Centrifugal effects in a Bose-Einstein condensate in the time-orbiting-potential magnetic trap

A. B. Kuklov,[1,2] N. Chencinski,[1] A. M. Levine,[1] W. M. Schreiber,[1] and Joseph L. Birman[2]
[1]*Department of Applied Sciences, The College of Staten Island, City University of New York, Staten Island, New York 10314*
[2]*Department of Physics, The City College, City University of New York, New York, New York 10031*
(Received 8 February 1996; revised manuscript received 11 July 1996)

Single-particle states in the atomic trap employing the rotating magnetic field are found using the full time-dependent instantaneous trapping potential. These states are compared with those of the effective time-averaged potential. We show that the trapping is possible when the frequency of the rotations exceeds some threshold. Slightly above this threshold the weakly interacting gas of the trapped atoms acquires the properties of a quasi-one-dimensional system in the frame rotating together with the field. The role of the atom-atom interaction in changing the ideal gas solution is discussed. We show that in the limit of large numbers of particles the rotating field whose frequency is appropriately modulated can be utilized as a driving force principally for the center-of-mass motion as well as for the angular momentum $L=2$ normal modes of the Bose condensate. A mechanism of quantum evaporation forced by the rotating field is analyzed. [S1050-2947(97)03001-1]

PACS number(s): 03.75.Fi, 05.30.Jp, 32.80.Pj, 67.90.+z

I. INTRODUCTION

The recently developed methods [1–4] for storing atoms at very low densities and temperatures open up opportunities for studying the role of the atom-atom interaction in macroscopic quantum phenomena [5]. The problems of Bose-Einstein condensate formation [6–8], the dynamical response of the condensate in the trap [9], and the interaction of the condensate with light [10] now can be investigated experimentally.

The zero-dimensional geometry and the small size of the atomic traps restrict the direct observation of the most spectacular effects known from the history of superfluidity of ^{4}He (see, e.g., in, [11]). Therefore, devising alternative practical methods for probing the condensate in the atomic traps becomes of crucial importance. In this regard the recent suggestion [12] to analyze the rotational properties of the trapped atomic cloud appears to be very promising. As it was pointed out in Ref. [13], an analysis of quantum evaporation from the condensate can yield valuable information about the interatomic interaction. Therefore, adopting this analysis to the trapped gases is highly desirable. The very recent theoretical [14] and experimental [15] analyses of a Bose-Einstein condensate undergoing variations of the trapping potential address the long-standing question regarding the coherent versus dissipative behavior of a many-body system.

The trap [16] where the Bose-Einstein condensation of Rb atoms was initially achieved [1] utilizes a rapidly rotating magnetic field (RMF). This field, if averaged over the rotational period, creates an effective static oscillator potential [which is called the time-orbiting potential (TOP) [16]] and reduces the escape of atoms from the trap due to spin-flip effects [16]. Other traps [2–4] do not rely on the RMF. A main assumption made about the RMF is that as long as the frequency of rotations ω is much larger than the frequency ω_0 of oscillations in the TOP, the trapped atoms are not disturbed by the time variations of the instantaneous potential. Accordingly, the results [17,18] obtained for the Bose-Einstein condensation in a static parabolic potential can be applied to this case as well.

Generally speaking, the RMF should transfer energy and angular momentum to the condensate. Therefore, the RMF can be viewed as a possible tool for studying the dynamical response of the condensate. In this sense, addressing the problem of the exact description of the quantum atomic states in the trap employing the RMF, rather than relying on the time-averaging procedure [16], appears to be quite important.

In this paper we study various aspects of the RMF: we find the exact single-particle states in the trap [16] without relying on the time-averaging procedure; it is shown that the RMF, if properly modulated, should excite selectively some modes of the condensate; we derive the Ginzburg-Gross-Pitaevskii (GGP) equation taking into account the effects of the quantum evaporation induced by the RMF. As an application, the decay rate of the condensate due to the RMF is calculated in the case of steady rapid rotations of the RMF.

The outline of the paper is as follows. In Sec. II we find the exact eigenenergies and eigenstates for a single particle in the trap [16]. This solution is obtained in the frame rotating together with the RMF. The properties of the solution as a function of ω are analyzed. In Sec. III we consider the limit of large numbers of particles in the condensate and analyze the nondissipative interaction between the RMF and the normal modes of the condensate. In Sec. IV the GGP equation with dissipation due to the RMF is derived under certain approximations. The quasistatic solution for the rate of the centrifugal evaporation of the condensate is derived in the limit of high ω.

II. SINGLE-PARTICLE STATES IN THE ROTATING FRAME

In our analysis of the behavior of a single atom trapped by the magnetic field **B** we follow the approximation that the atomic spin orientation is parallel to **B** [16]. Then the effective potential energy of the atom seeking the low field is

essentially the Zeeman energy $U=|\mu_B\mathbf{B}|$, where μ_B stands for the Bohr magneton (we ignore the nuclear magnetic moment). The magnetic field of the trap [16] consists of the static quadrupolar part $\mathbf{B}_q$ having axial symmetry with respect to the z axis and the RMF $\mathbf{B}_b$ rotating in the (x,y) plane. Representing these explicitly, one finds the components

$$B_{qx}=B_q'x,\quad B_{qy}=B_q'y,\quad B_{qz}=-2B_q'z,$$
$$B_{bx}=-B_b\cos(\omega t),\quad B_{by}=-B_b\sin(\omega t),\quad B_{bz}=0, \tag{1}$$

where B_q',B_b stand for the constant gradient of the quadrupolar field and the constant amplitude of the RMF, respectively. Given Eq. (1), the potential energy

$$U=|\mu_B(\mathbf{B}_q+\mathbf{B}_b)| \tag{2}$$

depends on time. As suggested in Ref. [16], for high ω the time dependence can be effectively averaged over, which results in the TOP potential [16]. In general, the time dependence in U should result also in the nonadiabatic exchange of energy between the external field and the atoms in the trap. However, as it will become clear from the following, for a time-independent ω and noninteracting atoms, no such exchange occurs between the RMF and the atoms in the trap. In fact, in the frame rotating together with the RMF, the single-particle Hamiltonian becomes time independent, ensuring that the atom once prepared in the pure state (in the rotating frame) will exist forever in such a state. Going to the rotating frame implies the coordinate transformation

$$x''=\cos[\theta(t)]x+\sin[\theta(t)]y,$$
$$y'=-\sin[\theta(t)]x+\cos[\theta(t)]y, \tag{3}$$

where $\theta(t)$ stands for the angle between $\mathbf{B}_b$ and the x direction. In the case of steady rotations, $\theta(t)=\omega t$. The transformation (3) results in (2) rewritten in the time-independent form as

$$U=|\mu_B B_q'|\sqrt{(x''-x_0)^2+y'^2+4z^2},\quad x_0=\frac{B_b}{B_q'}. \tag{4}$$

Since the effective size of the atomic cloud is much less than x_0 [16], one can expand (4) in terms of $1/x_0$. This gives, for the first two terms (linear and quadratic),

$$U=\frac{\omega_{0y}^2y'^2}{2}+\frac{\omega_{0z}^2z^2}{2}-x'', \tag{5}$$

where we have omitted the unimportant constant $|x_0|$; the notations $\omega_{0y}^2=1/|x_0|$ and $\omega_{0z}^2=4/|x_0|$ are introduced and the units of energy and length are employed as

$$\varepsilon_0=\frac{\hbar^2}{Ml_0^2},\quad l_0=\frac{\hbar^{2/3}}{(M|\mu_B B_q'|)^{1/3}}, \tag{6}$$

respectively. In (6), M stands for the atomic mass. Note that in the rotating frame the stiffness of the potential along the x'' coordinate is zero. This would imply that no states localized around the origin exist. However, as we will see below, the finite kinetic energy of the particle changes this conclusion for sufficiently large ω.

In the rotating frame the kinetic energy acquires the Coriolis term $-\omega L$, where L is the z component of the angular momentum operator and $\omega=\dot\theta$. This can be seen directly by means of implementing the coordinate transformation (3) in the Schrödinger equation written in the laboratory frame. Consequently, taking into account (5), one finds the single-particle Schrödinger equation ($\hbar=1$) in the rotating frame

$$i\partial_t\psi=H_\omega\psi,$$
$$H_\omega=-\tfrac{1}{2}\Delta+i\omega(x''\partial_{y'}-y'\partial_{x''})+U. \tag{7}$$

We consider first the case $\omega=\text{const}$. Note that Eqs. (5) and (7) represent a quadratic form that can be diagonalized explicitly (see Appendix A). Prior to solving it let us eliminate the linear term $-x''$ from (5). This can be accomplished by the transformation

$$\psi\Rightarrow\exp\left(-i\frac{y'}{\omega}\right)\psi(x',y',z),\quad x'=x''+\frac{1}{\omega^2}, \tag{8}$$

which results in Eq. (7) being rewritten as

$$i\partial_t\psi=H_\omega'\psi,$$
$$H_\omega'=-\frac{1}{2}\Delta+i\omega(x'\partial_{y'}-y'\partial_{x'})+\frac{\omega_{0y}^2y'^2}{2}+\frac{\omega_{0z}^2z^{2\prime}}{2}. \tag{9}$$

For the case $|\omega|<\omega_{0y}$, no discrete states localized near the origin $x'=y'=z'=0$ exist in the trap. Accordingly, we will not analyze this case any more. For $|\omega|>\omega_{0y}$ such states do exist. Their eigenenergies are (see Appendix A)

$$\varepsilon_{mnl}=\omega_+m-\omega_-n+\omega_{0z}l,$$
$$\omega_\pm=\omega\sqrt{1+\eta^2/2\pm\eta\sqrt{2+\eta^2/4}},\quad \eta=\omega_{0y}/\omega, \tag{10}$$

where m,n,l are integer non-negative quantum numbers and the energy of the state with $m=n=l=0$ is set equal to zero. The normalized eigenfunctions are (see Appendix A)

$$\psi_{mnl}(x',y',z)=\frac{(\omega_{0z}\omega_1\omega_2)^{1/4}}{\pi^{3/4}\sqrt{2^{m+n+l}m!n!l!}}\times e^{-\Xi_0}\left[\frac{\partial^{m+n+l}}{\partial t_1^m\partial t_2^n\partial t_3^l}e^{\Xi}\right]_{t_1=t_2=t_3=0}, \tag{11}$$

where we have introduced

$$\Xi_0=\frac{\omega_1 x'^2}{2}+\frac{\omega_2 y'^2}{2}+\frac{\omega_{0z} z^2}{2}-i\gamma_0 x' y',$$

$$\Xi=\frac{\alpha^2-\alpha^{-2}}{2}(t_1^2-t_2^2)-t_3^2-2\nu t_1 t_2+2\sqrt{\omega_{0z}}t_3 z$$
$$+\sqrt{2}[\alpha^{-1}\sqrt{\omega_1}x'-i\nu\alpha\sqrt{\omega_2}y']t_1$$
$$+\sqrt{2}[\nu\alpha\sqrt{\omega_1}x'+i\alpha^{-1}\sqrt{\omega_2}y']t_2. \quad (12)$$

In Eq. (12) the parameters are

$$\gamma_0=\omega\frac{\omega_2-\omega_1}{\omega_2+\omega_1},\quad \omega_2=\sqrt{1-\eta^2}\omega_1,$$

$$\eta=\frac{\omega_{0y}}{\omega},\quad \nu=\mathrm{sgn}(\omega),$$

$$\omega_1=\sqrt{8-8\sqrt{1-\eta^2}-4\eta^2-\eta^4}\,\eta^{-2}\omega,$$

$$\alpha=\frac{\left(1-\frac{\eta^2}{4}-\frac{\eta}{4}\sqrt{8+\eta^2}\right)^{1/2}}{\left(1+\frac{\eta^2}{2}-\frac{\eta}{2}\sqrt{8+\eta^2}\right)^{1/4}(1-\eta^2)^{1/8}}. \quad (13)$$

Note that the parameter α as a function of ω (or η) has the property $\alpha(-\eta)\alpha(\eta)=1$. In the limit $\eta\to 0$ ($\omega\to\infty$) one obtains $\alpha=1$, and the eigenfunctions (11) and (12) as well as the eigenenergies (10) become exactly those characterizing the TOP [16] as seen from the rotating frame. In particular, the spectrum acquires the form

$$\varepsilon'_{mnl}=\varepsilon^{(\mathrm{lab})}_{mnl}-\omega L,$$
$$\varepsilon^{(\mathrm{lab})}_{mnl}=\omega_0(m+n)+\omega_{0z}l,\quad L=n-m, \quad (14)$$

where $\varepsilon^{(\mathrm{lab})}_{mnl}$ are the eigenenergies of the TOP oscillator potential [16] viewed from the laboratory frame, with $\omega_0=\omega_{0y}/\sqrt{2}$ being the frequency of the oscillations in the (x,y) plane; L stands for the axial component of the angular momentum.

For large ω, Eq. (14) is an approximation of the exact expression (10). The corrections due to the finiteness of ω_0/ω turn out to be of the order of $(\omega_0/\omega)^3$, so that one can effectively ignore these even if ω is only a few times larger than ω_0. Consequently, for such ω we will employ Eq. (14) instead of the exact form (10).

In the limit $|\eta|\to 1$ from below (that is $\omega\to\omega_{0y}$ from above), the solution (10)–(13) acquires features characteristic of a quasi-one-dimensional system. Indeed, taking this limit in Eqs. (10)–(13), one finds

$$\omega_+=\omega_1=\sqrt{3}\omega_{0y},\quad \omega_-=\frac{1}{3}\omega_2\approx\sqrt{\frac{2}{3}(1-|\eta|)}\,\omega_{0y}. \quad (15)$$

This expression, together with Eqs. (11) and (12), implies that the typical extension in the y' direction diverges as $(1-|\eta|)^{-1/4}\to\infty$. Accordingly, the excitation spectrum (10) becomes characterized by the soft mode whose energy ω_- goes to zero. This implies that in this region of ω the low-energy dynamical response of the system of atoms should exhibit the one-dimensional (1D) behavior. In this paper we will not focus on the properties of such a 1D atomic gas.

Note that the spectrum (10) and its limiting form (14) have no lower bound. For the model Hamiltonian whose potential is the axially symmetric TOP [16], this is a pure consequence of the coordinate transformation (3) because this Hamiltonian conserves angular momentum. Accordingly, no instability with respect to a spontaneous growth of angular momentum $L=\hbar(n-m)$ (in physical units) in (14) is expected to occur. In contrast, the Hamiltonian (9) does not conserve angular momentum. This implies that under certain conditions such an instability could be realized.

Another interesting feature of the solution (11) and (12) is the common phase factor $\exp(i\gamma_0 x'y')$. Its magnitude is controlled by the asymmetry of the eigenfunctions in the (x,y) plane (in addition to the squeezing in the z direction). Also, this factor implies a very specific pattern for the velocity $\mathbf{v}=\nabla\,\mathrm{Im}[\ln(\psi_{00l})]$ at the levels with $m=n=0$. Employing (11) and (12) for $m=n=0$ one finds

$$v_r=\gamma_0 r\sin(2\theta),\quad v_\theta=\gamma_0 r\cos(2\theta) \quad (16)$$

for the radial and polar components of $\mathbf{v}$, respectively [in the polar coordinates $x'=r\cos(\theta), y'=r\sin(\theta)$]. This expression exhibits quadrupolar symmetry. As long as particles are condensing into the state with $m=n=l=0$ they will form the current pattern characterized by (16). This pattern can be thought of as two pairs of vortices of opposite vorticity coupled together.

Above we have shown that no nonadiabatic energy exchange occurs between the RMF and the ideal gas in the trap. In Sec. IV we will show that the interaction between particles changes this situation.

III. CONDENSATE CONTAINING LARGE NUMBERS OF PARTICLES IN THE ROTATING FRAME

In this section we will analyze the case of the condensate containing large numbers of particles in the presence of the RMF. The condensate wave function Φ obeys the GGP equation [19]. For the single-particle Hamiltonian (7) in the rotating frame, this equation is

$$i\partial_t\Phi=(H_\omega-\mu)\Phi+u_0|\Phi|^2\Phi, \quad (17)$$

where $u_0>0$ is the interaction constant and μ stands for the chemical potential. Following the approach [9], we will derive approximate hydrodynamical equations for the condensate in the presence of the RMF. We denote

$$\Phi=\sqrt{\rho}e^{i\phi},\quad \int d\mathbf{x}\,\rho=N_c, \quad (18)$$

where ρ, ϕ, and N_c are the density, the phase, and the total number of particles in the condensate, respectively. Substituting (18) into (17), one arrives at the expressions

$$\dot{\rho}-\omega(x''\partial'_y\rho-y'\partial_{x''}\rho)+\nabla(\rho\nabla\phi)=0, \quad (19a)$$

$$\dot{\phi}-\omega(x'\partial'_y\phi-y'\partial_{x''}\phi)=\frac{\Delta(\sqrt{\rho})}{2\sqrt{\rho}}-\frac{1}{2}(\nabla\phi)^2+\mu-U-u_0\rho. \tag{19b}$$

in the rotating frame. The main approximation made in the limit of large N_c is that the term proportional to $\Delta(\sqrt{\rho})$ in Eq. (19b) can be neglected [18,9].

In the limit $\omega\to\infty$, one expects to obtain a solution of (19) that is close to that characterizing the TOP [18,9]. In order to see it, one should separate a rotationally invariant part from the total potential (5). Specifically,

$$U=U_{\rm TOP}+\delta U, U_{\rm TOP}=\frac{\omega_0^2}{2}(x''^2+y'^2)+\frac{\omega_{0z}^2}{2}z^2,$$

$$\delta U=-\frac{\omega_0^2}{2}(x''^2-y'^2)-x''. \tag{20}$$

For the sake of convenience we will omit all primes ($x''\to x, y'\to y$) from the coordinates, implying that we are working in the frame connected with the RMF unless otherwise stated. Note that $U_{\rm TOP}$ is the time-averaged potential (TOP) derived in Ref. [16]. The term δU describes the deviations of the instantaneous potential (5) from $U_{\rm TOP}$. It is not strictly obvious that δU can be treated as a small correction to $U_{\rm TOP}$. However, the exact results obtained above for the single-particle Hamiltonian show that this is true in the limit of large ω at least. Below we will show that if $\omega\gg\omega_0$, corrections caused by δU remain small for large N_c as well.

To zeroth order with respect to δU, one obtains from Eq. (19) the solution

$$\phi^{(0)}=0, \quad \rho^{(0)}=\frac{1}{u_0}(\mu-U_{\rm TOP}), \tag{21}$$

which is valid inside the droplet whose radius is determined by the condition $\rho^{(0)}=0$ [18,9]. We represent $\rho=\rho^{(0)}+\delta\rho$ [9], where $\delta\rho$ is a small correction due to δU. Correspondingly, we ignore the term $\nabla(\delta\rho\nabla\phi)$ in Eq. (19a). Linearizing Eqs. (19) in $\delta\rho,\phi$ [9], one obtains

$$\delta\dot{\rho}-\omega(x\partial_y\delta\rho-y\partial_x\delta\rho)+\nabla(\rho^{(0)}\nabla\phi)=0, \tag{22a}$$

$$\dot{\phi}-\omega(x\partial_y\phi-y\partial_x\phi)=-u_0\delta\rho-\delta U. \tag{22b}$$

Note that these equations are close analogs to those obtained in [9] for a trapping oscillator potential that is axially symmetric. The additional feature of Eqs. (22) is the term δU, which plays the role of an external force. Later we will see that this term, under certain conditions, can resonantly excite the condensate normal modes with the angular momenta $L=1,2$.

A particular solution corresponding to the symmetry of the driving term δU (20) can be taken in the form

$$\delta\rho=\rho'_2(x^2-y^2)+2\rho''_2xy+\rho'_1x+\rho''_1y,$$
$$\phi=\phi'_2(x^2-y^2)+2\phi''_2xy+\phi'_1x+\phi''_1y, \tag{23}$$

where $\rho'_l,\rho''_l,\phi'_l,\phi''_l$, $l=1,2$ are the time-dependent amplitudes of the dipole ($l=1$) and the quadrupole ($l=2$) harmonics. Substitution of (21) and (23) into (22) yields

$$\dot{\phi}_l+il\omega\phi_l+u_0\rho_l=\left(\frac{\omega_0^2}{2}\right)^{l-1},$$
$$\dot{\rho}_l+il\omega\rho_l-l\frac{\omega_0^2}{u_0}\phi_l=0 \tag{24}$$

for the complex amplitudes

$$\rho_l=(\rho'_l+i\rho''_l)e^{il\omega t}, \quad \phi_l=(\phi'_l+i\phi''_l)e^{il\omega t}. \tag{25}$$

Note that the dipole amplitudes $l=1$ describe essentially the center-of-mass motion of the whole atomic cloud in the trap [9].

If the RMF frequency ω does not change in time, one obtains the steady solutions ($\dot{\rho}=\dot{\phi}=0$)

$$\phi'_1=0, \quad \phi''_1=\frac{\omega}{\omega_0^2-\omega^2}, \quad \rho'_1=\frac{\omega_0^2}{u_0(\omega_0^2-\omega^2)}, \quad \rho''_1=0 \tag{26}$$

and

$$\phi'_2=0, \quad \phi''_2=-\frac{\omega_0^2}{4\omega\left(1-\frac{\omega_0^2}{2\omega^2}\right)},$$
$$\rho'_2=\frac{\omega_0^2}{2u_0(2\omega^2-\omega_0^2)}, \quad \rho''_2=0. \tag{27}$$

This implies that in the limit $\omega\gg\omega_0$ the corrections due to the RMF to the zeroth-order solution (21) [9,18] are small.

Note that Eqs. (27) and (23) indicate that the phase factor $\exp(i\gamma_0xy)$, discussed in Sec. II for the ideal gas situation, is not affected much by the interaction as long as $\omega\gg\omega_0$. Indeed, comparing Eqs. (11)–(13) with Eqs. (23) and (27), one finds that the parameter γ_0 in Eqs. (12), (13), and (16) must be replaced by $2\phi''_2=\gamma_0+0((\omega_0/\omega)^3)$. When $\omega\to\omega_0$, the solutions (26) and (27) based on the condition $\rho^{(0)}\gg\delta\rho$ become no longer valid.

We now consider the case when ω depends on time. For concreteness, we assume that the frequency ω of the RMF is modulated as

$$\omega=\overline{\omega}+\lambda\sin(\omega' t), \tag{28}$$

where $|\overline{\omega}|\gg|\lambda|$ and ω' are constants. Accordingly, one finds that the angle between the RMF and the x axis in the laboratory frame is

$$\theta(t)=\overline{\omega t}+\frac{\lambda}{\omega'}[1-\cos(\omega' t)]. \tag{29}$$

Employing the ansatz

$$\phi_l=e^{-il\theta(t)}\tilde{\phi}_l, \quad \rho_l=e^{-il\theta(t)}\tilde{\rho}_l, \tag{30}$$

one obtains from Eqs. (24)

$$\tilde{\phi}_l = \frac{u_0}{l\omega_0^2}\dot{\tilde{\rho}}_l, \tag{31a}$$

$$\ddot{\tilde{\rho}}_l + l\omega_0^2\tilde{\rho}_l = \frac{\omega_0^{2l}}{u_0}e^{il\theta(t)}. \tag{31b}$$

These equations indicate that the resonance condition on ω' in (28) is different for the dipole ($l=1$) and quadrupole ($l=2$) harmonics. Indeed, given (29) and expanding the right-hand side of Eq. (31b) in the small quantity $|\lambda/\omega'| \ll 1$, we get

$$e^{il\theta(t)} \approx e^{il\overline{\omega}t}\left(1 + \frac{il\lambda}{\omega'}(1-\cos\omega' t)\right). \tag{32}$$

Then one obtains that the resonance with the dipole harmonic occurs when the modulating frequency ω' obeys the condition

$$\omega' = \omega_1' = \overline{\omega} \pm \omega_0. \tag{33}$$

Equation (31) yield the resonance condition for the quadrupolar harmonic

$$\omega' = \omega_2' = 2\overline{\omega} \pm \sqrt{2}\omega_0, \tag{34}$$

where $\sqrt{2}\omega_0$ stands for the frequency of the lowest quadrupolar harmonic of the trapped condensate with large N_c [9]. Note that $\omega_2' - \omega_1' \approx \overline{\omega} \gg \omega_0$.

In the following, we will show that the preceding analysis based on the GGP equation does not take into account quantum processes of the creation of pairs out of the condensate by the RMF. These lead to forced evaporation of the condensate even for zero temperature and steady rotations of the magnetic field. As a consequence, the hydrodynamical equations (22) will acquire a dissipative contribution.

IV. CENTRIFUGAL INSTABILITY IN THE MANY-BODY APPROACH

In our previous analysis, we neglected quantum fluctuations of the condensate. These fluctuations in the conventional stable condensate can be thought of as the virtual creation and absorption of pairs. In this regard we note that the spectrum (10) has no lower bound, so that the condensate could be unstable with respect to the real creation of pairs even though the RMF is steady. Correspondingly, the GGP equation (17) can acquire a dissipative part.

Consider first the case $\omega \to \infty$. The term proportional to ω in Eq. (14) is a direct consequence of the Galilean transformation into the rotating frame. Indeed, the limit $\omega \to \infty$ in Eqs. (10)–(13) ensures that the term $(n-m)$ is the projection of the angular momentum L on the z axis, so that the ω-dependent part in (14) is exactly the Coriolis contribution $-\omega L$. This implies that no instability should develop because the absence of the lower bound for the spectrum is purely a frame of reference effect. Nevertheless, the condensate can be considered as being potentially able to gain high values of L. In this regard we can employ the rotating frame reasoning [19] (see also [11], Chap. 6) for the vortex creation in the rotating vessel containing a superfluid. In the frame connected to the vessel rotating with the frequency ω around its axis, the vortex energy is $E_v - \omega L_v$, where E_v and L_v stand for the vortex energy in the laboratory frame and the vortex angular momentum, respectively. The vortex can be created spontaneously if the Coriolis energy exceeds E_v. However, this argument does not indicate what the probability for developing this centrifugal instability is. In fact, in the case of the perfectly symmetric vessel this probability is essentially zero. To make the vortex creation real, the vessel must have some irregularities on the walls breaking the rotational symmetry so that the angular momentum of the vessel could be transferred to the vortex (or vortices).

Returning to our case, we can see that in the case $\omega \to \infty$ the eigenfunctions (11) and (12) of the trap [16] are approaching those of the effective time-averaged Hamiltonian (the TOP [16]), which is axially symmetric. Therefore, no centrifugal instability of the condensate is expected to occur in this limit. In other words, no energy exchange between the RMF and the condensate happens in the limit $\omega = \infty$.

For finite ω, the functions (11) and (12) are not eigenfunctions of the operator L. Consequently, the difference $n-m$ can no longer be interpreted as the eigenvalue of L. Accordingly, the effective vessel can be thought of as having a symmetry-breaking deformation, which in turn implies that the energy and the angular momentum can now be given up to the pairs leaving the condensate into the highly excited states whose energies are $\hbar\omega \gg \hbar\omega_0$ (in physical units). In this regard one should distinguish two cases: (1) $\hbar\omega \gg \mu$ and (2) $\hbar\omega \leqslant \mu$. In case (1) the pair escapes into states lying far from those effectively involved in the formation of the interacting condensate. Accordingly, the pair escape process can be treated as an incoherent step in the condensate evaporation. In contrast, in case (2) the escape states with the energies $\approx \hbar\omega$ are to be renormalized strongly because of the presence of the condensate. This implies that the multipair processes become significant. Correspondingly, the centrifugal instability should be interpreted as a coherent process of vortex formation. In this paper we will not analyze this case.

The process of the escape of pairs represents the nonresonant quantum evaporation of the condensate induced by the RMF. We emphasize the crucial role of the interatomic interaction for realization of this centrifugal evaporation. The rotating observer sees this process as follows: two atoms in the condensate interact with each other. As a consequence, they jump to a new pair of single-particle states characterized by large quantum numbers, so that their total energy is conserved (the large $\varepsilon_{mnl}^{(\mathrm{lab})}$ is compensated by the Coriolis term). Correspondingly, the rotating observer interprets this event as a nearly elastic escape of the pair from the condensate. Note that if the eigenfunctions (11) were eigenfunctions of angular momentum, there would be a selection rule requiring that the angular momentum of the interacting pair not change in the transition. Correspondingly, referring to Eq. (14), one sees that no instability would occur. In fact, this is not the case for finite ω and instability could occur.

To describe the centrifugal instability effect, we proceed to derive a damping term in the GGP equation for the condensate wave function Φ. The many-body Hamiltonian in the rotating frame is

$$H=\int d\mathbf{x}\left[\Psi^{\dagger}(H_{\omega}-\mu)\Psi+\frac{u_0}{2}\Psi^{\dagger}\Psi^{\dagger}\Psi\Psi\right], \quad (35)$$

where primes are omitted from the coordinates and the Bose operators $\Psi^{\dagger},\Psi$ obey the usual Bose commutation rule. The Heisenberg equation is

$$i\partial_t\Psi=(H_{\omega}-\mu)\Psi+u_0\Psi^{\dagger}\Psi\Psi. \quad (36)$$

Taking into account the explicit form (7) for H_{ω}, one finds from (36) the current conservation condition

$$\partial_t(\Psi^{\dagger}\Psi)+\nabla\cdot\mathbf{J}=0, \quad (37)$$

where the current operator $\mathbf{J}$ in the rotating frame is defined as

$$\mathbf{J}=\frac{1}{2i}[\Psi^{\dagger}(\nabla-\mathbf{A})\Psi],$$

$$A_x=-i\omega y, \quad A_y=i\omega x, \quad A_z=0. \quad (38)$$

As usual, the condensate wave function is defined by $\Phi=\langle\Psi\rangle$. In what follows we assume that the noncondensate part $\Psi'=\Psi-\Phi$ obeys the standard Bose commutation relation, which, strictly speaking, implies that the excited eigenstates are not orthogonal to the condensate. However, it can be shown that for our purpose of deriving the dissipative part of the GGP this approximation gives the same result as if we employed the modified commutation relation [20]. From Eq. (36), one finds [21,20,22]

$$\partial_t\Phi=(H_{\omega}-\mu)\Phi+u_0|\Phi|^2\Phi + u_0[\Phi^*\langle\Psi'\Psi'\rangle+2\Phi\langle\Psi'^{\dagger}\Psi'\rangle+\langle\Psi'^{\dagger}\Psi'\Psi'\rangle] \quad (39)$$

and

$$i\partial_t\Psi'=(H_{\omega}-\mu)\Psi'+u_0[\Phi^*(\Psi'\Psi'-\langle\Psi'\Psi'\rangle) + 2\Phi(\Psi'^{\dagger}\Psi'-\langle\Psi'^{\dagger}\Psi'\rangle)+2\Phi^*\Phi\Psi' + \Phi^2\Psi'^{\dagger}+\Psi'^{\dagger}\Psi'\Psi'-\langle\Psi'^{\dagger}\Psi'\Psi'\rangle]. \quad (40)$$

The condensate wave function Φ is normally viewed as an external classical field in Eq. (40) for the noncondensate part. Employing the Keldysh technique [23], the system (39) and (40) can be expressed in terms of the joint dynamics of Φ and the population numbers of the excitations. In general, this procedure is very complicated [8] (see also the generalized density-functional approach [24,22]). However, under certain conditions it becomes possible to eliminate the averages from Eq. (39). Specifically, we will make several assumptions and approximations: (a) the population numbers of the excited states are zero; (b) the pairs escaping from the condensate due to the centrifugal effect escape from the trap as well; (c) the terms leading to powers higher than the third in Φ and Φ^* in the effective GGP equation are omitted; (d) in Eq. (40) only the terms linear in Ψ' and $\Psi'^{\dagger}$ are retained in accordance with the Bogolubov approximation [19,20].

Assumption (a) excludes the normal component from the analysis. Assumption (b) ensures that no normal component is building up in the highly excited levels due to the centrifugal escape of the pairs. Note that (b) is reasonable for high ω and in the presence of the radio-frequency scalpel, which provides the evaporative cooling (see, e.g., in [25,5]). Given (a) and (b), we avoid the necessity to analyze the dynamics of the normal component. Finally, from Eqs. (39) and (40) under (a)–(d) we obtain the GGP equation with the dissipation term included (see Appendix B)

$$i\partial_t\Phi=(H_{\omega}-\mu)\Phi+u_0|\Phi|^2\Phi-iu_0^2\Phi^*\int dt' \times\int d\mathbf{x}'G^{(r)2}(\mathbf{x}t,\mathbf{x}'t')\Phi^2(\mathbf{x}'t'), \quad (41)$$

where $G^{(r)}(\mathbf{x}t,\mathbf{x}'t')$ is defined in (B7). Note that if the total number of atoms in the trap were conserved, it would not be possible to obtain the dissipation term in (41) in closed form [6]. Below it will be shown explicitly that the last term in Eq. (41) would have been zero if either the single-particle Hamiltonian H_{ω} conserved angular momentum or if the single-particle excitation spectrum were positively defined.

Multiplying (41) by Φ and adding the complex conjugate of the resulting expression, one obtains the generalized current conservation condition

$$\dot{\rho}+\mathrm{div}\mathbf{J}=-S,$$

$$S=u_0^2\int dt'\int d\mathbf{x}'G^{(r)2}(\mathbf{x}t,\mathbf{x}'t')\rho(\mathbf{x}t)\rho(\mathbf{x}'t') \times\exp\{-2i[\phi(\mathbf{x}t)-\phi(\mathbf{x}'t')]\}+\mathrm{c.c.}, \quad (42)$$

where the representation (18) is employed. This equation corresponds to Eq. (22a) modified to include the dissipation caused by the centrifugal evaporation.

Integration of Eq. (42) over the whole space yields

$$\dot{N}_c=-2u_0^2\int dt'd\mathbf{x}\,d\mathbf{x}'\mathrm{Re}[G^{(r)2}(\mathbf{x}t,\mathbf{x}'t')]\rho(\mathbf{x}t)\rho(\mathbf{x}'t') \times\exp\{-2i[\phi(\mathbf{x}t)-\phi(\mathbf{x}'t')]\}, \quad (43)$$

where we have used the second relation in Eq. (18). Note, that in this equation the integral depends on both the density ρ as well as the phase ϕ of the condensate. Accordingly, the coherence of the condensate could be tested by analyzing the rate of the quantum vaporization induced by the RMF. Elsewhere, we will consider this possibility in greater detail. Presently, let us calculate the quasistatic decay rate assuming that N_c is not large so that one can employ the ideal gas ansatz $\Phi=\sqrt{N_c}\psi_{000}$. Note, however, that the applicability of Eqs. (41)–(43) is not limited by the requirement of small N_c ($a\sqrt{\omega_0}N_c\leqslant 1$ [18], where a stands for the scattering length). Later we will derive a condition [see (B5)] allowing one to employ the variational approach [18] for calculating Φ, with $G^{(r)}$ taken unrenormalized, in Eq. (43).

Making use of Eqs. (11) and (12), we find

$$\dot{N}_c(t) = -2u_0^2 \int dt' K(t-t') N_c(t) N_c(t'), \tag{44}$$

where

$$K(t-t') = \widetilde{\theta}(t-t') \sum_{1,2} |M_{12}|^2 \mathrm{Re}\, e^{-i(\varepsilon_1+\varepsilon_2)(t-t')},$$

$$M_{12} = \int d\mathbf{x}\, \psi_1 \psi_2 \psi_{000}^{*2}, \tag{45}$$

and the summation is performed over the final states of the escaping pair.

As will be seen below, in the limit $\omega \to \infty$ the escape rate is much smaller than the typical time scale in the trap corresponding to ω_0. Consequently, one can employ the quasistatic approximation that the time dependence of N_c is slow. We set $N_c(t') = N_c(t)$ in (44) and after the time integration rewrite it as

$$\dot{N}_c = -\chi N_c^2, \tag{46}$$

where

$$\chi = 2\pi u_0^2 \sum_{[1],[2]} |M_{12}|^2 \delta(E_{12}), \tag{47}$$

and the quantity $\tau = 1/\chi N_c$ can be considered as an effective lifetime of the condensate. In Eq. (47), the notation $E_{12} = \varepsilon_1 + \varepsilon_2$ for the escaping pair energy is introduced and $\delta(\xi)$ stands for the delta function. This expression accounts quantitatively for the centrifugal effect discussed above. One can see that the conditions $M_{12} \neq 0$ and $E_{12} = 0$, where explicitly

$$\begin{aligned} E_{12} &= \omega_0(m_1+m_2+n_1+n_2) + \omega_{0z}(l_1+l_2) \\ &\quad - \omega(n_1+n_2-m_1-m_2) \\ &= 0, \end{aligned} \tag{48}$$

can be satisfied simultaneously because the eigenfunctions (11) and (12) utilized in (45) and (47) are not the eigenfunctions of the angular momentum operator. In what follows, we will show that in the limit of large ω, the dominant contribution in (47) comes from the states with the quantum numbers

$$m_1 \approx m_2 \approx l_1 \approx l_2 \approx n_1 \approx n_2 \approx \frac{\omega}{\omega_0} \gg 1 \tag{49}$$

corresponding to a pair leaving the condensate into the states characterized by large quantum displacements. Returning to the laboratory frame, this simply means that two atoms absorb the energy $2\hbar\omega$ (in physical units) from the RMF so that this energy is approximately equally distributed between them. As a result, the pair is transferred to highly excited states. The radio-frequency scalpel [5,25] is assumed to eventually remove this pair ensuring condition (a) of zero population of the excited states.

In order to calculate M_{12} explicitly, we employ the representation (compare with [20], Chap. 15)

$$\begin{aligned} &\psi_{m_1 n_1 l_1}(\mathbf{x}_1) \psi_{m_2 n_2 l_2}(\mathbf{x}_2) \\ &\quad = \sum_{p,k,q} g_p^{m_1 m_2} g_k^{n_1 n_2} g_q^{l_1 l_2} \psi_{pkq}(\mathbf{R}) \psi_{p'k'q'}(\mathbf{r}), \end{aligned} \tag{50}$$

where the notations $p' = m-p$, $k' = n-k$, $q' = l-q$, and

$$\begin{aligned} g_p^{ab} &= \sum_{p'} (-1)^{p+p'} \\ &\quad \times \frac{\sqrt{a!b!p!(a+b-p)!}}{2^{(a+b)/2} p'!(a-p')!(p-p')!(b-p+p')!}, \end{aligned}$$

$$\mathbf{R} = \frac{\mathbf{x}_1+\mathbf{x}_2}{\sqrt{2}}, \quad \mathbf{r} = \frac{\mathbf{x}_1-\mathbf{x}_2}{\sqrt{2}},$$

$$m = m_1+m_2, \quad n = n_1+n_2, \quad l = l_1+l_2 \tag{51}$$

are employed. The summations in Eqs. (50) and (51) run over all integer non-negative numbers obeying the condition that all the numbers under the signs of the factorial are non-negative as well. The relations (50) and (51) were derived from the explicit representation (11) and (12) for the eigenfunctions. Employing Eqs. (50) and (51) in Eqs. (45) and (47), we obtain for Eq. (47)

$$\begin{aligned} \chi &= \frac{\pi u_0^2}{4} \sum_{m,n,l} |\psi_{mnl}(0)|^2 |\psi_{000}(0)|^2 \\ &\quad \times \delta(\omega_0(m+n) + \omega_{0z} l - \omega(n-m)). \end{aligned} \tag{52}$$

Note that χ is exactly zero for the case $\alpha = 1$ (or $\omega \to \infty$) in Eqs. (11)–(13). As mentioned above, no escape of the pairs occurs in the case when the RMF is so rapidly rotating that the effective trapping (TOP) potential becomes axially symmetric. In the limit of large but still finite ω one finds from Eq. (13)

$$\alpha^2 - \alpha^{-2} = -\left(\frac{\omega_0}{\omega}\right)^3 + o\left(\frac{\omega_0}{\omega}\right)^5, \tag{53}$$

which implies that the first term only should be kept in (53) and that the exponent e^{Ξ} in Eq. (11) can be expanded in terms of the smallness of ω_0/ω. This expansion represents the eigenfunction (11) in terms of the harmonics of the angular momentum operator. Thus one finds that each eigenfunction can be effectively characterized by the following terms: (i) the harmonic of the angular momentum operator with the angular momentum $(n-m)$, (ii) two functions with the momenta $(n-m)\pm 2$ whose weight is proportional to $(\omega_0/\omega)^3 \ll 1$, and (iii) the other angular harmonics with the relative weight containing powers of ω_0/ω higher than the third. This implies that in Eq. (47) it is enough to consider the contribution due to the lowest term. Physically, this term corresponds to an absorption of the energy $2\hbar\omega$ and the angular momentum 2 by a pair of atoms escaping from the condensate. Finally, one finds

$$\chi=\frac{1}{4}a^2\omega_0^2\omega_{0z}\left(\frac{\omega_0}{\omega}\right)^6\sum_{m,l}\frac{(2l)!(m+1)(m+2)}{2^{2l}(l!)^2}$$
$$\times\delta(2\omega_0 m+2\omega_{0z}l-2\omega). \quad (54)$$

A simple analysis shows that most of the contribution to the sum (54) comes from the region of high m,l [see (49)]. Accordingly, we replace the summation in (54) by integration. Finally, in the chosen limit and chosen units (6) we find

$$\chi=\epsilon a^2\omega_0^2\left(\frac{\omega_0}{\omega}\right)^{7/2}, \quad \epsilon=\frac{2^{7/4}}{15\sqrt{\pi}}\approx 0.13. \quad (55)$$

This expression indicates that the centrifugal escape rate is extremely sensitive to the RMF frequency ω. For the parameters employed experimentally in Ref. [1], the estimate of (55) gives a very small number (the corresponding lifetime is about $\tau\approx 10^6$ s for $N_c=1000$), implying that the centrifugal vaporization can be effectively ignored as a cause for the condensate escape from the trap. However, with decrease of the ratio ω/ω_0, the vaporization rate increases strongly. In the case $\omega\to\sqrt{2}\omega_0$ from above, the approximation (53) we employed is no longer valid. Accordingly, the exact expressions (11)–(13) for the eigenfunctions should be utilized in (52). This means that the escaping pairs acquire higher (even) angular momenta. As a result, the lifetime of the condensate can become very short.

Equations (39) and (40) can be analyzed for the case of a nonsteady RMF. Especially interesting appears to be the case when the frequency modulated RMF excites resonantly the quadrupolar harmonic of the condensate [see the condition (34)]. Generally, it is natural to expect that this resonance would result in the increase of the vaporization rate as a function of the modulating frequency ω'. In the future, we will consider this case in greater detail.

In this paper we do not consider other collisional mechanisms of losses (see, e.g., [26]) from the magnetic traps because these mechanisms are not expected to be very sensitive to the RMF frequency ω. Correspondingly, one can make the centrifugal decay dominant by means of varying ω.

Note, however, that the single-spin (noncollisional) reversals due to the energy transfer from the RMF to the atomic spins turn out to be very sensitive to ω. As mentioned in [1], these processes should be suppressed when ω is smaller than the Larmor frequency $\omega_L=|\mu_B B_b|/\hbar$. Such a spin flip event is followed by the escape of the atom from the trap. An analysis of this process can be carried out in the rotating frame as well. The full Hamiltonian including the spin degrees of freedom becomes time independent for $\omega=\text{const}$ and acquires the additional Coriolis term $-\omega S_z$, where S_z stands for the z component of the spin matrix. Accordingly, the runaway states should also be classified in the rotating frame. It can be shown that a dominant channel for the spin flip and the following escape corresponds to a transition, with the orbital angular momentum of the escaping particle changed by one. The energy carried away by the atom is approximately $\Delta=\hbar(\omega_L-\omega)$ for $\omega\ll\omega_L$. In this regard we comment that the probability of this escape process is significantly suppressed because of the smallness of the matrix element taken between states localized in the trap and the runaway states. Simple estimates show that this matrix element has a structure $\sim\exp(-\Delta/\hbar\omega_0)$ for $\Delta/\hbar\omega_0\gg 1$. Taking into account that $\omega_L>10^7$ Hz, $\omega_0\approx 10^2-10^3$ Hz, and $\omega\approx 100\omega_0$ [1,16], one can see that the nonadiabatic spin flips can be ignored, if compared with the centrifugal losses.

V. CONCLUSION

The atomic trap [16,1] bears features absent in the static traps [2–4]. These features can be accounted for in the frame rotating together with the RMF. For large frequencies of rotation of the RMF, the exact eigenenergies and eigenfunctions of the trap [16,1] approach those characterizing the time-averaged potential TOP [16] having axial symmetry. For frequencies close to the threshold below which the trapping is impossible, the eigenstates lose their axial symmetry and become elongated in the direction perpendicular to the RMF (in its frame of reference). Very close to the threshold a gas of trapped atoms acquires properties of an essentially 1D system.

Due to the asymmetry introduced by the RMF, the atom-atom interaction results in the induced evaporation of the Bose-Einstein condensate. The time scale for this evaporation is very sensitive to the RMF frequency of rotation. For high frequencies, the lifetime of the condensate increases as a large power of ω. Close to the trapping threshold the lifetime shortens considerably, implying that the 1D gas formed in the trap [1,16] in this situation is a strongly interacting system.

The frequency-modulated RMF can be utilized as a driving force selectively exciting the condensate normal modes. In the limit of large numbers of atoms, when Stringari's hydrodynamical approximation is valid, two modes can be excited by the RMF whose frequency of rotation is appropriately modulated. The first is a dipole mode accounting for the center-of-mass motion. The second mode that can be excited by the RMF is the lowest quadrupolar harmonic. The resonance conditions for the RMF modulation period depend on the averaged RMF frequency, in addition to the eigenfrequencies of the harmonics. The effect of the quantum evaporation induced by the RMF opens up a channel for dissipation of the condensate normal modes.

ACKNOWLEDGMENTS

This research was supported by grants from The City University of New York PSC-CUNY Research Award Program.

APPENDIX A: SOLVING THE EIGENPROBLEM IN THE ROTATING FRAME

The Hamiltonian (9) for the function $\psi'(x',y',z)=\exp(i\omega x''y')\psi$ can be written as

$$H'_\omega=H_{xy}+H_z, \quad H_z=-\frac{1}{2}\partial_z^2+\frac{\omega_{0z}^2}{2}z^2,$$

$$H_{xy}=-\frac{1}{2}(\partial_x^2+\partial_y^2)+\frac{\omega_{0y}^2-\omega^2}{2}y^2+\frac{3}{2}\omega^2x^2+2i\omega x\partial_y, \quad (A1)$$

where the unimportant constant $1/2\,\omega^2$ and the primes from the coordinates are omitted. The eigenfunctions $\varphi_l(z)$ of H_z are the well-known oscillator states. These are represented in (11) and (12) by the generating function of the Hermite polynomials [see the auxiliary variable t_3 in (11) and (12)] so that the $\psi'(x,y,z)=\varphi_l(z)\psi'_{mn}(x,y)$. Performing the Fourier transform

$$\widetilde{\psi}(x,p)=\int d\widetilde{y}\,e^{-ip\widetilde{y}}\psi'(x,\widetilde{y}),\quad \widetilde{y}=\sqrt{|\omega^2-\omega_{0y}^2|}y, \tag{A2}$$

one finds for (A1)

$$H_{xy}=-\frac{1}{2}\partial_x^2+\frac{s}{2}\partial_p^2+\frac{\omega^2-\omega_{0y}^2}{2}p^2+\frac{3}{2}\omega^2x^2 -2\omega\sqrt{|\omega^2-\omega_{0y}^2|}xp, \tag{A3}$$

where $s=\mathrm{sgn}(\omega^2-\omega_{0y}^2)$.

In the case $|\omega|<\omega_{0y}$ the Hamiltonian (A3) can be diagonalized by implementation of a real rotation in the (x,p) plane. However, no discrete states exist in this case because the effective potential of (A3) turns out to have a saddlelike shape, with the kinetic part being positively defined. In the opposite limit ($s=1$) the discrete states do exist. The diagonalization can be achieved by means of the Lorentz transformation

$$x=\cosh(\vartheta)\xi+\sinh(\vartheta)p',$$
$$p=\sinh(\vartheta)\xi+\cosh(\vartheta)p', \tag{A4}$$

leaving the kinetic part $-\frac{1}{2}(\partial_x^2-\partial_p^2)$ invariant. In terms of the new variables (ξ,p'), (A3) acquires the form

$$H_{xy}=-\frac{1}{2}\partial_\xi^2+\frac{\omega_+^2}{2}\xi^2-\left[-\frac{1}{2}\partial_{p'}^2+\frac{\omega_-^2}{2}p'^2\right], \tag{A5}$$

where $\omega_\pm$ are given in Eq. (10) and the angle ϑ satisfies the equation

$$\tanh(2\vartheta)=\frac{4\nu\sqrt{1-\eta^2}}{4-\eta^2}, \tag{A6}$$

with ν,η defined in Eq. (13). The resulting spectrum of the total Hamiltonian is given by Eq. (10). The eigenfunctions of (A5), expressed in terms of the ξ,p' variables, can be converted into the x,p coordinates by means of the relations (A4). Finally, performing the inverse of the Fourier as well as the scaling transforms (A2), one finds the normalized eigenfunctions (11)–(13).

APPENDIX B: DERIVATION OF THE GGP EQUATION WITH THE DISSIPATION DUE TO THE RMF

Under assumptions (a)–(d), Eqs. (39) and (40) simplify considerably. We need to find the lowest-order term that contributes to the imaginary part of Eq. (39). In Eq. (39), the second term in the square brackets does not contribute to the imaginary part, so we omit it. The last term in the square brackets of Eq. (39) produces the imaginary part. However, it can be shown that it is proportional to the population numbers of the excited states. Consequently, we omit this term also and rewrite (39) as

$$i\partial_t\Phi=(H_\omega-\mu)\Phi+u_0|\Phi|^2\Phi+u_0\Phi^*G(\mathbf{x}t,\mathbf{x}t), \tag{B1}$$

where the equal-time anomalous Green's function [21,20] is defined as $G(\mathbf{x}t,\mathbf{x}'t)=\langle\Psi'(\mathbf{x}t)\Psi'(\mathbf{x}'t)\rangle$ (the overall factor i is omitted here and below in the Green's-functions definitions [21,20]). Note that because the interaction potential in(35) is chosen in the $\delta(\mathbf{x})$ form, Eq. (B1) contains G with $\mathbf{x}=\mathbf{x}'$. The equation for G can be obtained from Eq. (40). In order to accomplish this, we will employ the Bogolubov approximation (d). Accordingly, multiplying (40) by Ψ' and taking the average, one finds

$$\begin{aligned} i\partial_t G(\mathbf{x}t,\mathbf{x}'t) &= (H_\omega-\mu)_{\mathbf{x}}G(\mathbf{x}t,\mathbf{x}'t)+(H_\omega-\mu)_{\mathbf{x}'}G(\mathbf{x}t,\mathbf{x}'t) \\ &+u_0\{2[\Phi^*(\mathbf{x}t)\Phi(\mathbf{x}t)+\Phi^*(\mathbf{x}'t)\Phi(\mathbf{x}'t)]G(\mathbf{x}t,\mathbf{x}'t) \\ &+[\Phi^2(\mathbf{x}'t)+\Phi^2(\mathbf{x}t)][\rho(\mathbf{x}t,\mathbf{x}'t)+\tfrac{1}{2}\delta(\mathbf{x}-\mathbf{x}')]\}, \end{aligned} \tag{B2}$$

where

$$\rho(\mathbf{x}t,\mathbf{x}'t)=\langle\Psi'^{\dagger}(\mathbf{x}t)\Psi'(\mathbf{x}'t)\rangle \tag{B3}$$

stands for the density matrix of the excitations. In Eq. (B2) the notation $(\,)_{\mathbf{x}}$ means that the single-particle Hamiltonian H_ω acts on the coordinate $\mathbf{x}$. These equations should be supplemented by ones for the normal Green's functions. However, conditions (a) and (b) of zero population of the excited states imply that one can set $\rho=0$ in Eq. (B2). Furthermore, as long as one is interested only in deriving the imaginary contribution to (B1) to lowest order with respect to u_0, significant simplification can be achieved. Indeed, for $\omega\gg\omega_0$, only the high-energy part of the spectrum of the normal excitations contributes to the imaginary part of G in Eq. (B1). Correspondingly, one can neglect the effect of the condensate on this part of the spectrum [see case (1) discussed above]. The condition when this assumption is valid can be formulated in terms of the smallness of the first term in the curly brackets of Eq. (B2) if compared with ωG. In other words,

$$\hbar|\omega|\gg u_0|\Phi|^2 \tag{B4}$$

in physical units. Employing the variational approach [18], one can estimate $|\Phi|^2$ and obtain from (B4)

$$a\sqrt{\omega_0}N_c\leqslant 0.034\left(\frac{\omega}{\omega_0}\right)^{5/2}, \tag{B5}$$

where we have employed the representation $u_0=4\pi a$ for the interaction constant u_0 in terms of the scattering length a in the units (6). Actually, for the parameters of the trap [1] the estimate for N_c gives $N_c\leqslant 10^6$. If this condition holds, the

anomalous Green's function G can be found by iteration with respect to u_0, with the zeroth-order approximation being zero. Assuming that (B4) [or (B5)] is valid, we finally find

$$G(\mathbf{x}t,\mathbf{x}t) = -iu_0 \int dt' \int d\mathbf{x}' G^{(r)2}(\mathbf{x}t,\mathbf{x}'t')\Phi^2(\mathbf{x}'t'), \tag{B6}$$

where the retarded Green's function [20]

$$G^{(r)}(\mathbf{x}t,\mathbf{x}'t') = \tilde{\theta}(t-t')\sum_1 e^{-i\varepsilon_1(t-t')}\psi_1(\mathbf{x})\psi_1^*(\mathbf{x}') \tag{B7}$$

is expressed explicitly in terms of the single-particle eigenfunctions (11) and (12) and the eigenvalues (10), with the summation performed over all the single-particle quantum numbers indicated as 1. In (B7), $\tilde{\theta}(\tau)$ denotes the step function. Finally, substitution of Eqs. (B6) and (B7) into Eq. (B1) yields Eq. (41).

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