant, and (α_1, α_2) alone are varied, as found in Ref. [27]. Writing γ_n as the line integral of a vector potential [11], it is evident that this latter case (λ = const) implies a vector potential **A** with components $(\alpha_2, -\alpha_1, 0)$ along the α_1, α_2, and λ directions. The corresponding ''magnetic field'' $\mathbf{V} = \nabla_{\mathbf{R}} \times \mathbf{A}$ is therefore a *uniform* field along the λ direction; the Berry phase is thus equal, in magnitude, to twice the area enclosed by the loop in the (α_1, α_2) plane. On the other hand, if λ is *also* varied along with α_1 and α_2, the vector potential $\mathbf{A} = (\alpha_2, -\alpha_1, -\alpha_1\alpha_2)$. It is interesting to note how the variation in λ gets coupled to the displacement parameters α_1 and α_2. The field **V** now involves a singular source over and above the earlier uniform field: a line singularity (''antivortex'') along the λ axis, with winding number equal to -1.

(iii) *α = const*: In this case (which includes $\alpha = 0$, or no displacement) we have an n-dependent Berry phase, and therefore a nonzero $\Delta\theta$. This remains true if λ is also kept constant and only β_1, β_2 are varied [27]. Then

$$\gamma_n = -\left(n+\frac{1}{2}\right) \oint \sinh^2|\beta| d(\arg\beta) \tag{33}$$

corresponding to a magnetic field normal to the β plane of magnitude $(n+\frac{1}{2})\sinh(2|\beta|)/|\beta|$. We note also that a nonvanishing γ_n occurs if β_1 and λ alone are varied, provided the *imaginary* part β_2 of the squeezing parameter β is nonzero. In this connection, it is useful to note that the Hamiltonian $\hbar\omega S(\beta,\beta^*)(a^\dagger a + 1/2) S^\dagger(\beta,\beta^*)$ corresponding to pure squeezing can be written, in terms of the original oscillator operators q and p, as $Ap^2 + B(pq+qp) + Cq^2$, with

$$A = \frac{1}{2m}\left[\cosh^2|\beta| + \sinh^2|\beta| + 2\frac{\beta_1}{|\beta|}\cosh|\beta|\sinh|\beta|\right], \tag{34}$$

$$B = -\frac{\beta_2}{|\beta|}\cosh|\beta|\sinh|\beta|, \tag{35}$$

$$C = \frac{m\omega^2}{2}\left[\cosh^2|\beta| + \sinh^2|\beta| - 2\frac{\beta_1}{|\beta|}\cosh|\beta|\sinh|\beta|\right]. \tag{36}$$

IV. ANHOLONOMIES FOR MULTIPHOTON SQUEEZED COHERENT STATES

We turn now to the application of our formalism to Hamiltonians based on multiphoton coherent states. To be specific, we consider the eigenstates of the square of the annihilation operator. We begin with the observation [8] that the commutation relation

$$\left[\frac{1}{2}(1+a^\dagger a)^{-1}a^2, a^{\dagger 2}\right] = 1 \tag{37}$$

is valid on the *even* subspace $\mathcal{H}_1 = \{\text{span}\,|2n\rangle; n = 0,1,\ldots\}$ of $\mathcal{H}_0$. (It is in fact valid on $\mathcal{H}_0 - \text{span}\,|1\rangle$, but for our present purposes we restrict our attention to $\mathcal{H}_1$). Comparing Eq. (37) with Eq. (2), we identify the raising and lowering operators $G^\dagger$ and XG according to

$$G^\dagger = a^{\dagger 2}, \quad X = \tfrac{1}{2}(1+a^\dagger a)^{-1}. \tag{38}$$

The ''Hamiltonian'' $G^\dagger XG$ itself is easily verified to have matrix elements identical to those of $a^\dagger a/2$, as one might have anticipated. However, it is not this Hamiltonian in which we are interested, but rather in the anholonomies associated with its deformations that have generalized coherent and/or squeezed states (eigenstates of XG) as their ground states.

We therefore define the corresponding displacement operator

$$D(\alpha,\alpha^*) = \exp(\alpha G^\dagger - \alpha^* XG). \tag{39}$$

The state $D|0\rangle$ is then an eigenstate of XG with eigenvalue α. The next step is to attempt to construct a Hamiltonian $D(G^\dagger XG)D^{-1}$ whose ground state would be the coherent state $D|0\rangle$ (rather than the vacuum $|0\rangle$), so that we may proceed as in Sec. III to investigate the associated anholonomies. Unfortunately, the displacement operator in Eq. (39) is no longer unitary, so that $D G^\dagger XGD^{-1}$ is not Hermitian. It is evident that the problem arises because the raising operator $G^\dagger = a^{\dagger 2}$ and its conjugate a^2 do not satisfy the commutation relation $[G, G^\dagger] = 1$; rather, it is the commutator $[XG, G^\dagger]$ that is equal to unity. One way out is to make a different identification of G and X than that made in Eq. (38), and we shall consider this possibility subsequently. For the present, we note that there is another approach, based on squeezing rather than displacement:

The squeezing operator $S(\beta,\beta^*)$ defined in Eq. (16) can be expanded [31] in the normal-ordered form

$$S(\beta,\beta^*) = (\cosh|\beta|)^{-1/2}\exp\left(\frac{a^{\dagger 2}\beta}{2|\beta|}\tanh|\beta|\right) \times \left(\sum_{r=0}^{\infty}\frac{(\operatorname{sech}|\beta|-1)^r}{r!}a^{\dagger r}a^r\right) \times \exp\left(-\frac{a^2\beta^*}{2|\beta|}\tanh|\beta|\right). \tag{40}$$

With the help of this expansion, we may establish that

$$\frac{1}{2}(1+a^{\dagger}a)^{-1}a^{2}S(\beta,\beta^{*})\,|0\rangle=\left[\frac{\beta}{2|\beta|}\tanh|\beta|\right]S(\beta,\beta^{*})\,|0\rangle. \tag{41}$$

In other words, the squeezed vacuum

$$|0,\beta\rangle\equiv S(\beta,\beta^{*})\,|0\rangle \tag{42}$$

is also a generalized coherent state (an eigenstate of the lowering operator XG). Moreover, S is unitary. We may therefore construct the deformed, Hermitian Hamiltonian (restoring the appropriate constants)

$$H_{S}=\frac{1}{2}\hbar\omega S(\beta,\beta^{*})[\,a^{\dagger 2}(1+a^{\dagger}a)^{-1}a^{2}+1]\,S^{\dagger}(\beta,\beta^{*}). \tag{43}$$

The ground state of this Hamiltonian is the state $|0,\beta\rangle$ defined in Eq. (42):

$$H_{S}\,|0,\beta\rangle=\tfrac{1}{2}\hbar\omega\,|0,\beta\rangle. \tag{44}$$

The raising and lowering operators for this system are

$$G^{\dagger}(\mathbf{R})=S(\beta,\beta^{*})\,a^{\dagger 2}\,S^{\dagger}(\beta,\beta^{*}) \tag{45}$$

and

$$X(\mathbf{R})G(\mathbf{R})=\tfrac{1}{2}S(\beta,\beta^{*})(1+a^{\dagger}a)^{-1}a^{2}\,S^{\dagger}(\beta,\beta^{*}), \tag{46}$$

respectively. The excited state $|2n,\beta\rangle$ $(n=1,2,\ldots)$ of H_S is obtained by applying $[\,G^{\dagger}(\mathbf{R})]^{n}$ to $|0,\beta\rangle$, and the corresponding eigenvalue is $\hbar\omega(n+1/2)$.

We may now consider the Berry phase acquired by the state $|2n,\beta\rangle$under the adiabatic, cyclic variation of the four parameters m,ω,β_1, and β_2 in H_S (the first two being implicitly contained in a and $a^{\dagger}$ as before). The answer, in fact, may be written down directly from our earlier results once we recognize that $|2n,\beta\rangle$ is *also* given by

$$|2n,\beta\rangle=S(\beta,\beta^{*})\,|2n\rangle, \tag{47}$$

i.e., raising with $(\,G^{\dagger})^{n}$ and squeezing with S can be performed in either order. The Berry phase is therefore precisely $\gamma_{2n}^{(S)}$ where $\gamma_{n}^{(S)}$ is given by Eq. (32), and the rest of the discussion proceeds as before.

Finally, let us return to the Hamiltonian (or number operator)

$$N'=\tfrac{1}{2}\,a^{\dagger}a=\tfrac{1}{2}\,a^{\dagger 2}(1+a^{\dagger}a)^{-1}a^{2}, \tag{48}$$

which, in the space $\mathcal{H}_1$, has eigenvalues $0,1,2,\ldots$. The question is whether we can write N' in the form $a_1{}^{\dagger}a_1$ where $a_1{}^{\dagger}$ and a_1 are the corresponding raising and lowering operators and, *moreover*, $[a_1,a_1{}^{\dagger}]=1$ in $\mathcal{H}_1$. This would avoid the problem encountered earlier, which arose because $[G,G^{\dagger}]$ was not equal to the unit operator. Now, since $(1+a^{\dagger}a)^{-1}$ is a bounded positive operator, there exists (according to the square-root lemma [32]) a unique positive bounded operator $(1+a^{\dagger}a)^{-1/2}$ whose square is $(1+a^{\dagger}a)^{-1}$. We may therefore write

$$N'=a_1{}^{\dagger}a_1 \tag{49}$$

with

$$a_1=2^{-1/2}(1+a^{\dagger}a)^{-1/2}a^{2},\ a_1{}^{\dagger}=2^{-1/2}\,a^{\dagger 2}(1+a^{\dagger}a)^{-1/2}, \tag{50}$$

and $[a_1,a_1{}^{\dagger}]=1$ in $\mathcal{H}_1$. (It is clear [33] that $(a_1,\mathcal{H}_1)$ is isomorphic to $(a,\mathcal{H}_0)$, each of these constituting an irreducible representation of the basic commutation relation $[F,F^{\dagger}]=1)$. The procedure followed in Sec. III for the original oscillator Hamiltonian may now be repeated, unaltered: *unitary* displacement and squeezing operators

$$D_1(\alpha,\alpha^{*})=\exp(\alpha\,a_1{}^{\dagger}-\alpha^{*}a_1) \tag{51}$$

and

$$S_1(\beta,\beta^{*})=\exp\left(\frac{\beta\,a_1{}^{\dagger 2}-\beta^{*}a_1^{2}}{2}\right) \tag{52}$$

may be used to deform N', corresponding squeezed coherent states constructed, and their anholonomies derived, exactly as in Sec. III. It is also evident that the same process can be repeated in the (isomorphic) subspaces $\mathcal{H}_2\supset\mathcal{H}_3\supset\cdots$, where $\mathcal{H}_k=\{\text{span}\,|2^{k}n\rangle\}$, by defining the operators a_k, $a_k{}^{\dagger}$ in $\mathcal{H}_k$ recursively, according to

$$a_k=2^{-1/2}(1+a^{\dagger}{}_{k-1}a_{k-1})^{-1/2}a_{k-1}^{2}, \tag{53}$$

so that $[a_k,a_k{}^{\dagger}]=1$ in $\mathcal{H}_k$.

V. CONCLUSIONS

In this paper we have shown that, for an arbitrary Hamiltonian with equally spaced, nondegenerate eigenvalues, the geometric phase γ_n of the nth eigenstate is a linear function of n. Crucial to the derivation of this result is the existence of raising and lowering operators (connecting the different states) that satisfy a definite algebra. Using the above formalism, the geometric phase was calculated both for generalized squeezed coherent states and for a class of multiphoton coherent states.

A natural question that arises is whether our approach can be extended to the case of Hamiltonians with unequally spaced levels. Although the existence (and construction) of appropriate raising and lowering operators is not immediately obvious in the general case, one possible avenue of approach is the factorization method [34] and its recent extensions, particularly in the context of supersymmetric quantum mechanics [35]. This question is presently under investigation.

Finally, it turns out to be possible to construct coherent states (and generalized coherent states) for a class of Hamiltonians that are strictly isospectral to the harmonic oscillator [36]. While certain classes of these states are essentially unitarily equivalent to those obtained from the original oscillator Hamiltonian, other classes of coherent states can be constructed, via supersymmetry transformations, that are not unitarily equivalent to the original ones. The geometric phases associated with such states are also under investigation, and the results will be reported elsewhere.

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APPENDIX

We outline the steps leading to the explicit formulas given in Eqs. (30)–(32) for the Berry phase γ_n corresponding to the squeezed, displaced oscillator Hamiltonian H of Eq. (18).

To evaluate γ_0, given by Eq. (24), we work in the position representation, in which

$$\gamma_0 = i \oint d\mathbf{R}\cdot\left[\int dx\,\psi_0{}^*(x;\mathbf{R})\nabla_{\mathbf{R}}\psi_0(x;\mathbf{R})\right], \tag{A1}$$

where $\mathbf{R}$ stands for the set of six parameters $\{m,\omega,\alpha=\alpha_1+i\alpha_2,\beta=\beta_1+i\beta_2\}$. The ground state wave function ψ_0 is found in a straightforward manner, and is given by

$$\psi_0(x;\mathbf{R}) = \left(\frac{m\omega}{2\pi\hbar}\right)^{1/4}\exp\left(-\frac{v_1^2}{4u_1}\right)\exp\left(-\frac{1}{2}ux^2 - vx\right), \tag{A2}$$

with

$$u = u_1 + iu_2 = \left(\frac{m\omega}{2\hbar}\right)\left(\frac{|\beta| - i\beta_2 \sinh 2|\beta|}{|\beta|\cosh 2|\beta| + \beta_1 \sinh 2|\beta|}\right), \tag{A3}$$

$$v = v_1 + iv_2 = \left(\frac{2m\omega}{\hbar}\right)^{1/2}\left(\frac{-\alpha_1|\beta| + i(\alpha_1\beta_2 - \alpha_2\beta_1)\sinh 2|\beta| - i\alpha_2|\beta|\cosh 2|\beta|}{|\beta|\cosh 2|\beta| + \beta_1\sinh 2|\beta|}\right). \tag{A4}$$

Next, we calculate the partial derivatives of ψ_0 with respect to the six parameters (it is convenient to consider the logarithmic derivative of ψ_0), substitute these in Eq. (A1) and carry out the (Gaussian) integrals over x, to arrive at the result

$$\gamma_0 = \oint\left[(\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2 - \alpha_1\alpha_2 d\lambda) + \frac{1}{2}\left(\frac{\sinh|\beta|}{|\beta|}\right)^2(\beta_2 d\beta_1 - \beta_1 d\beta_2) - \frac{1}{2}\frac{\beta_2}{|\beta|}\sinh|\beta|\cosh|\beta|\,d\lambda\right], \tag{A5}$$

where $\lambda = \ln(m\omega)$ as defined in Eq. (29).

We must now compute γ_1 from Eq. (27). Using the representation given in Eq. (28) for the operator $G(\mathbf{R})$ (and remembering that m and ω occur in the expressions for a and $a^\dagger$), we find

$$[G(\mathbf{R}),(\nabla_{\mathbf{R}} G^\dagger(\mathbf{R}))]\cdot d\mathbf{R} = i\frac{\beta_2}{|\beta|}\sinh|\beta|\cosh|\beta|\,d\lambda - i\left(\frac{\sinh|\beta|}{|\beta|}\right)^2 \times(\beta_2 d\beta_1 - \beta_1 d\beta_2)]. \tag{A6}$$

There is no operator dependence left in this expression because $[a,\,a^\dagger]=1$. Moreover, since $\psi_0(x;\mathbf{R})$ is normalized to unity, Eq. (27) becomes

$$\gamma_1 = \gamma_0 + \oint\left[\left(\frac{\sinh|\beta|}{|\beta|}\right)^2(\beta_2 d\beta_1 - \beta_1 d\beta_2) - \frac{\beta_2}{|\beta|}\sinh|\beta|\cosh|\beta|\,d\lambda\right]. \tag{A7}$$

Substitution of Eqs. (A5) and (A7) in the general formula for γ_n [Eq. (13)] yields the results quoted in Eqs. (30)–(32).

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