Unified and standardized procedure to solve various nonlinear Jaynes-Cummings models

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In this article we present a simple and unified procedure to solve various nonlinear two-level Jaynes-Cummings models. By establishing their similarity to the model describing a spin-$\frac{1}{2}$ particle in a magnetic field, we obtain the standardized forms, appropriate uniformly for all Jaynes-Cummings models, of eigenvalues, eigenstates, evolution, and atomic inversion operators. In this way, we show that the analytical solution of any single-mode and two-mode nonlinear Jaynes-Cummings model can easily be obtained. We also apply this procedure to a three-mode Jaynes-Cummings model. [S1050-2947(97)07806-2]

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I. INTRODUCTION

Exact solvability of full quantum mechanical models plays a critically important role in the field of light-atom interactions for the study of purely quantum features, such as collapse and revival of Rabi oscillations, because it permits access to regimes that are incompatible with perturbation theory and that embrace most long-time low-loss near-resonance phenomena and include the domain of few-photon strong fields, in which atomic response can be large even if the field is extremely weak by conventional measures [1]. Over the last two decades, there has been intensive study [2,3] on the solvable Jaynes-Cummings (JC) model and its various extensions, such as intensity-dependent coupling constants [4], two-photon or multiphoton transitions [5], and two- [2,6–12] or three- [1,13,14] cavity modes for three-level atoms. These three-level models can be transformed into effective two-level models either by exact transformation [11] or approximating methods in large detuning cases [1,6,7,10,13]. After so many years, the JC model together with its variants are still under intense investigation and new results and new variants are still being reported. In addition, these models have found their new applications in laser trapping and cooling of atoms [15] and quantum-nondemolition measurements [16]. In view of their importance, it seems worthwhile to seek a simple and unified solving procedure and put the results into standardized forms so that much labor can be saved in solving new variants.

We shall present such a procedure in this paper. It is shown that all two-level Jaynes-Cummings models can be put into a form closely resembling the model describing a spin-$\frac{1}{2}$ particle in a magnetic field. In this way, the eigenvalues can usually be quickly obtained and the analytical expressions of eigenvalues, eigenstates, evolution, and atomic inversion operators can be put into standardized forms. This procedure also reveals that any single-mode and two-mode nonlinear model is analytically solvable. In Sec. II, we illustrate the essential points of this procedure by considering the JC model, which is a linear model, while, at the same time, we will derive the standardized expressions for eigenvalues, eigenstates, evolution, and atomic inversion operators that are suitable not only for this linear model but also for its nonlinear singlemode and multimode extensions. In Sec. III, we first show that any single-mode nonlinear JC model can be solved analytically and the results can be put into closed forms. We then explicitly express the results for a most general form of the single-mode nonlinear JC model. In Sec. IV, we show that this procedure is still a powerful method in the case of two- and three-cavity modes. In fact, we show that two-mode JC models can also be solved analytically and the results can be put into closed forms. Section V gives concluding remarks.

II. STANDARDIZED FORMS

To illustrate the main points of the solving procedure, we first consider the simplest model, that is, the JC model, although solving it directly is also very simple. Aside from an illustrative purpose, we also intend to generalize these results to obtain standardized expressions in this section that are also suitable for nonlinear models. The JC model reads

$$H=\tfrac{1}{2}\Delta\sigma_z+\omega\left(a^\dagger a+\tfrac{1}{2}\sigma_z\right)+H_{\rm int}, \tag{2.1a}$$

$$H_{\rm int}=g(a\sigma_{+-}+a^\dagger\sigma_{-+}), \tag{2.1b}$$

where Δ denotes the detuning, a and $a^\dagger$ are the creation and annihilation operators of the photon, subscripts $+$ and $-$ denote atomic levels $|+\rangle$ and $|-\rangle$, the σ are the usual atomic transition operators satisfying $\sigma_{jk}\sigma_{mn}=\sigma_{jn}\delta_{mk}$, and $\sigma_{--}+\sigma_{++}=1$, $\sigma_z=\sigma_{++}-\sigma_{--}$ and $\sigma_{++}=(1+\sigma_z)/2$, $\sigma_{++}=(1-\sigma_z)/2$.

We now establish the similarity between the JC model and the one describing a spin-$\frac{1}{2}$ particle in a magnetic field. Noting that

$$(a\sigma_{+-}+a^\dagger\sigma_{-+})^2=a^\dagger a+\tfrac{1}{2}(1+\sigma_z)=a^\dagger a+\sigma_{++},$$

which obviously commutes with $(a\sigma_{+-}+a^\dagger\sigma_{-+})$, we can rewrite the interaction Hamiltonian as

$$H_{\rm int}=\tfrac{1}{2}\Omega\sigma_x, \tag{2.2}$$

where $\Omega=g\sqrt{(2H_{\rm int}/g)^2}$ and $\sigma_x=2H_{\rm int}/\Omega$, that is,

$$\Omega=2g\sqrt{a^\dagger a+\sigma_{++}}=2g\sqrt{a^\dagger a+\tfrac{1}{2}(1+\sigma_z)}, \tag{2.3}$$

$$\sigma_x=\frac{a\sigma_{+-}+a^\dagger\sigma_{-+}}{\sqrt{a^\dagger a+\sigma_{++}}}. \tag{2.4}$$

It is easy to check that $\sigma_x,\sigma_y=i\sigma_x\sigma_z=i(-a\sigma_{-+}+a^\dagger\sigma_{+-})/\sqrt{a^\dagger a+\sigma_{++}}$ and $\sigma_z=\sigma_{++}-\sigma_{--}$ satisfy the well known Pauli operators' commutation and anticommutation relations $\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}$ and $[\sigma_k,\sigma_l]=2i\epsilon_{klm}\sigma_m$. Note that the operator Ω commutes with the three Pauli operators. Substituting Eq. (2.2) into Eq. (2.1), we arrive at

$$H=\omega\left(a^\dagger a+\tfrac{1}{2}\sigma_z\right)+\tfrac{1}{2}\overline{\Omega}\sigma_{\overline{x}}, \tag{2.5}$$

where

$$\overline{\Omega}=\sqrt{\Delta^2+\Omega^2}=2\left[\left(\frac{\Delta}{2}\right)^2+g^2(a^\dagger a+\sigma_{++})\right]^{1/2}$$
$$=2\left\{\left(\frac{\Delta}{2}\right)^2+g^2\left[a^\dagger a+\frac{1}{2}(1+\sigma_z)\right]\right\}^{1/2}, \tag{2.6}$$

$$\sigma_{\overline{x}}=\frac{\Delta}{\overline{\Omega}}\sigma_z+\frac{\Omega}{\overline{\Omega}}\sigma_x\equiv\cos(\theta)\sigma_z+\sin(\theta)\sigma_x. \tag{2.7}$$

Obviously, $(\sigma_{\overline{x}})^2=1$ and eigenvalues of operator $\sigma_{\overline{x}}$ are ±1. This form of Hamiltonian is identical to the model for a spin-$\frac{1}{2}$ particle in a magnetic field B along the $\overline{x}$ direction with its magnitude proportional to the quantity $\overline{\Omega}$, except that B and Ω are now operators, not c numbers. The quantity $\overline{\Omega}$ (or Ω in the case of zero detuning $\Delta=0$) is nothing but the Rabi operator (its eigenvalues give all the frequencies in the Rabi oscillations of the atomic inversion) for the JC model and the gyration frequency (operator) for the latter model. This similarity between the two models permits us to obtain the energy eigenvalues and solutions of the evolution operator and other operators, such as the atomic inversion, as well as the expression of the Rabi operator, quickly. Let us consider the eigenvalues and eigenvectors first.

First of all, note that the three operators $\overline{\Omega},\sigma_{\overline{x}}$, and $(a^\dagger a+1/2\sigma_z)$ in Eq. (2.5) are mutually commutative and hence represent three constants of motion. Two of them, say, $(a^\dagger a+\sigma_{++})$ and $\sigma_{\overline{x}}$, are independent. To obtain energy eigenvalues, we only need to know the eigenvalues (denoted by N hereafter) of $(a^\dagger a+\sigma_{++})$, since $\sigma_{\overline{x}}$ has eigenvalues $m_{\overline{x}}=\pm1$. The relation between N and photon number n is $N=n$ for the state $|n,-\rangle$ (the atom in the ground state and light field having n photons); $N=n+1$ for the state $|n,+\rangle$ (the atom in the excited state and light field having n photons). It is pointed out that the number N is a conserved quantity, while the photon number is not. For instance, suppose the system is initially in the state $|n,-\rangle$. The atom can absorb a photon and make a transition to the excited state $|+\rangle$ so that the system's state becomes $|n-1,+\rangle$. The photon number changes by a unit, while N does not change during this process and the inverse one, i.e., the corresponding radiating photon process. Obviously, we have $N=0,1,2,\ldots$ and, therefore, immediately obtain the energy eigenvalues as follows:

$$E_{N,m_{\overline{x}}}=\omega\left(N-\frac{1}{2}\right)+\frac{m_{\overline{x}}}{2}\overline{\Omega}_N$$
$$=\omega\left(N-\frac{1}{2}\right)+m_{\overline{x}}\left[g^2N+\left(\frac{\Delta}{2}\right)^2\right]^{1/2}, \tag{2.8}$$

where $N=0,1,2,\ldots$, $m_{\overline{x}}=\pm1$, and $m_{\overline{x}}$ only takes -1 when $N=0$. Consequently, the system's energy levels manifest a doublet structure with the doublets' energy differences equal to the nonzero eigenvalues of a Rabi operator, while the ground state is a singlet state corresponding to $N=0$. The Rabi operator Ω has zero eigenvalue as $N=0$, and hence there is some ambiguity in the definitions of the operators σ_x and σ_y, since their denominator Ω becomes zero as $N=0$, although their numerators are also equal to zero in this case. This ambiguity is closely related to the singlet structure of the ground state. It is pointed out that the state corresponding to $\Omega=0$ represents the state without coupling between the atom and the field, since $H_{\rm int}=0$ in this case, and a corresponding energy eigenvalue [expressed in Eq. (2.8) as $N=0$, $m_{\overline{x}}=-1$) and eigenstate are easily obtained. It is worthwhile to mention that one of the advantages of this solving procedure is that the energy eigenvalues can quickly be obtained without needing to have detailed knowledge of the energy eigenstates.

Suppose $|n\rangle$ denotes Fock states and $|\pm\rangle$ are atomic states and satisfy $\sigma_z|\pm\rangle=\pm|\pm\rangle$. Denoting $|n;\pm\rangle\equiv|n\rangle\otimes|\pm\rangle$, let $|\Omega_N;\pm\rangle$ represent the common eigenvectors of Rabi operator Ω (or $\overline{\Omega}$) and operator σ_z, satisfying

$$\sigma_z|\Omega_N;\pm\rangle=\pm|\Omega_N;\pm\rangle\quad\Omega|\Omega_N;\pm\rangle=\Omega_N|\Omega_N;\pm\rangle,$$

where $\Omega_N=2g\sqrt{N}$. We see from Eq. (2.6) that $|\Omega_{N=0};-\rangle=|n=0;-\rangle$ and

$$|\Omega_N;+\rangle=|n=N-1;+\rangle;|\Omega_N;-\rangle=|n=N;-\rangle,$$
$$N=1,2,\ldots. \tag{2.9}$$

Our purpose is to express energy eigenstates or the common eigenvectors ($|\Omega_N;m_{\overline{x}}\rangle$) of the Rabi operator Ω (or $\overline{\Omega}$) and operator $\sigma_{\overline{x}}$ in terms of $|\Omega_N;\pm\rangle$. Since we want to obtain the expressions of energy eigenstates appropriate for this model, and for other models as well, the following derivation and the style of expressing results are rather general and are a little more complicated than direct calculation and the direct expression of results for this particular model. Using Eq. (2.7) and the relation

$$\sigma_{\overline{x}}(1+\sigma_{\overline{x}}\sigma_z)=(\sigma_{\overline{x}}+\sigma_z)=(1+\sigma_{\overline{x}}\sigma_z)\sigma_z,$$

one sees that

$$|\Omega_N;m_{\overline{x}}=1\rangle\sim(1+\sigma_{\overline{x}}\sigma_z)|\Omega_N;+\rangle$$

and $|\Omega_N;m_{\overline{x}}=-1\rangle$ can be obtained by its orthogonality to the former vector. The results are

$$|\Omega_N;m_{\overline{x}}=1\rangle=\cos\left(\frac{\theta_N}{2}\right)|\Omega_N;+\rangle+\sin\left(\frac{\theta_N}{2}\right)|\Omega_N;-\rangle, \tag{2.10a}$$

$$|\Omega_N;m_{\bar{x}}=-1\rangle=\cos\left(\frac{\theta_N}{2}\right)|\Omega_N;-\rangle-\sin\left(\frac{\theta_N}{2}\right)|\Omega_N;+\rangle, \tag{2.10b}$$

where $N=1,2,3,\ldots$ (we need not consider $N=0$, since $|\Omega_{N=0};-\rangle=|n=0;-\rangle$ is already the energy eigenstate) and

$$\sin\left(\frac{\theta_N}{2}\right)=\left(\frac{\bar{\Omega}_N-\Delta}{2\bar{\Omega}_N}\right)^{1/2}=\frac{\bar{\Omega}_N-\Delta}{\sqrt{(\bar{\Omega}_N-\Delta)^2+\Omega_N^2}}, \tag{2.11a}$$

$$\cos\left(\frac{\theta_N}{2}\right)=\left(\frac{\bar{\Omega}_N+\Delta}{2\bar{\Omega}_N}\right)^{1/2}=\frac{\Omega_N}{\sqrt{(\bar{\Omega}_N-\Delta)^2+\Omega_N^2}}, \tag{2.11b}$$

where $\bar{\Omega}_N=\sqrt{\Delta^2+\Omega_N^2}$ and $N=1,2,3,\ldots$. In the derivation of Eq. (2.10), we have utilized the relation $\sigma_x|\Omega_N;\pm\rangle=|\Omega_N;\mp\rangle$. (The general result is $\sigma_x|\Omega_N;\pm\rangle=c^{\pm1}|\Omega_N;\mp\rangle$, where constant $c=\exp(i\beta)$ and the value of real constant β depends on the choice of the phase factor of σ_x. In our case, we have chosen that the Rabi operator Ω [Eq. (2.3)] has the same sign as coupling parameter g and hence fixed the phase factor of σ_x, such that $c=1$.)

The dynamics in Heisenberg's scheme is represented by the transformations $A\equiv A(0)\rightarrow A(t)=U(t)AU^{\dagger}(t)$, where $U(t)$ is the system's evolution operator and has the form

$$\begin{aligned}U(t)&=\exp(iHt)=\exp(i\bar{H}_0t)\exp\left(i\frac{1}{2}\bar{\Omega}\sigma_{\bar{x}}t\right)\\&=\exp(i\bar{H}_0t)\left[\cos\left(\frac{\bar{\Omega}t}{2}\right)+i\sigma_{\bar{x}}\sin\left(\frac{\bar{\Omega}t}{2}\right)\right]\\&=\exp(i\bar{H}_0t)\left[\cos\left(\frac{\bar{\Omega}t}{2}\right)+i\left(\frac{2H_{\rm int}}{\bar{\Omega}}+\frac{\Delta}{\bar{\Omega}}\sigma_z\right)\sin\left(\frac{\bar{\Omega}t}{2}\right)\right],\end{aligned} \tag{2.12}$$

where $\bar{H}_0=\omega(a^{\dagger}a+1/2\sigma_z)$ denotes the free Hamiltonian H_0 for zero detuning. The time evolution of the atomic inversion operator has the form

$$\begin{aligned}\sigma_z(t)=&\left[\frac{\Delta^2}{\bar{\Omega}^2}+\frac{\Omega^2}{\bar{\Omega}^2}\cos(\bar{\Omega}t)\right]\sigma_z\\&+\frac{2H_{int}\sigma_z}{\bar{\Omega}}\left\{\frac{\Delta}{2\bar{\Omega}}[1-\cos(\bar{\Omega}t)]+i\sin(\bar{\Omega}t)\right\}\\&+\frac{H_{\rm int}\Delta}{\bar{\Omega}^2}[1-\cos(\bar{\Omega}t)].\end{aligned} \tag{2.13}$$

Equations (2.10), (2.12), and (2.13) are very general in that they are suitable not only for this model, but also for other nonlinear JC models, because in their derivations we have only utilized the general properties of three Pauli operators (i.e., we do not use their relations to this particular model) and because the Rabi operator commutes with three Pauli operators, which is true for this linear and other nonlinear JC models. Let us explain this point. The Rabi operator Ω has the form $\sqrt{f+g\sigma_z}$ where f and g are two functions of field variables. Therefore, it obviously commutes with σ_z. Ω also commutes with $\sigma_x=2H_{\rm int}/\Omega$, since $\Omega\sim\sqrt{H_{\rm int}^2}$ obviously commutes with $H_{\rm int}$, and so does the operator $\sigma_y=i\sigma_x\sigma_z$. In Eq. (2.12), we have also utilized the fact that $\bar{H}_0$ commutes with Ω. We put the eigenvalues of $H'_{\rm int}\equiv H-\bar{H}_0$ into the general form as follows:

$$E'_{N,m_{\bar{x}}}=\tfrac{1}{2}m_{\bar{x}}\sqrt{\Omega_N^2+\Delta^2}, \tag{2.14}$$

where $m_{\bar{x}}=\pm1$ and $m_{\bar{x}}$ only takes -1 when $\Omega_N=0$.

Equations (2.10), (2.12), (2.13), and (2.14) are the central results of this section. They are the standardized forms for energy eigenstates, evolution operator, atomic inversion operator, and energy eigenvalues, respectively, and they are suitable for any two-level nonlinear JC models. The final results of the solution to any given model can easily be obtained from these standardized forms as long as one obtains the expressions of a Rabi operator's eigenvalues and common eigenvectors of a Rabi operator and σ_z. In all cases, obtaining them is much easier than obtaining directly the eigenvalues and eigenstates of H. As a matter of fact, little effort is needed to obtain them in many cases, as will be seen in the next two sections.

III. SINGLE-MODE NONLINEAR JC MODELS

Let us first show that any single-mode nonlinear model can be solved analytically. In this case, we have one atom and one field mode, and hence we only have two independent constants of motion that completely determine the energy eigenvalues and eigenstates. However, we have known three constants of motion, Ω, $\sigma_{\bar{x}}$, and $N=a^{\dagger}a+\alpha\sigma_{++}$, where the value of integer α is different for different models. This implies that Ω is the function of two independent quantities N and $\sigma_{\bar{x}}$ (in fact, we only need to consider the situation where Ω is independent of $\sigma_{\bar{x}}$, although it is unimportant in this general proof). Consequently, to obtain the solution to any given model, we only need to know the eigenvalues of operator N (obviously, $N=0,1,2,\ldots$) and the common eigenvectors of N and σ_z, which are certainly easy to express in terms of Fock states. Once this is done, the solution to this model is given by Eqs. (2.10), (2.12), (2.13), and (2.14).

We now illustrate this point by considering the following single-mode nonlinear JC model,

$$H=\frac{E_{+-}}{2}\sigma_z+\omega a^{\dagger}a+H_{\rm int}, \tag{3.1a}$$

$$H_{\rm int}=g[a^{\dagger k}f(a^{\dagger}a)\sigma_{-+}+f(a^{\dagger}a)a^k\sigma_{+-}], \tag{3.1b}$$

where k can be any positive integer and f is any reasonable function of photon number operator and satisfies $f^{\dagger}=f$, that is, Hermitian operator. This kind of Hamiltonian describes intensity-dependent (characterized by function f) k-photon transitions and it is the most general form of single-mode JC-type model. It becomes the Buck-Sukumar model [4] when $k=1$ and $f(x)=\sqrt{x+1}$ by noting identities $a^{\dagger}f(a^{\dagger}a)$

$=f(a^\dagger a-1)a^\dagger$ and $f(a^\dagger a)a=af(a^\dagger a-1)$. It is easily shown that the constant of motion N in this case has the form $N=a^\dagger a+k\sigma_{++}$ and

$$(H_{\rm int}/g)^2=F(n-k)+[F(n)-F(n-k)]\sigma_{++},$$

where $n=a^\dagger a$ and

$$F(n)=(n+k)(n+k-1)\cdots(n+1)f(n)f(n).$$

As long as $F(n)$ can be expanded as a Taylor series, one can use the identities

$$(n+k\sigma_{++})^l=n^l+[(n+k)^l-n^l]\sigma_{++},\quad l=0,1,2,3,\ldots$$

to obtain $(H_{\rm int}/g)^2=F(N-k)$ or

$$\Omega=2gf(n-k)\sqrt{N(N-1)(N-2)\cdots(N-k+1)},\qquad (3.2)$$

where $N=a^\dagger a+k\sigma_{++}$. For simplicity, we shall hereafter use N to denote operator $a^\dagger a+k\sigma_{++}$ and its eigenvalues, and use n to denote photon operator $a^\dagger a$ and its eigenvalues. Similar to the explanations in Sec. II, the N is a conserved quantity, while n is not, and their relation is $N=n+k$ for the state $|n,+\rangle$ and $N=n$ for the state $|n,-\rangle$. Obviously, $N=0,1,2,3,\ldots$ and the energy eigenvalues are

$$E_{N,m_{\bar{x}}}=-\frac{1}{2}k\omega+\omega N+m_{\bar{x}}\left\{[gf(N-k)]^2N(N-1)\times(N-2)\cdots(N-k+1)+\left(\frac{\Delta}{2}\right)^2\right\}^{1/2},\qquad (3.3)$$

where $\Delta=E_{+-}-k\omega$ is also the detuning in this case, $N=0,1,2,3,\ldots$, $m_{\bar{x}}=\pm1$, and $m_{\bar{x}}$ only takes -1 when $N=0,1,\ldots,k-1$.

Substituting the expression (3.2) of Ω, detuning $\bar{\Delta}=E_{+-}-k\omega$, $H_0=-1/2k\omega+\omega N$, and the expression $\bar{\Omega}=\sqrt{\Omega^2+\Delta^2}$ into Eqs. (2.12) and (2.13), we then obtain the expressions of the evolution and the atomic inversion operators, respectively, for this model. We now consider the energy eigenstates. The eigenvalue of the Rabi operator Ω is zero when $N=0,1,\ldots,k-1$, and in these situations corresponding energy levels are ones without effective coupling between the atom and the field and manifest a singlet structure. The corresponding eigenstates are $|n\rangle\otimes|-\rangle$, $n=0,1,2,\ldots,k-1$. The rest levels manifest doublet structure. Using the expressions of Ω and $N=a^\dagger a+k\sigma_{++}$, one may easily see that the common eigenvectors of Ω and σ_z are

$$|\Omega_N;+\rangle=|n=N-k;+\rangle;|\Omega_N;-\rangle=|n=N;-\rangle,$$
$$N=k,k+1,k+2,\ldots,\qquad (3.4)$$

where $|n;\pm\rangle=|n\rangle\otimes|\pm\rangle$, and $|n\rangle$ and $|\pm\rangle$ denote Fock states and atomic states, respectively. Substituting them into Eq. (2.10) and using

$$\bar{\Omega}_N=\sqrt{\Omega_N^2+\Delta^2},$$
$$\Omega_N=2gf(N-k)\sqrt{N(N-1)(N-2)\cdots(N-k+1)},$$

one then obtains all the rest energy eigenstates. We have now completely solved this very general single-mode nonlinear JC model, two particular cases of which [$k=1$, $f(x)=\sqrt{(x+1)}$ and any positive k, but taking $f(x)\equiv1$, the detuning Δ is taken to be zero in both of these cases] have been solved by Buck and Sukumar by another method [4]. Our solutions for these two particular cases are easily shown to be identical to theirs. For instance, substituting $\Delta=0$, $k=1$, $f(x)=\sqrt{(x+1)}$ into Eqs. (3.12) and (2.13), we obtain the Rabi operator (its eigenvalues are Rabi frequencies) and the atomic inversion (its average represents atomic inversion), respectively, as follows:

$$\Omega=2gf(N-1)\sqrt{N}=2g(a^\dagger a+\sigma_{++}),$$
$$\sigma_z(t)=\cos(\Omega t)\sigma_z-i\frac{2g}{\Omega}[af(a^\dagger a-1)\sigma_{+-}-f(a^\dagger a-1)a^\dagger\sigma_{-+}]\sin(\Omega t),$$

where $\sigma_{++}=(\sigma_z+1)/2$. Except for different notation, these two expressions are identical to Eqs. (7) and (8) of the first paper of Ref. [4], respectively. Before ending this section, we mention that a straightforward generation of this model in the coupling parameter g can be any function of operator $N=a^\dagger a+k\sigma_{++}$ and its solution is the same as the above one as long as one replaces g in the above results by g_N.

IV. MULTIMODE NONLINEAR JC MODELS

In this section, we shall apply the general solving procedure and the standardized forms in Sec. II to Raman coupling models with two and three modes. We begin with the two-mode Raman model.

A. Two-mode Raman model

In dealing with two-mode Raman-type processes, one can consider a three-level system of energies E_1, E_2, and E_3 in the Λ configuration interacting with a pump ω_1 and a Stokes mode ω_2 [6,7,10,12]. The Hamiltonian of the system is written as [6,10–12]

$$H=\sum_{i=1}^{3}E_i\sigma_{ii}+\hbar\omega_1a_1^\dagger a_1+\hbar\omega_2a_2^\dagger a_2+\hbar g_1(a_1\sigma_{31}+a_1^\dagger\sigma_{13})+\hbar g_2(a_2\sigma_{32}+a_2^\dagger\sigma_{23}),\qquad (4.1)$$

where symbols $a_j(j=1,2)$ represent the field operators of modes 1 and 2, $\sigma_{ii}=|i\rangle\langle i|$ are the level occupation numbers and $\sigma_{ij}=|i\rangle\langle j|(i\neq j)$ are the transition operators from levels j to i. Levels 3 and 1 (2) are coupled by a dipole-coupling constant $g_1(g_2)$. There is no direct coupling between levels 1 and 2. The quantities Δ_1 and Δ_2 denote detunings given by $\Delta_j=(E_3-E_j)/\hbar-\omega_j$, $j=1,2$. This three-level problem is usually reduced to an effective two-level one by using adiabatic elimination [6,7] or evaluating transformation perturbatively [10] under the large detuning assumption. It has recently been proven by one of the authors [11] that this three-level problem can be exactly transformed into a two-level

problem, regardless of whether the detunings are large or small. The corresponding two-level Hamiltonian reads [6,7,10,11]

$$H=\tfrac{1}{2}E_{+-}\sigma_z+\omega_1 a_1^\dagger a_1+\omega_2 a_2^\dagger a_2+H_{\rm int}, \tag{4.2a}$$

$$H_{\rm int}=g[a_1^\dagger a_2\sigma_{-+}+a_2^\dagger a_1\sigma_{+-}], \tag{4.2b}$$

where the frequency difference $\omega_1-\omega_2$ is roughly equal to the energy difference E_{+-} of the two atomic levels. The Rabi operator Ω is easily obtained and has the form

$$\Omega=2g\sqrt{(n_1+\sigma_{++})(n_2+\sigma_{--})}, \tag{4.3}$$

which is the function of two constants of motion, $N_1=a_1^\dagger a_1+\sigma_{++}$ and $N_2=a_2^\dagger a_2+\sigma_{--}$. The total photon number

$$n_t=n_1+n_2=N_1+N_2-1$$

is also a constant of motion. Obviously, eigenvalues of N_j are $N_1,N_2=0,1,2,3,\ldots$, and hence energy eigenvalues are

$$E_{N_1,N_2,m_x}=-\frac{\omega_1+\omega_2}{2}+\omega_1 N_1+\omega_2 N_2 +m_{\bar{x}}\left[g^2N_1N_2+\left(\frac{\Delta}{2}\right)^2\right]^{1/2}, \tag{4.4}$$

where $\Delta=E_{+-}-(\omega_1-\omega_2)$ and $m_{\bar{x}}=\pm1$. However, $\Omega=0$ (or interaction Hamiltonian $H_{\rm int}=0$) as N_1 or N_2 is zero; the corresponding energy eigenvalues are easily found to be

$$E=\left\{\begin{array}{ll}\omega_2(N_2-1)-\dfrac{E_{+-}}{2} & \text{for } N_1=0\\[2ex] \omega_1(N_1-1)+\dfrac{E_{+-}}{2} & \text{for } N_2=0\end{array}\right\},$$

which represent two singlets and are also the two states without effective coupling between the atom and field modes. The two eigenvectors corresponding to these two energy eigenvalues are, respectively,

$$|n_1=0,n_2=N_2-1\rangle\otimes|-\rangle,\quad |n_1=N_1-1,n_2=0\rangle\otimes|-\rangle,$$

where $|n_1,n_2\rangle$ denote Fock states. These two vectors are also the common eigenvectors of Ω and σ_z. The other common eigenvectors of Ω and σ_z are easily seen to be

$$|\Omega_{N_1,N_2};+\rangle=|n_1=N_1-1,n_2=N_2;+\rangle,$$

$$|\Omega_{N_1,N_2};-\rangle=|n_1=N_1,n_2=N_2-1;-\rangle, \tag{4.5}$$

where $|n_1,n_2;\pm\rangle=|n_1,n_2\rangle\otimes|\pm\rangle$. Substituting them into Eq. (2.10), we obtain the rest-energy eigenvectors, expressed in terms of Fock states and atomic states, as follows:

$$|N_1,N_2;m_{\bar{x}}=1\rangle =\left(\frac{\overline{\Omega}_{N_1,N_2}+\Delta}{2\overline{\Omega}_{N_1,N_2}}\right)^{1/2}|n_1=N_1-1,n_2=N_2;+\rangle +\left(\frac{\overline{\Omega}_{N_1,N_2}-\Delta}{2\overline{\Omega}_{N_1,N_2}}\right)^{1/2}|n_1=N_1,n_2=N_2-1;-\rangle, \tag{4.6a}$$

$$|N_1,N_2;m_{\bar{x}}=-1\rangle =\left(\frac{\overline{\Omega}_{N_1,N_2}+\Delta}{2\overline{\Omega}_{N_1,N_2}}\right)^{1/2}|n_1=N_1,n_2=N_2-1;-\rangle -\left(\frac{\overline{\Omega}_{N_1,N_2}-\Delta}{2\overline{\Omega}_{N_1,N_2}}\right)^{1/2}|n_1=N_1-1,n_2=N_2;+\rangle \tag{4.6b}$$

where $\Omega_{N_1,N_2}=2g\sqrt{N_1N_2}$, $\Delta=E_{+-}-(\omega_1-\omega_2)$, and $\overline{\Omega}_{N_1,N_2}=\sqrt{\Omega_{N_1,N_2}^2+\Delta^2}$.

Substituting $H_0=-(\omega_1+\omega_2)/2+\omega_1N_1+\omega_2N_2$, $\Omega=2g\sqrt{N_1N_2}$, $\Delta=E_{+-}-(\omega_1-\omega_2)$, and $\overline{\Omega}=\sqrt{\Omega^2+\Delta^2}$ into Eqs. (2.12) and (2.13), one then obtains the evolution and atomic inversion operators.

Last, we point out that the two-mode JC models are in fact solvable analytically. The argument is similar to the one in the single-mode situation. Generally speaking, there are three independent constants of motion, $\sigma_{\bar{x}}$, $N_1=a_1^\dagger a_1+\alpha\sigma_{++}$, and $N_2=a_2^\dagger a_2+\beta\sigma_{--}$, where the values of integers α, β depend on models. For instance, $\alpha=\beta=k$ for the following model, describing $2k$-photon transitions,

$$H=\tfrac{1}{2}E_{+-}\sigma_z+\omega_1 a_1^\dagger a_1+\omega_2 a_2^\dagger a_2+H_{\rm int}, \tag{4.7a}$$

$$H_{\rm int}=g[a_1^{\dagger k}a_2^k f(n_1,n_2)\sigma_{-+}+f(n_1,n_2)a_2^{\dagger k}a_1^k\sigma_{+-}], \tag{4.7b}$$

where f satisfies $f^\dagger=f$, $n_j=a_j^\dagger a_j$ and k can be any positive integer. This fact implies that another constant-of-motion Rabi operator Ω can be expressed as the function of the two quantities N_1,N_2. Consequently, to obtain the energy eigenvalues, one only needs to know the eigenvalues of N_1,N_2, which are obviously $N_1,N_2=0,1,2,\ldots$, since another quantum number $m_{\bar{x}}$ is known to take ±1. The energy eigenvectors are $|N_1,N_2;m_{\bar{x}}\rangle$, which are expressed in Eq. (2.10) in terms of the common eigenvectors of N_1,N_2, and σ_z, while the common eigenvectors of N_1,N_2, and σ_z are obviously easy to obtain and express in terms of Fock states. We now explicitly show that the Rabi operator for the above, rather general model can indeed be expressed as the function of N_1,N_2. Using the expression of the interaction Hamiltonian, one sees that $(H_{\rm int}/g)^2=F(n_1-k,n_2+k)\sigma_{--}+F(n_1,n_2)\sigma_{++}$, where

$$F(n_1,n_2)=(n_1+k)(n_1+k-1)\cdots(n_1+1)n_2(n_2-1)\cdots \times(n_2-k+1)f(n_1,n_2)f(n_1,n_2),$$

where use has been made of the identities $G(n)a^\dagger = a^\dagger G(n+1)$, $G(n)a = aG(n-1)$. As long as F can be expanded as a Taylor series, one can use identity

$$(n_1-k)^l(n_2+k)^m\sigma_{--}+n_1^l n_2^m\sigma_{++}=(n_1-k+k\sigma_{++})^l(n_2+k\sigma_{--})^m, l,m=0,1,2,\ldots$$

to obtain $(H_{\rm int}/g)^2 = F(n_1-k+k\sigma_{++}, n_2+k\sigma_{--})$ or

$$\Omega = 2g\sqrt{N_1N_2(N_1-1)(N_2-1)\cdots(N_1-k+1)(N_2-k+1)}\,f(N_1-k,N_2), \tag{4.8}$$

where $N_1=n_1+k\sigma_{++}$, $N_2=n_2+k\sigma_{--}$. The relations between N_j and n_j are $N_1=n_1+k$, $N_2=n_2$ for the atomic state $|+\rangle$ and $N_1=n_1$, $N_2=n_2+k$ for the atomic state $|-\rangle$. It is pointed out that N_j are conserved quantities while n_j are not.

B. Three-mode nonlinear JC model

In this subsection, we consider a Raman-type model proposed by Wang, Puri, and Eberly [1] and Wu [14],

$$H=\omega_p N+E_{+-}J_z+H_{\rm int}, \tag{4.9a}$$

$$H_{\rm int}=(g_1a_S^\dagger a_P+g_2a_P^\dagger a_A)\sigma_{+-}+(g_1a_P^\dagger a_S+g_2a_A^\dagger a_P)\sigma_{-+}, \tag{4.9b}$$

where subscripts P, S, and A represent pump, Stokes, and anti-Stokes modes, respectively, a and $a^\dagger$ are the creation and annihilation operators for the corresponding modes, $J_z = n_A-n_S+\frac{1}{2}(\sigma_{++}-\sigma_{--})$, and $N=n_P+n_S+n_A$ is the total photon number. We have ignored the detuning term and the Stark shift term. When g_1 or g_2 is zero, the model becomes the one discussed in the last subsection. We shall consider the case of $g_1g_2\neq 0$ here.

It is noted that one needs four independent constants of motion to characterize the energy eigenvalues and eigenstates in this case. We have known three of them, σ_x (or $\sigma_{\bar{x}}$ as detuning term and Stark shift term are included), $N=n_P+n_S+n_A$, and $J_z=n_A-n_S+\frac{1}{2}(\sigma_{++}-\sigma_{--})$. No other simple constant of motion has been found yet for the general case of $g_1\neq g_2$. This implies that the eigenvalue problem of Rabi operator Ω (and therefore the eigenvalue problem of the Hamiltonian) is not simple compared with one- and two-mode situations, since the Rabi operator here cannot express the function of simple constants of motion unless one has succeeded in finding the fourth simple constant of motion. This is the reason why no one, to the best of our knowledge, has solved this model for the general case of $g_1\neq g_2$. Wang, Puri, and Eberly have solved this model by another method for $g_1=g_2$ and $J_z=-\frac{1}{2}$. One of us [14] has solved it for $g_1=g_2$ by the procedure presented in this paper, which is considerably simpler than their method. In addition, the solution to the model including detuning term and Stark shift term for any possible J_z is obtained [14]. For the purpose of completeness, we cite the main points here.

In the situation where $g_1=g_2=g$, the interaction Hamiltonian can be put into the form [1,14],

$$H_{\rm int}=\frac{g}{\sqrt{2}}\,(L_-\sigma_{+-}+L_+\sigma_{-+}), \tag{4.10}$$

where L_- and L_+ are the lowering and raising operators of the angular momentum L. The quantity L^2 is the fourth simple constant of motion in this situation. The relations between angular momentum and field variables are as follows:

$$L_+=L_x+iL_y=\sqrt{2}(a_Sa_P^\dagger+a_A^\dagger a_P), \tag{4.11a}$$

$$L_-=L_+^\dagger=\sqrt{2}(a_Pa_S^\dagger+a_P^\dagger a_A), \tag{4.11b}$$

$$L_z=a_A^\dagger a_A-a_S^\dagger a_S, \tag{4.11c}$$

$$L^2=(n_A-n_S)^2+(n_A+n_S)(2n_P+1)+2np+2(a_P^2a_A^\dagger a_S^\dagger + a_P^{\dagger 2}a_Sa_A). \tag{4.11d}$$

The Rabi operator has the form

$$\Omega=g\sqrt{2}(L^2-J_z^2+\tfrac{1}{4})^{1/2}, \tag{4.12}$$

which is the function of the two constants of motion, L^2 and J_z. The eigenvalues of these two quantities are very easy to obtain and their common eigenvectors are also easy to find. Then, the solution to the model can be obtained by the standardized forms in Sec. II. The details are referred to in Ref. [14].

V. SUMMARY

In this paper, we have presented a simple procedure to analytically solve various nonlinear two-level JC models. By establishing the similarity between the various JC models and the time for a spin-$\frac{1}{2}$ particle in the presence of a magnetic field, we have obtained the standardized forms for energy eigenvalues and eigenvectors, evolution operator, and the time evolution of atomic inversion operator, which are suitable for all the two-level models. These standardized forms allow one to quickly obtain the solutions as long as one can obtain the eigenvalues and the common eigenvectors of the Rabi operator and σ_z, which turn out to be very simple in many cases. We have shown by this procedure that one- and two-mode nonlinear JC models are generally solvable analytically and their solutions are easily obtained by determining the corresponding solutions of simple operators σ_z and $N=a^\dagger a+k\sigma_{++}$ (or $N_1=a_1^\dagger a_1+k\sigma_{++}$ and $N_2=a_2^\dagger a_2+k\sigma_{--}$ in two-mode cases). As examples of this solving procedure, we have explicitly discussed any intensity-dependent one- and two-mode models describing k- or $2k$-photon transitions, respectively. Finally, we have briefly discussed a three-mode Raman model and explained

that this procedure is still simpler than other methods in three-mode situations.

According to our results, we find that all the models show a singlet+doublet level structure. The singlets correspond to the zero eigenvalue of a Rabi operator or interaction Hamiltonian and represent the states without effective coupling between the field modes and the atom, very similar to the ground state in the usual JC model. And models have many more doublets with the energy difference between each doublet given by one of the eigenvalues of the Rabi operator, while these eigenvalues also determine all the oscillating frequencies of the atomic inversion. These properties shall resemble those of the doublets in the usual JC model

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