Einstein-Podolsky-Rosen-Bohm experiment with relativistic massive particles

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(Received 12 March 1996)

Two aspects of the relativistic version of the Einstein-Podolsky-Rosen-Bohm (EPRB) experiment with massive particles are discussed: (a) a possibility of using the experiment as an implicit test of a relativistic center-of-mass concept, and (b) influence of the relativistic effects on degree of violation of the Bell inequality. The nonrelativistic singlet state average $\langle\psi|\mathbf{a}\cdot\boldsymbol{\sigma}\otimes\mathbf{b}\cdot\boldsymbol{\sigma}|\psi\rangle=-\mathbf{a}\cdot\mathbf{b}$ is relativistically generalized by defining spin via the relativistic center-of-mass operator. The corresponding EPRB average contains relativistic corrections which are stronger in magnitude than standard relativistic phenomena such as the time delay, and can be measured in Einstein-Podolsky-Rosen-Bohm-type experiments with relativistic massive spin-$\frac{1}{2}$ particles. The degree of violation of the Bell inequality is shown to depend on the velocity of the pair of spin-$\frac{1}{2}$ particles with respect to the laboratory. Experimental confirmation of the relativistic formula would indicate that for relativistic nonzero-spin particles centers of mass and charge do not coincide. The result may have implications for quantum cryptography based on massive particles. [S1050-2947(97)00201-1]

PACS number(s): 03.65.Bz, 03.30.+p

I. INTRODUCTION

Contemporary applications of the Einstein, Podolsky, and Rosen (EPR) correlations [1,2] and the Bell inequality [3,4] range from purely philosophical problems to quantum cryptography, computation and teleportation. In the cryptographic scheme proposed by Ekert [5] Alice and Bob test for eavesdropping by measuring the average of the ''Bell observable''

$$c(\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}')=\langle\psi|\hat{\mathbf{a}}\otimes\hat{\mathbf{b}}|\psi\rangle+\langle\psi|\hat{\mathbf{a}}\otimes\hat{\mathbf{b}}'|\psi\rangle+\langle\psi|\hat{\mathbf{a}}'\otimes\hat{\mathbf{b}}|\psi\rangle-\langle\psi|\hat{\mathbf{a}}'\otimes\hat{\mathbf{b}}'|\psi\rangle, \qquad (1)$$

where $\hat{\mathbf{a}}$, etc., are ''yes–no'' observables (say, signs of spin for electrons, or planes of polarization for photons). Quantum mechanics predicts that for some choices of such observables one can obtain $|c(\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}')|=2\sqrt{2}$. In an ideal situation a result of the form $|c(\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}')|<2\sqrt{2}$ indicates that at least some pairs of particles were not prepared in the singlet state and this indicates an eavesdropper.

Practical applicability of quantum cryptographic protocols crucially depends on detector efficiencies. In typical Bell-type photon pair experiments the efficiencies were smaller than 20%. The advent of solid-state photodiodes provides efficiencies of detection which are much higher [6] but still far from ideal.

An almost ideal experimental scheme has been recently discussed by Fry, Walther, and Li [7] who propose to replace photons with massive particles (pairs of ^{199}Hg atoms). Detection efficiency is then at least 95% and can be pushed to more than 99%. An obvious drawback of such a communication channel is that it is slow. To make it faster one might be tempted to use relativistic velocities.

It will be shown below that for high velocities one may expect a surprising effect: The amount of violation of the Bell inequality may decrease with growing velocity of the spin-$\frac{1}{2}$ particles. Alice and Bob must therefore additionally know the velocity distribution of the particle beam. Otherwise they may be confused and ''detect an eavesdropper'' even though the particles remain in a pure zero-helicity singlet state. The effect is related to the old problem described already in 1930 by Schrödinger [9,10]. As is widely known Schrödinger examined the behavior of the coordinate operator $\mathbf{x}$ associated with Dirac's equation and discovered the oscillatory motion he called the *Zitterbewegung*. The *Zitterbewegung* takes place with respect to the *center-of-mass* position operator $\mathbf{x}_A$ and this is the operator which should be used to define a physically meaningful spin operator. The situation is not typical only of the Dirac equation and is not associated with the presence of negative-energy solutions as one is sometimes led to believe. The so-called new Dirac equation generalized by Mukunda, van Dam, and Biedenharn [11] admits only positive-energy solutions, but the *Zitterbewegung* is present and the associated center-of-mass operator is algebraically identical to this implied by Schrödinger's analysis of the Dirac equation [12]. The analysis presented in [11] shows clearly that in order to obtain a physically consistent model of an extended hadron one has to proceed in the way identical to the one chosen in this paper: First define the center-of-mass operator $\mathbf{Q}$, then introduce the angular momentum $\mathbf{L}=\mathbf{Q}\times\mathbf{P}$, and finally define spin by $\mathbf{S}=\mathbf{J}-\mathbf{L}$.

In what follows I use a group representation formulation, elements of which can be found in the 1965 papers by Fleming [13]. The group theoretic approach has the advantage of being applicable to any physical system whose symmetry group is the Poincaré group, or whose symmetry group contains the Poincaré group as a subgroup.

II. RELATIVISTIC SPIN OPERATORS

Let us begin with generators of the unitary, infinite-dimensional-irreducible representation of the Poincaré group

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corresponding to a nonzero mass m and spin j. Their standard form is [14]

$$\mathbf{J}=\frac{\hbar}{i}\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}+\mathbf{s}, \tag{2}$$

$$\mathbf{K}=\pm\left(|p_0|\frac{\hbar}{i}\frac{\partial}{\partial\mathbf{p}}-\frac{\mathbf{p}\times\mathbf{s}}{mc+|p_0|}\right), \tag{3}$$

$$\mathbf{P}=\mathbf{p}, \tag{4}$$

$$P_0=p_0=\pm\sqrt{\mathbf{p}^2+m^2c^2}. \tag{5}$$

Here $\mathbf{s}$ denotes finite-dimensional angular momentum matrices corresponding to the $(2j+1)$-dimensional representation D^j of the rotation group. Similar forms are obtained if one uses the hadronic representation introduced in [11].

The Poincaré group has two Casimir operators: The squared mass and the square of the Pauli-Lubanski vector W^μ. The latter operator written in the above representation is

$$W^\mu=(W^0,\mathbf{W})=(\mathbf{P}\cdot\mathbf{J},P_0\mathbf{J}-\mathbf{P}\times\mathbf{K}) \tag{6}$$

$$=[\mathbf{p}\cdot\mathbf{s},p_0(\mathbf{n}\cdot\mathbf{s})\mathbf{n}\pm mc\mathbf{s}_\perp], \tag{7}$$

where $\mathbf{n}$ is the unit vector pointing in the momentum direction and

$$\mathbf{s}_\perp=\mathbf{s}-(\mathbf{n}\cdot\mathbf{s})\mathbf{n}.$$

The center-of-mass position operator which generalizes to any representation the operator $\mathbf{x}_A$ of Schrödinger is

$$\mathbf{Q}=-\tfrac{1}{2}(P_0^{-1}\mathbf{K}+\mathbf{K}P_0^{-1}) \tag{8}$$

$$=i\hbar\frac{\partial}{\partial\mathbf{p}}-i\hbar\frac{\mathbf{p}}{2p_0^2}+\frac{\mathbf{p}\times\mathbf{s}}{|p_0|(mc+|p_0|)}. \tag{9}$$

This operator extends naturally also to massless fields and can be shown to be uniquely (up to subtleties with domains of unbounded operators) derived from symmetry considerations in the case of the Maxwell field [15,16]. In the Maxwell field case, formula (9) can be regarded as defining a connection on a light cone. A parallel transport with respect to this connection can be shown to generate a Berry phase [17,18].

Orbital angular momentum and spin corresponding to $\mathbf{Q}$ were given by Pryce and Fleming [13,19]

$$\mathbf{L}=\mathbf{Q}\times\mathbf{P}=\frac{\hbar}{i}\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}+\frac{|p_0|-mc}{|p_0|}[\mathbf{s}-(\mathbf{n}\cdot\mathbf{s})\mathbf{n}],$$

$$\mathbf{S}=\mathbf{J}-\mathbf{L}=\frac{mc}{|p_0|}\mathbf{s}+\left(1-\frac{mc}{|p_0|}\right)(\mathbf{n}\cdot\mathbf{s})\mathbf{n}$$

$$=\sqrt{1-\beta^2}\mathbf{s}_\perp+(\mathbf{n}\cdot\mathbf{s})\mathbf{n}=\mathbf{W}/p_0. \tag{10}$$

$\boldsymbol{\beta}=\mathbf{n}|\mathbf{v}|/c$, where $\mathbf{v}=c^2\mathbf{p}/p_0$ is a velocity of the particle. Equation (10) shows that relativistic spin is closely related to the Pauli-Lubanski vector. Projection of spin in a direction given by the unit vector $\mathbf{a}$ commutes with the Hamiltonian P_0 and equals

$$\mathbf{a}\cdot\mathbf{S}=\left[\frac{mc}{|p_0|}\mathbf{a}+\left(1-\frac{mc}{|p_0|}\right)(\mathbf{n}\cdot\mathbf{a})\mathbf{n}\right]\mathbf{s} \tag{11}$$

$$=[\sqrt{1-\beta^2}\mathbf{a}_\perp+\mathbf{a}_\parallel]\cdot\mathbf{s}=:\boldsymbol{\alpha}(\mathbf{a},\mathbf{p})\mathbf{s}. \tag{12}$$

The latter equality defines the vector $\boldsymbol{\alpha}(\mathbf{a},\mathbf{p})$ whose length is

$$|\boldsymbol{\alpha}(\mathbf{a},\mathbf{p})|=\frac{\sqrt{(\mathbf{p}\cdot\mathbf{a})^2+m^2c^2}}{|p_0|}=\sqrt{1+(\boldsymbol{\beta}\cdot\mathbf{a})^2-\beta^2}.$$

The eigenvalues of $\mathbf{a}\cdot\mathbf{S}$ are therefore

$$\lambda_a=j_3\hbar\sqrt{1+(\boldsymbol{\beta}\cdot\mathbf{a})^2-\beta^2}, \tag{13}$$

where $j_3=-j,\ldots,+j$. The eigenvalues of the Pauli-Lubanski vector projections are $\omega_a=p_0\lambda_a$. In the infinite-momentum or massless limit the eigenvalues of the relativistic spin in a direction perpendicular to $\mathbf{p}$ vanish, which can be regarded as a consequence of the Lorentz flattening of the moving particle [in these limits $\mathbf{S}=(\mathbf{n}\cdot\mathbf{s})\mathbf{n}$]. Projection of spin on the momentum direction is equal to the helicity, i.e., $\mathbf{p}\cdot\mathbf{S}=\mathbf{p}\cdot\mathbf{s}$ for any $\mathbf{p}$, and $\mathbf{S}=\mathbf{s}$ in the rest frame $(\mathbf{p}=0)$. Bacry [20] observed that a nonrelativistic limit of $\mathbf{Q}$ leads to a correct form of the spin-orbit interaction in the Pauli equation if one uses potentials $V(\mathbf{Q})$ instead of $V(\mathbf{x})$ [21]; an analogous effect was described in [22] where the internal angular momentum of the *Zitterbewegung* leads to spin with the correct $g=2$ factor. An algebraic curiosity is the fact that the components of $\mathbf{S}$ satisfy an algebra which is $so(3)$ in the rest frame and formally contracts to the Euclidean $e(2)$ in the infinite-momentum or massless limit, and thus provides an interesting alternative explanation of the privileged role played by the Euclidean group in the theory of massless fields [23,24].

In spite of all these facts suggestng that both $\mathbf{Q}$ and $\mathbf{S}$ are natural candidates for physical observables no experimental tests distinguishing them from other definitions of position and spin have been proposed so far. Obviously, it is not easy to test directly $\mathbf{Q}$ which, representing the center of mass, may be expected to couple to the gravitational field. The spin operator, on the other hand, is responsible for the magnetic moment and should couple to the electromagnetic field which is much stronger.

Consider now a Stern-Gerlach-type measurement involving spin-$\frac{1}{2}$ relativistic particles and assume that $\mathbf{S}$ is the physical internal angular momentum which is measured in this experiment [25]. Assume also that we have two spin-$\frac{1}{2}$ particles in a singlet state (total helicity equals zero) and propagating in the same direction with identical momenta $\mathbf{p}$ (to be more precise one should take wave packets in momentum space, but for simplicity assume that they are sufficiently well localized around momenta $\mathbf{p}$, so that we can approximate them by plane waves)

$$|\psi\rangle=\frac{1}{\sqrt{2}}\left(\left|+\frac{1}{2},\mathbf{p}\right\rangle\left|-\frac{1}{2},\mathbf{p}\right\rangle-\left|-\frac{1}{2},\mathbf{p}\right\rangle\left|+\frac{1}{2},\mathbf{p}\right\rangle\right). \tag{14}$$

The kets $|\pm\frac{1}{2},\mathbf{p}\rangle$ form the *helicity* basis. Consider the binary operators $\hat{\mathbf{a}}=\mathbf{a}\cdot\mathbf{S}/|\lambda_a|$, $\hat{\mathbf{b}}=\mathbf{b}\cdot\mathbf{S}/|\lambda_b|$. Their eigenvalues are ±1. The relativistic corrections that arise are those resulting

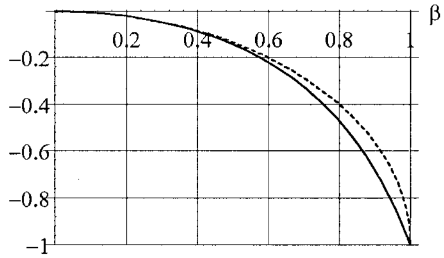

FIG. 1. Average (18) (solid) as compared to $[1-\beta^2]^{1/2}-1$ (dotted). The EPRB average varies with β faster than proper time. Relativistic corrections described by Eqs. (15) and (18) are caused by both the Lorentz contraction and the Møller shift of the center of mass. Their experimental verification would provide an indirect proof that the noncommuting position operator (9) is physically well defined.

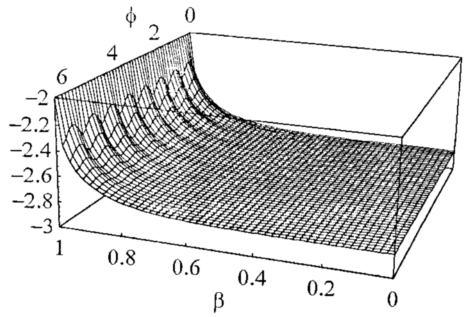

FIG. 2. For $\beta=0$ we obtain the maximal violation and no violation for $\beta\to 1$. Alice and Bob may be confused and ''detect'' an eavesdropper even if the state is pure singlet, but β is close to 1. Spins of ultrarelativistic spin-$\frac{1}{2}$ particles are ''almost classical'' and are either almost parallel or antiparallel to their momenta.

from the modification of the spin direction as ''seen'' by a measuring device. The average of the relativistic EPR-Bohm-Bell operator is

$$\langle\psi|\hat{\mathbf{a}}\otimes\hat{\mathbf{b}}|\psi\rangle=-\frac{\mathbf{a}\cdot\mathbf{b}-\beta^2\mathbf{a}_\perp\cdot\mathbf{b}_\perp}{\sqrt{1+\beta^2[(\mathbf{n}\cdot\mathbf{a})^2-1]}\sqrt{1+\beta^2[(\mathbf{n}\cdot\mathbf{b})^2-1]}}. \tag{15}$$

There are several interesting particular cases of formula (15). First, if $\mathbf{a}=\mathbf{a}_\perp$, $\mathbf{b}=\mathbf{b}_\perp$ then

$$\langle\psi|\hat{\mathbf{a}}\otimes\hat{\mathbf{b}}|\psi\rangle=-\mathbf{a}\cdot\mathbf{b}, \tag{16}$$

which is the nonrelativistic result. This case will never occur in a realistic experiment since localization of detectors will lead to a momentum spread. If $\mathbf{a}\cdot\mathbf{n}\neq 0$, $\mathbf{b}\cdot\mathbf{n}\neq 0$ then in the ultrarelativistic case $\beta^2=1$

$$\langle\psi|\hat{\mathbf{a}}\otimes\hat{\mathbf{b}}|\psi\rangle=-\frac{(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})}{|\mathbf{a}\cdot\mathbf{n}||\mathbf{b}\cdot\mathbf{n}|}=\pm 1 \tag{17}$$

independently of the choice of **a**, **b**. It is easy to intuitively understand this result: In the ultrarelativistic limit projections of spin in directions perpendicular to the momentum vanish for both particles and spins are (anti-) parallel to the momentum. The most striking case occurs if **a** and **b** are perpendicular and the nonrelativistic average is 0. Let $\mathbf{a}\cdot\mathbf{b}=0$, $\mathbf{a}\cdot\mathbf{n}=\mathbf{b}\cdot\mathbf{n}=1/\sqrt{2}$. Then

$$\langle\psi|\hat{\mathbf{a}}\otimes\mathbf{b}|\psi\rangle=-\frac{\beta^2}{2-\beta^2}. \tag{18}$$

This average is 0 in the rest frame ($\beta=0$) and -1 for $\beta=1$. Any observable deviation from 0 in an EPR-Bohm type experiment would be an indication that the operators **S** and **Q** are physically correct observables and that massive spin-$\frac{1}{2}$ particles are extended in the sense that centers of mass and charge do not coincide. Figure 1 shows that Eq. (18) describes a relativistic effect that is even stronger than the Lorentz contraction or the time delay (both are proportional to $\sqrt{1-\beta^2}$). One peculiarity of **Q** is that its components do not commute for nonzero spins. An uncertainty principle guarantees therefore that such a particle cannot be localized at a point [29], or is extended in some nonclassical sense, a property that cannot be without implications for the renormalization and self-energy problems. The definition of **Q** implies also that the center of mass does not transform as a spatial component of a four vector. This apparently counter-intuitive result agrees however with the classical analysis of Møller [8,13] who showed that the center of mass of a spinning classical body is not a component of a four vector. These interesting properties seem unavoidable and can be proved in various ways at both quantum and classical levels (for their classical derivations see [11,27]).

Consider now the vectors $\mathbf{a}=(1/\sqrt{2},1/\sqrt{2},0)$, $\mathbf{a}'=(-1/\sqrt{2},1/\sqrt{2},0)$, $\mathbf{b}=(0,1,0)$, $\mathbf{b}'=(1,0,0)$ leading to the maximal violation of the Bell inequality in nonrelativistic domain. Figure 2 shows the dependence of the average (1) on β and ϕ where $\boldsymbol{\beta}=(\beta\cos\phi,\beta\sin\phi,0)$. Figure 3 shows the average (1) for $\boldsymbol{\beta}=\beta(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)$ as a function of the spherical angles and for $\beta=0.99$ and $\beta=0.95$.

These results show clearly that the information about the degree of violation of the Bell inequality is not sufficient for

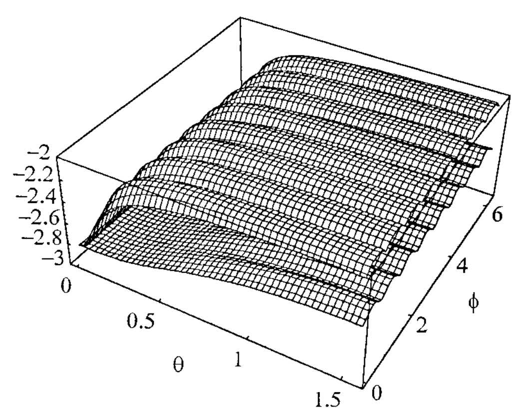

FIG. 3. The average (1) for $\beta=0.99$ (upper) and $\beta=0.95$ (lower). $\theta=0$ corresponds to particles moving perpendicularly to measuring devices (maximal violation). For $\theta=\pi/2$ we have the situation from Fig. 2.

determining purity of a massive two-particle zero-helicity state. Additionally one has to know the momentum distribution of the particle beam.

III. PAULI-LUBANSKI VECTOR VS SPIN

The relation between W^μ and $\mathbf{S}$ is similar to the one between the four-velocity u^μ and the three-velocity $\boldsymbol{\beta}$. The Casimir operator $W_\mu W^\mu$ equals $(mc)^2 j(j+1)$ if an irreducible representation of the Poincaré group is considered. For this reason it is typical to define the spin four-vector as

$$w^\mu = W^\mu/(mc) = \left[\mathbf{u}\cdot\mathbf{s}, \frac{p_0}{mc}(\mathbf{n}\cdot\mathbf{s})\mathbf{n} \pm s_\perp\right], \tag{19}$$

where u^μ is the four velocity. In the rest frame $p_0 = \pm mc$ and $w^\mu = \pm(0,\mathbf{s})$ which seems to justify this choice. For a moving particle the eigenvalues of $\mathbf{a}\cdot\mathbf{w}$ are $\lambda_a p_0/mc$ where λ_a denote the respective eigenvalues of $\mathbf{a}\cdot\mathbf{S}$. The eigenvalues of $\mathbf{a}\cdot\mathbf{w}$ therefore tend to $\pm\infty$ in the infinite-momentum limit which is unphysical for a spin observable. Nothing of that kind occurs if one divides $\mathbf{W}$ by *energy* and *not by mass* which again selects our spin operator as a candidate for a physical observable.

Nevertheless, irrespective of this subtlety, the relativistic EPRB average is the same for both $\mathbf{S}$ and $\mathbf{w}$ since we consider a ''yes-no'' observable which is obtained by *normalization* of eigenvalues to ± 1. This is another reason to believe that the discussed suppression of degree of the Bell inequality violation is a physical phenomenon that should be observable in experiments with massive particles.

IV. COMPARISON WITH THE DIRAC EQUATION

Just for the sake of completeness let us compare the general formulas to the analogous calculations performed for the Dirac electrons. The Pauli-Lubanski vector is

$$W^0 = \mathbf{p}\cdot\mathbf{s} \tag{20}$$

$$\mathbf{W} = \tfrac{1}{2}(\mathbf{s}H + H\mathbf{s}), \tag{21}$$

where H is the Dirac free Hamiltonian and

$$\mathbf{s} = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

is the spinor part of the generator of rotations. The relativistic spin operator is therefore equal to

$$\begin{aligned}\mathbf{S} &= \mathbf{W}H^{-1} = \tfrac{1}{2}(\mathbf{s} + \Lambda\mathbf{s}\Lambda) \\ &= \Pi_+\mathbf{s}\Pi_+ + \Pi_-\mathbf{s}\Pi_- .\end{aligned} \tag{22}$$

Here Λ is the sign-of-energy operator and $\Pi_\pm$ project on states of given signs of energy. It follows that $\mathbf{S}$ is the so-called even part of the spinor part of the generator of rotations. This operator commutes with H and, hence, can be used for analyzing the EPRB experiment [30]. The explicit form of this observable in units with $\hbar = 1$ and $c = 1$ is

$$\mathbf{S} = \frac{m^2}{p_0^2}\mathbf{s} + \frac{\mathbf{p}^2}{p_0^2}(\mathbf{n}\cdot\mathbf{s})\mathbf{n} + \frac{im}{2p_0^2}\mathbf{p}\times\boldsymbol{\gamma}. \tag{23}$$

$\boldsymbol{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$ where γ^k are Dirac matrices. The eigenvalues of $\mathbf{a}\cdot\mathbf{S}$ are given by Eq. (13) with $j_3 = \pm 1/2$. The corresponding positive-energy eigenstates in the standard representation are

$$\Psi^a_\pm = \begin{pmatrix} \sqrt{|p_0| + m}\left((|\lambda_a| + \tfrac{1}{2}\,\mathbf{a}\cdot\mathbf{n})w_\pm \pm \dfrac{m\mathbf{a}\cdot\mathbf{n}}{2|p_0|}w_\mp\right) \\ \sqrt{|p_0| - m}\left(\pm(|\lambda_a| + \tfrac{1}{2}\,\mathbf{a}\cdot\mathbf{n})w_\pm - \dfrac{m\mathbf{a}\cdot\mathbf{n}}{2|p_0|}w_\mp\right) \end{pmatrix}$$

where $w_\pm$ satisfies $\mathbf{n}\cdot\boldsymbol{\sigma}w_\pm = \pm w_\pm$.

I have remarked that a positive verification of the relativistic center-of-mass concept would indicate that nonzero-spin relativistic particles are extended. The example of the Dirac equation illustrates this idea. Consider again the spinor part of the generator of rotations $\mathbf{s}$. It does not commute with H and satisfies in the Heisenberg picture the precession equation

$$\dot{\mathbf{s}} = \boldsymbol{\omega}\times\mathbf{s}, \tag{24}$$

where $\boldsymbol{\omega} = -2c\gamma^5\mathbf{p}/\hbar$. For massive fields $\boldsymbol{\omega}$ does not commute with H and, hence, can be decomposed into even and odd parts. The even part is

$$\boldsymbol{\Omega} = \frac{c^2 + mc^3\boldsymbol{\gamma}\cdot\mathbf{n}/|\mathbf{p}|}{c^2 + m^2c^4/\mathbf{p}^2}\boldsymbol{\omega}.$$

$\boldsymbol{\Omega}$ reduces to $\boldsymbol{\omega}$ in both massless and infinite-momentum limits. A Hamiltonian of a particle moving with velocity $\mathbf{v} = c\boldsymbol{\beta}$ can now be expressed as

$$H = \left(1 + \frac{m^2c^4}{c^2\mathbf{p}^2}\right)\boldsymbol{\Omega}\cdot\mathbf{S} = \boldsymbol{\beta}^{-2}\boldsymbol{\Omega}\cdot\mathbf{S} = \boldsymbol{\Omega}'\cdot\mathbf{S}, \tag{25}$$

where each of the operators appearing in H is even and commuting with H. The form (25) is analogous to the one discussed in [31]. The limiting form $H = \boldsymbol{\omega}\cdot\mathbf{S}$ is characteristic of all massless fields, where for higher spins Eq. (24) is still valid, but angular velocities for a given momentum are smaller the greater the helicity.

The new form of the Hamiltonian leads to the following observation. Notice that for massless fields the Hamiltonian can be written in either of the following two forms:

$$H = \boldsymbol{\omega}\cdot\mathbf{S} \tag{26}$$

or

$$H = \mathbf{c}\cdot\mathbf{p} = \mathbf{v}\cdot\mathbf{p}, \tag{27}$$

where $\mathbf{v}$ is the velocity operator for a general massless field ($c\boldsymbol{\alpha}$ in case of the Dirac equation) and $\mathbf{c} = (\mathbf{v}\cdot\mathbf{p})\mathbf{p}/\mathbf{p}^2$ is its even part. We recognize here the classical mechanical rule for a transition from a pointlike description to the extended-objectlike one: linear momentum goes into angular momentum, linear velocity into angular velocity, and vice versa. The third part of this rule (mass moment of inertia) can be naturally postulated as follows:

$$H = m_k\mathbf{c}^2 = I_k\boldsymbol{\omega}^2, \tag{28}$$

where Eq. (28) defines the kinetic mass (m_k) and the kinetic moment of inertia (I_k) of the massless field. The explicit form of I_k for massless fields of helicity $\lambda = m - n$ [corresponding to the (m,n) spinor representation of $SL(2,C)$] is, in ordinary units,

$$I_k = \frac{\lambda\hbar \mathbf{p}\cdot\mathbf{S}}{c\mathbf{p}^2}. \tag{29}$$

The equation

$$I_k = m_k r_\lambda^2 \tag{30}$$

characteristic, by the way, of circular strings (here with mass m_k) defines some radius which is equal to

$$r_\lambda = \frac{\hbar\lambda}{|\mathbf{p}|}, \tag{31}$$

which can be expressed also as a form of the uncertainty principle

$$|\mathbf{p}| r_\lambda = \hbar\lambda. \tag{32}$$

The center-of-mass commutation relation

$$[Q_k, Q_l] = -i\hbar \epsilon_{klm} S_m / p_0^2$$

leads, in the massless case, to the uncertainty relations of the type

$$\Delta Q_1 \Delta Q_2 \geqslant \hbar^2 |\lambda| / (2\langle |\mathbf{p}|^2 \rangle |) = \langle r_\lambda^2 \rangle / |2\lambda|.$$

It is remarkable that the same radius occurs naturally in the twistor formulation of massless fields [28]. It is known that although spin-0 twistors can be represented geometrically by null straight lines, this does not hold for spin λ, $\lambda \neq 0$, twistors [28]. Instead of the straight line we get a congruence of twisting, null, shear-free world lines, the so-called Robinson congruence. A three-dimensional projection of this congruence consits of *circles*, whose radii are given exactly by our formula (31) (cf. the footnote on p. 62 in [28]). The circles propagate with velocity of light in the momentum direction and rotate in the right- or left-handed sense depending on the sign of helicity. The same construction can be performed for the massive Dirac particle if one uses $\mathbf{\Omega}'$.

V. SUMMARY

The main idea advocated in this paper can be summarized as follows. Consider *some* procedure leading to a measurement of a nonrelativistic spin [25]. This procedure is based on a black box giving results ''yes'' or ''no'' for spins equal to, respectively, $+\hbar/2$ and $-\hbar/2$. In the nonrelativistic domain the particles enter the device ''slowly.'' Imagine now that for some reasons we decide to use faster particles. The measured average may vary with the growing (average) velocity of the particle beam and, obviously, some result will be obtained. The question is how to calculate the result of such an experiment assuming that the procedure measures the spin itself and not the total angular momentum. Many different definitions of relativistic spins exist in literature but all of them are momentum dependent [32]. Calculations based on the definition which seems the most physical (via the relativistic center of mass) show that relativistic corrections are nontrivial. Their strength can be regarded as a combined influence of two independent relativistic phenomena: The Lorentz contraction and the Møller shift of the center of mass of a spinning body. The same result is obtained if one uses the spin operator defined via the Pauli-Lubanski vector. The effect can be in principle measured and will have to be taken into account in quantum cryptographic tests for eavesdropping if fast massive particles will be used for a key transfer.

ACKNOWLEDGMENTS

I am grateful to Ryszard Horodecki for suggesting the problem, Vasant Natarajan for information concerning experiments, and Gerald Kaiser for extensive discussions. The paper is a part of the KBN Project 2P30B03809.

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[24] Typically the Euclidean structure of massless fields is interpreted in the language of finite dimensional and nonunitary representations. It can be shown that the $SE(2)$ structure is associated with a cylindrical geometry: The cylinder is parallel to momentum and the group contains rotations around and translations along the cylinder. The translations are gauge transformations of a vector potential. The relativistic spin algebra discussed in this paper provides an explanation of the contraction in terms of relativistic deformations of an extended spinning particle and the corresponding deformation of the spin algebra. The analogy between the two approaches is, however, even deeper: The noncommuting position operator allows for localizations of the Maxwell fields on curves [15] and the classical phase-space picture derived in [27] leads to world tubes rather than world lines. An analogous phenomenon is found in the twistor formulation where instead of world lines the massless fields are interepreted in terms of the Robinson congruence of twisting null world lines [28]. The cylindrical approach is described in Y. S. Kim and M. E. Noz, Phys. Rev. D **15**, 335 (1977); Y. S. Kim, Phys. Rev. Lett. **63**, 348 (1989); D. Han, Y. S. Kim, and D. Son, Am. J. Phys. **54**, 818 (1986); Phys. Lett. **131B**, 327 (1983); D. Han and Y. S. Kim, Am. J. Phys. **49**, 348 (1981); J. J. van der Bij, H. van Dam, and Y. J. Ng, Physica **116A**, 307 (1982); D. Han, Y. S. Kim, and D. Son, Phys. Rev. D **26**, 3717 (1982); Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986); Y. S. Kim and E. P. Wigner, J. Math. Phys. **28**, 1175 (1987); **31**, 55 (1990).
[25] The standard textbook description of the Stern-Gerlach experiment is, strictly speaking, incorrect as it involves a ''magnetic'' field **B** which does not satisfy the Maxwell equation $\nabla\cdot\mathbf{B}=0$ [2,26]. This difficulty can be circumvented by saying that since spin is a well defined observable there must exist a procedure for its measurement. The question to what kind of observables (i.e., self-adjoint operators) there correspond operational measurement procedures is open. I assume here that a physically well defined observable is measurable *in principle*.
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[30] The equivalence between the relativistic spin operator and the even part of the spinor part of the generator of rotations may lead to the naive conclusion that a measurable EPRB average could be simply $\langle\psi|\widetilde{a}\otimes\widetilde{b}|\psi\rangle$, where $\widetilde{a}=\mathbf{a}\cdot\mathbf{s}/(\hbar/2)$, etc., provided the electromagnetic fields used during the measurement are sufficiently weak so that they do not couple particle to antiparticle states, and ψ is a positive-energy state [33]. This assertion is wrong since the average uses the incorrectly normalized ''yes-no'' observable: The correct normalization is $\mathbf{a}\cdot\mathbf{s}/|\lambda_a|$ that is by eigenvalues of the relativistic operator.
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[33] I should mention here another argument appearing sometimes in similar contexts. It is claimed that the second quantization eliminates problems with negative energies so no decomposition of operators into ''even'' and ''odd'' parts is really physical. Notice, however, that we have obtained the even spin of the Dirac particle as a by product of the analysis which started from unitary representations of the Poincaré group. These are precisely the representations that are used in quantum-field theory and the notion of the Pauli-Lubanski vector ''survives'' second quantization. The same concerns the relativistic spin introduced above.