Resonant interaction between identical atoms including recoil

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The emission spectrum is calculated for a system of two identical atoms, having relative coordinate **r**. Initially one atom is in an excited state and the other in its ground state. Both atom and field variables are fully quantized, so that the calculation automatically includes effects related to both retardation and atomic recoil on the absorption or emission of radiation. It is shown that for $k_0 r \gg 1$ (where $k_0 = 2\pi/\lambda$ and λ is the wavelength associated with the resonant excited- to ground-state transition), the emission spectrum can consist of a triplet. The frequency separation between the components of the triplet can be interpreted in terms of the recoil the atoms undergo individually on emitting or absorbing radiation. For $k_0 r \ll 1$, the atoms emit as a composite system and the recoil (while not resolvable) is that associated with a ''molecule'' of mass $2m$, where m is the mass of one of the atoms. [S1050-2947(97)01506-0]

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I. INTRODUCTION

In a recent experimental *tour de force*, Devoe and Brewer [1] observed the variation in the decay rate of a two-ion system as a function of the separation of the ions. This paper was preceded by a theoretical paper by Brewer [2] on two-ion superradiance that contained a survey of the literature on the theory of two-atom superradiance. Of particular relevance to the present discussion are the articles by Milonni and Knight [3] and Power [4]. Milonni and Knight consider the evolution of a system of two identical atoms, one of which starts in an excited state and the other in its ground state. They concentrate on the limit in which the atoms are separated by a fixed distance that is larger than the wavelength of the excited- to ground-state transition and obtain solutions that reflect the retardation effects associated with the finite transmission time it takes for radiation emitted by one atom to influence the other atom. Their calculation treats the motion of the atoms (or, more precisely, the lack of motion of the atoms) classically. At the other extreme, Power presents a fully quantized calculation of transition rates and energy shifts in order to assess the role played by the motion of the atoms in modifying the superradiant decay. Although the starting point of the theory is a fully quantized Hamiltonian for the atoms and the (vacuum) field that, in principle, contains all effects related to retardation and atomic recoil on the emission of radiation, there is no explicit mention of retardation effects in the paper and effects related to recoil are neglected.

It is the purpose of this paper to present a fully quantized calculation of the emission spectrum from a system of two identical atoms in which the effects of retardation and recoil are considered explicitly. The two atoms are represented by wave packets that are separated initially by some average distance r_0, with one of the atoms in its excited state and the other in its ground state. The goal of the calculation is to answer questions of the following nature. (i) How does recoil modify the emission spectrum? (ii) Do the atoms recoil as a composite system or as a system of two individual atoms?

In attempting to answer these questions, a formalism is developed in which retardation and recoil enter the equations in an intuitively obvious fashion. Both coordinate and momentum state representations are used. The paper is organized as follows. In Sec. II, the problem is defined, the notation is established, and the equations of motion are obtained. The limiting cases of $k_0 r \gg 2\pi$ and $k_0 r \ll 1$ are discussed in Secs. III and IV, respectively, where $k_0 = 2\pi/\lambda$, λ is the wavelength of the ground- to excited-state transition, and r is the interatomic separation. In each of these limits, the emission spectrum is evaluated and its dependence on atomic recoil is analyzed. The results are summarized in Sec. V. The paper contains an appendix in which the use of the Weisskopf-Wigner approximation is justified and questions related to causality are explored.

II. EQUATIONS OF MOTION AND SOLUTION

The system under consideration is shown in Fig. 1. Two identical atoms are separated by a distance that is large compared to the spatial extent of the electronic wave function of each of the atoms. Each atom has a ground state g and excited state e that are separated in frequency by ω_0. Initially, atom 1 is excited and atom 2 is in its ground state (the atoms can be considered as distinguishable owing to their relatively large separation). As a result of the atoms' interaction with the vacuum radiation field, the system evolves into a super-

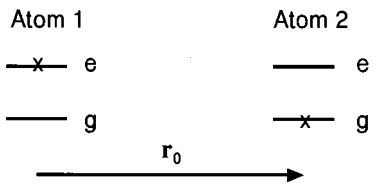

FIG. 1. Schematic representation of the initial conditions for the problem under discussion. Atom 1 is in its excited state, atom 2 in its ground state, and there are no photons in the field. The center-of-mass motion of the atoms is described by a plane-wave state, while the relative motion is described by a wave packet that is centered at $\mathbf{r} = \mathbf{r}_0$ at $t = 0$, having an average relative momentum equal to zero. The extent of the wave packet associated with the relative motion is much less than r_0.

position of three internal states: $|\alpha\rangle=|e_1,g_2;0\rangle$ in which atom 1 is excited, atom 2 in its ground state, and there are no photons in the field; $|\alpha'\rangle=|g_1,e_2;0\rangle$ in which atom 2 is excited, atom 1 in its ground state, and there are no photons in the field; and $|\beta\rangle=|g_1,g_2;\mathbf{k}_{\boldsymbol{\epsilon}}\rangle$ in which both atoms are in their ground states and a photon is emitted into a mode having wave vector $\mathbf{k}$ and polarization $\boldsymbol{\epsilon}$. States other than these enter the calculation as virtual states and can contribute to single-atom level shifts and to the van der Waals interaction between the atoms [3]: such effects are neglected in this work [5].

In order to describe the external state variables of the atoms, we follow the general procedure outlined by Power. The center of mass of atom i ($i=1,2$) is located at position $\mathbf{R}_i$ and the momentum canonical to the position variable is $\mathbf{P}_i$. The $\mathbf{R}_i$ and $\mathbf{P}_i$ are quantum-mechanical variables. Initially, the average separation of the wave packets of the two atoms is $\langle\mathbf{R}_2-\mathbf{R}_1\rangle=\mathbf{r}_0$. We are interested in mapping out the evolution of the system as a function of k_0r_0, where $k_0=\omega_0/c$. On the other hand, we are not concerned here with the modification of the internal state dynamics that results from an initial relative velocity of the two atoms since this has been discussed previously by Cooper and Stacey [6] and Power [4]. Consequently, it is assumed that $\langle\mathbf{P}_2-\mathbf{P}_1\rangle=\mathbf{0}$ initially.

Rather than work with the individual coordinates of the atoms, it is useful to introduce the variables

$$\begin{aligned}&\mathbf{R}=(\mathbf{R}_1+\mathbf{R}_2)/2,\quad \mathbf{P}=\mathbf{P}_1+\mathbf{P}_2,\\&\mathbf{r}=\mathbf{R}_2-\mathbf{R}_1,\quad \mathbf{p}=(\mathbf{P}_2-\mathbf{P}_1)/2,\end{aligned}\tag{1}$$

which imply that

$$\begin{aligned}&\mathbf{R}_1=\mathbf{R}-\tfrac{1}{2}\mathbf{r},\quad \mathbf{R}_2=\mathbf{R}+\tfrac{1}{2}\mathbf{r},\\&\mathbf{P}_1=\tfrac{1}{2}\mathbf{P}-\mathbf{p},\quad \mathbf{P}_2=\tfrac{1}{2}\mathbf{P}+\mathbf{p}.\end{aligned}\tag{2}$$

The vector $\mathbf{R}$ is the center-of-mass coordinate of the two-atom system and $\mathbf{P}$ is its conjugate momentum, while $\mathbf{r}$ is the relative coordinate of the two atoms and $\mathbf{p}$ is the momentum conjugate to $\mathbf{r}$.

The formalism developed below is quite general; however, the discussion in this article is aimed at a restricted set of wave functions and interatomic separations. It may help to put the problem in better perspective if these restricted sets are discussed at this point. The initial wave function is taken to be of the form

$$|\Psi(\mathbf{R},\mathbf{r},t=0)\rangle=\chi_0(\mathbf{R})\psi_0(\mathbf{r})|\alpha\rangle.\tag{3}$$

Although the center of mass of the two-atom system recoils as a result of spontaneous emission, the center-of-mass motion really is not a critical element of this problem. Without loss of generality, one can take $\chi_0(\mathbf{R})$ as a plane-wave state. With regard to the relative motion, it is assumed that

$$\begin{aligned}&\Delta r\ll r_0,\quad [\hbar k_0/m]/\Gamma\ll r_0,\\&[\hbar/(m\Delta r)]/\Gamma\ll r_0,\quad r_0\gg c/\Gamma,\end{aligned}\tag{4}$$

where m is the mass of one of the atoms, Δr is the spread associated with $\psi_0(\mathbf{r})$, and Γ is the excited-state decay rate. The first inequality corresponds to the requirement that the spread of the initial wave packet (for the relative motion) is much less than the interatomic separation, the second and third inequalities correspond to the requirements that the wave packet does not move or spread significantly during the lifetime of the excited state, and the last inequality corresponds to the requirement that the atoms can undergo excitation exchange in a time short compared to the excited-state lifetime. As long as we restrict the calculation to separations

$$10^4\gtrsim k_0r_0\gtrsim 0.1,$$

inequalities (4) are satisfied for typical atomic decay rates. Note that if $k_0r_0<1$, the momentum spread Δp must be greater than $\hbar k_0$; as a result, the Doppler width associated with this spread $k_0\Delta p/m$ is necessarily greater than the recoil frequency shift $\omega_{k_0}^r=\hbar k_0^2/2m$. As a consequence, it is impossible to resolve the recoil splitting if the atoms are separated by less than the wavelength $\lambda_0=2\pi/k_0$ of the resonant transition. On the other hand, for separations $r_0\gg\lambda_0$, it is possible to choose $\Delta r>\lambda_0$, $\Delta p<\hbar k_0$, enabling one to resolve the recoil shift for such separations.

In the dipole approximation, it is possible to neglect the variation of the electric-field amplitude on the electronic coordinates of each of the atoms and simply evaluate the field acting on atom i at position $\mathbf{R}_i$. In this limit, the Hamiltonian describing the two-atom system is

$$\begin{aligned}H=\frac{P^2}{4m}+\frac{p^2}{m}+H_1+H_2+\sum_{\mathbf{k}}\hbar\Omega_k a_{\mathbf{k}}^\dagger a_{\mathbf{k}}-\boldsymbol{\mu}_1\cdot\mathbf{E}\left(\mathbf{R}-\frac{1}{2}\mathbf{r}\right)\\-\boldsymbol{\mu}_2\cdot\mathbf{E}\left(\mathbf{R}+\frac{1}{2}\mathbf{r}\right),\end{aligned}\tag{5}$$

where

$$\mathbf{E}(\mathbf{R})=\sum_{\mathbf{k}}[g_{\mathbf{k}}\boldsymbol{\epsilon}_{\mathbf{k}}a_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{R})+g_{\mathbf{k}}^*\boldsymbol{\epsilon}_{\mathbf{k}}a_{\mathbf{k}}^\dagger\exp(-i\mathbf{k}\cdot\mathbf{R})],\tag{6}$$

H_i is the Hamiltonian for atom i, $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ are the destruction and creation operators for a field mode having propagation vector $\mathbf{k}$ and frequency $\Omega_k=kc$, $\boldsymbol{\mu}_i$ is the dipole moment operator of atom i, $g_{\mathbf{k}}=i\sqrt{\hbar\Omega_{\mathbf{k}}/2\varepsilon_0V}$, V is the quantization volume, and $\boldsymbol{\epsilon}_{\mathbf{k}}$ is the polarization vector for mode $\mathbf{k}$ (there are two independent values of $\boldsymbol{\epsilon}_{\mathbf{k}}$ for each $\mathbf{k}$).

To simplify matters, it is assumed that the ground and excited states of each of the atoms are nondegenerate (a prescription for generalizing the calculation to states of arbitrary angular momentum is given below). The wave function for the system is expanded as

$$\begin{aligned}|\Psi(\mathbf{R},\mathbf{r},t\rangle=(2\pi\hbar)^{-3}\int d\mathbf{P}\int d\mathbf{p}\\\times\sum_{\sigma=\alpha,\alpha',\beta}\exp[-i(E_\sigma+E_P+E_p)t/\hbar]\\\times\exp[i(\mathbf{P}\cdot\mathbf{R}+\mathbf{p}\cdot\mathbf{r})/\hbar]b_\sigma(\mathbf{P},\mathbf{p},t)|\sigma\rangle,\end{aligned}\tag{7}$$

where

$$E_P = P^2/4m, \quad E_p = p^2/m,$$

$E_\alpha = E_{\alpha'} = \hbar\omega_0$, and $E_\beta = \hbar\Omega_k$. It then follows from Schrodinger's equation that the state amplitudes $b_\sigma(\mathbf{P},\mathbf{p},t)$ evolve as

$$\dot{b}_\alpha(\mathbf{P},\mathbf{p},t) = (i\hbar)^{-1}\sum_{\mathbf{k}} g_k(-\boldsymbol{\mu}\cdot\boldsymbol{\epsilon}_{\mathbf{k}}) \times \exp\left[i\left((\omega_0-\Omega_k)+\frac{\mathbf{k}\cdot\mathbf{P}}{2m}-\frac{\mathbf{k}\cdot\mathbf{p}}{m}-\omega_k^r\right)t\right] \times b_\beta(\mathbf{P}-\hbar\mathbf{k},\mathbf{p}+\tfrac{1}{2}\hbar\mathbf{k},t), \tag{8a}$$

$$\dot{b}_{\alpha'}(\mathbf{P},\mathbf{p},t) = (i\hbar)^{-1}\sum_{\mathbf{k}} g_k(-\boldsymbol{\mu}\cdot\boldsymbol{\epsilon}_{\mathbf{k}}) \times \exp\left[i\left((\omega_0-\Omega_k)+\frac{\mathbf{k}\cdot\mathbf{P}}{2m}+\frac{\mathbf{k}\cdot\mathbf{p}}{m}-\omega_k^r\right)t\right] \times b_\beta(\mathbf{P}-\hbar\mathbf{k},\mathbf{p}-\tfrac{1}{2}\hbar\mathbf{k},t), \tag{8b}$$

$$\dot{b}_\beta(\mathbf{P},\mathbf{p},t) = (i\hbar)^{-1}g_k^*(-\boldsymbol{\mu}^*\cdot\boldsymbol{\epsilon}_{\mathbf{k}}) \times \exp\left[-i\left((\omega_0-\Omega_k)+\frac{\mathbf{k}\cdot\mathbf{P}}{2m}+\omega_k^r\right)t\right] \times \left[\exp\left(i\,\frac{\mathbf{k}\cdot\mathbf{p}}{m}\,t\right)b_\alpha(\mathbf{P}+\hbar\mathbf{k},\mathbf{p}-\tfrac{1}{2}\hbar\mathbf{k},t) + \exp\left(-i\,\frac{\mathbf{k}\cdot\mathbf{p}}{m}\,t\right)b_{\alpha'}(\mathbf{P}+\hbar\mathbf{k},\mathbf{p}+\tfrac{1}{2}\hbar\mathbf{k},t)\right], \tag{8c}$$

where

$$\omega_k^r = \frac{\hbar k^2}{2m} \tag{9}$$

is a recoil frequency and $\boldsymbol{\mu} = \langle e|\boldsymbol{\mu}_1|g\rangle = \langle e|\boldsymbol{\mu}_2|g\rangle$ is a matrix element. Equations (8) are written in a resonance or rotating-wave approximation; counterrotating or antiresonance terms would contribute to single-atom level shifts [3,5], but single-atom level shifts are not considered in this work.

When Eq. (8c) is formally integrated over time and inserted back into Eq. (8b), one obtains

$$b_{\alpha'}(\mathbf{P},\mathbf{p},t) = -\hbar^{-2}\sum_{\mathbf{k}} |g_k|^2|\boldsymbol{\mu}\cdot\boldsymbol{\epsilon}_{\mathbf{k}}|^2 \times \int_0^t dt' \exp[i(\omega_0-\Omega_k)(t-t')] \times [b_{\alpha'}(\mathbf{P},\mathbf{p},t') + \exp\{2i[(\mathbf{k}\cdot\mathbf{p}/m)-\omega_k^r]t'\} \times b_\alpha(\mathbf{P},\mathbf{p}-\hbar\mathbf{k},t')], \tag{10}$$

with a similar equation for $b_\alpha(\mathbf{P},\mathbf{p},t)$. A common factor

$$\exp\left[i\left(\frac{\mathbf{k}\cdot\mathbf{P}}{2m}+\frac{\mathbf{k}\cdot\mathbf{p}}{m}-\omega_k^r\right)(t-t')\right]$$

has been omitted from Eq. (10) since it leads to corrections of order $P/(mc)$ or $p/(mc)$, which can be neglected in the nonrelativistic limit. When the sum over $\mathbf{k}$ is converted to an integral using the prescription $\Sigma_k \to [V/(2\pi c)^3]\int\Omega_k^2 d\Omega_k d\Theta_k$, the first term on the right-hand side of Eq. (10) gives $-[(\Gamma/2)+iS_0]b_{\alpha'}(\mathbf{P},\mathbf{p},t)$, where $\Gamma = \frac{4}{3}[\mu^2/(4\pi\varepsilon_0\hbar c)](\omega_0^3/c^2)$ is the single-atom excited-state decay rate and S_0 is a single-atom level shift. Although single-atom shifts are neglected, it is useful to point out that, within the context of this nonrelativistic calculation, the shift diverges as ω_c^3, where ω_c is a some cutoff used for the Ω_k integration. The origin of the divergence can be traced to *virtual* transitions in which an excited-state atom emits an off-resonance photon and reabsorbs it. The virtual state lives for a time of order $|\Omega_k-\omega_0|^{-1}$. There is no natural cutoff placed on Ω_k in this process by the energy-time uncertainty principle; the larger the detuning, the shorter the lifetime of the virtual state. We return to this point shortly in considering the exchange of energy between the two atoms. One also notes that the center-of-mass momentum appears simply as a spectator variable in Eq. (10), enabling one to write

$$b_\sigma(\mathbf{P},\mathbf{p},t) = b_\sigma(\mathbf{p},t)[(2\pi\hbar)^{3/2}V^{-1/2}\delta(\mathbf{P}-\mathbf{P}_0)], \tag{11}$$

where $\mathbf{P}_0$ is the center-of-mass momentum associated with the initial plane-wave state of the center-of-mass motion.

In light of the above comments, one can rewrite Eq. (10) as

$$\dot{b}_{\alpha'}(\mathbf{p},t) = -\gamma b_{\alpha'}(\mathbf{p},t) - \gamma\left(\frac{3}{8\pi^2}\right)\omega_0^{-3}\sum_{\nu=1,2} \times \int_0^\infty \Omega_k^3 d\Omega_k \int d\Theta_k \frac{|\boldsymbol{\mu}\cdot\boldsymbol{\epsilon}_{\mathbf{k}}^{(\nu)}|^2}{\mu^2} \times \int_0^t dt' \exp[i(\omega_0-\Omega_k)(t-t')] \times \exp\{2i[(\mathbf{k}\cdot\mathbf{p}/m)-\omega_k^r]t'\}b_\alpha(\mathbf{p}-\hbar\mathbf{k},t'), \tag{12}$$

where

$$\gamma = \Gamma/2$$

and the sum is over any two independent polarization vectors for a given $\mathbf{k}$. It is seen that the relative momentum $(\mathbf{P}_2-\mathbf{P}_1) = 2\mathbf{p}$ changes by $2\hbar\mathbf{k}$ on the exchange of radiation between the two atoms. Atom 1 recoils with momentum $-\hbar\mathbf{k}$ on emitting the radiation and atom 2 recoils by $+\hbar\mathbf{k}$ on absorbing it. It is sometimes convenient to use a mixed coordinate-momentum representation rather than a pure momentum state representation. If $b_\alpha(\mathbf{p},t)$ is expanded as

$$b_\alpha(\mathbf{p},t) = (2\pi\hbar)^{-3/2}\int d\mathbf{r} \times \exp[-(i/\hbar)(\mathbf{p}\cdot\mathbf{r}-E_p t)]b_\alpha(\mathbf{r},t), \tag{13}$$

and substituted into the integrand of Eq. (12), one finds

$$\dot{b}_{\alpha'}(\mathbf{p},t) = -\gamma b_{\alpha'}(\mathbf{p},t) - (2\pi\hbar)^{-3/2}\int_0^t dt' \int d\mathbf{r} \times \exp(-i\mathbf{p}\cdot\mathbf{r}/\hbar)\exp(iE_p t'/\hbar) \; G(\mathbf{r},t-t')b_{\alpha}(\mathbf{r},t'), \tag{14}$$

where

$$G(\mathbf{r},\tau) = \left(\frac{3}{8\pi^2}\right)\gamma\omega_0^{-3}\int_0^\infty \Omega_k^3 d\Omega_k \exp[i(\omega_0-\Omega_k)\tau] \times \sum_{\nu=1,2}\int d\Theta_k \frac{|\boldsymbol{\mu}\cdot\boldsymbol{\epsilon}_{\mathbf{k}}^{\nu}|^2}{\mu^2}\exp(i\mathbf{k}\cdot\mathbf{r}), \tag{15}$$

and $k=\Omega_k/c$.

The sum over ν and integral over Θ_k can be carried out using a geometry in which the k_z axis is taken along $\hat{\mathbf{r}}$ and the polarization vectors are $\boldsymbol{\epsilon}_{\mathbf{k}}^{(1)} = \cos\theta_k\cos\phi_k\mathbf{i} + \cos\theta_k\sin\phi_k\mathbf{j} - \sin\theta_k\hat{\mathbf{r}}$ and $\boldsymbol{\epsilon}_{\mathbf{k}}^{(2)} = -\sin\phi_k\mathbf{i} + \cos\phi_k\mathbf{j}$. One obtains

$$G(\mathbf{r},\tau) = \tfrac{3}{2}\gamma\pi^{-1}\omega_0^{-3}\int_0^\infty \Omega_k^3 d\Omega_k \exp[i(\omega_0-\Omega_k)\tau] \times\left[\left(\frac{\sin kr}{kr}+\frac{\cos kr}{(kr)^2}-\frac{\sin kr}{(kr)^3}\right)\sin^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}) + 2\left(-\frac{\cos kr}{(kr)^2}+\frac{\sin kr}{(kr)^3}\right)\cos^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}})\right], \tag{16}$$

where

$$\sin^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}) = \frac{|\mu_x|^2+|\mu_y|^2}{\mu^2}, \quad \cos^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}) = \frac{|\mu_z|^2}{\mu^2}. \tag{16'}$$

In contrast to the integral that appears in the theory of single-atom decay, this integral is not divergent. There is a natural cutoff in the integral that results from the exchange of energy between the two atoms. Atom 1 can emit radiation at any frequency Ω_k, which is then reabsorbed by atom 2. However, the reabsorption cannot occur instantaneously as it could in the single-atom case. There is a time delay τ for absorption equal to r/c. For separations $r \gtrsim \lambda_0$ considered in this paper, conservation of energy limits the maximum frequency of the radiation exchanged between the two atoms to be of order $(\Omega_k-\omega_0) \lesssim \tau^{-1} \sim \omega_0$. The integral can be evaluated by: (i) writing $k=k_0+(k-k_0)$, where $k_0=\omega_0/c$, (ii) evaluating Ω_k at ω_0 and k at k_0, *except* in the arguments of the sin, cos, and exp functions, and (iii) extending the lower bound of the integral to $-\infty$. When this program is carried out using the fact that $t \geqslant t'$, one finds

$$G(\mathbf{r},\tau) = G(\mathbf{r})\,\delta(\tau - r/c), \tag{17}$$

where

$$G(\mathbf{r}) = \frac{3}{2}\gamma\left[\left(-\frac{ie^{ik_0r}}{k_0r}+\frac{e^{ik_0r}}{(k_0r)^2}+\frac{ie^{ik_0r}}{(k_0r)^3}\right)\sin^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}) + 2\left(-\frac{\mathbf{e}^{\mathbf{ik_0r}}}{(\mathbf{k_0r})^2}-\frac{\mathbf{ie}^{\mathbf{ik_0r}}}{(\mathbf{k_0r})^3}\right)\cos^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}})\right]. \tag{18}$$

When this result is substituted back into Eq. (14), one obtains

$$\dot{b}_{\alpha'}(\mathbf{p},t) = -\gamma b_{\alpha'}(\mathbf{p},t) - (2\pi\hbar)^{-3/2}\int d\mathbf{r}\; G(\mathbf{r}) \times \exp(-i\mathbf{p}\cdot\mathbf{r}/\hbar)\exp[iE_p(t-r/c)/\hbar] \times b_{\alpha}(\mathbf{r},t-r/c), \tag{19}$$

in which the retardation is indicated explicitly. The corresponding equation for $\dot{b}_{\alpha}$ is obtained by interchanging α and α'. A somewhat more rigorous evaluation of the integral in Eq. (16) is given in the Appendix, where it is pointed out that the use of the δ function in Eq. (17) is strictly valid only for $k_0r \gg 2\pi$ and $|ct/r-1| \gg 1/k_0r$. However, in the near zone $k_0r<1$, one can still use Eq. (17) since the difference between the retarded and actual times is of order $r/c < 1/k_0c \sim 1/\omega_0$ and can be neglected.

When $k_0r \lesssim 1$, Eq. (19) can be recast in a form that allows for a simple physical interpretation. Starting with the transformation to the $\mathbf{r}$ representation,

$$b_{\alpha}(\mathbf{r},t) = (2\pi\hbar)^{-3/2}\int d\mathbf{p}\; \exp(i\mathbf{p}\cdot\mathbf{r}/\hbar) \times \exp(-iE_pt/\hbar)b_{\alpha}(\mathbf{p},t), \tag{20}$$

one can differentiate Eq. (20) with respect to time and use Eq. (19), along with the fact that $E_pr/\hbar c \sim E_p/\hbar\omega_0 \ll 1$, to obtain

$$i\hbar\dot{b}_{\alpha'}(\mathbf{r},t) = -(\hbar^2/m)\nabla_r^2 b_{\alpha'}(\mathbf{r},t) - i\hbar\gamma b_{\alpha'}(\mathbf{r},t) - i\hbar G(r)b_{\alpha}(\mathbf{r},t-r/c). \tag{21}$$

If one defines

$$b_{\pm} = (b_{\alpha} \pm b_{\alpha'})/\sqrt{2}, \tag{22}$$

sets

$$G(\mathbf{r}) = \gamma(\mathbf{r}) + is(\mathbf{r}), \tag{23}$$

and uses Eq. (21) and the corresponding equation with α and α' interchanged, one finds

$$i\hbar\dot{b}_{\pm}(\mathbf{r},t) = -(\hbar^2/m)\nabla_r^2 b_{\pm}(\mathbf{r},t) \pm \hbar s(\mathbf{r})b_{\pm}(\mathbf{r},t-r/c) - i\hbar\gamma b_{\pm}(\mathbf{r},t) \mp i\hbar\gamma(\mathbf{r})b_{\pm}(\mathbf{r},t-r/c). \tag{24}$$

The equations are uncoupled in the ''$\pm$'' basis. If retardation is neglected, the equations become

$$i\hbar\dot{b}_{\pm}(\mathbf{r},t) = -(\hbar^2/m)\nabla_r^2 b_{\pm}(\mathbf{r},t) \pm \hbar s(\mathbf{r})b_{\pm}(\mathbf{r},t) - i\hbar\gamma_{\pm}(\mathbf{r})b_{\pm}(\mathbf{r},t), \tag{25}$$

where

$$\gamma_{\pm}(\mathbf{r}) = \gamma \pm \gamma(\mathbf{r}). \tag{26}$$

In this limit, the $\pm$ states move along their respective potential-energy curves as they undergo spontaneous decay with a rate that is a function of the interatomic separation.

III. EMISSION SPECTRUM FOR $k_0 r \gg 2\pi$

For large separations $k_0 r \gg 2\pi$, the dynamics of spontaneous emission is pretty much what one would expect, although there are a few interesting wrinkles. The initial state for the system has atom 1 excited and atom 2 in its ground state. For definiteness, the wave function for the relative coordinate at time $t=0$ is taken as

$$\psi_0(\mathbf{r}) = [\pi(\Delta r)^2]^{-3/4} \exp[-\tfrac{1}{2}(|\mathbf{r}-\mathbf{r}_0|/\Delta r)^2], \quad (27)$$

with the corresponding momentum-space wave function given by

$$b_\alpha(\mathbf{p},0) = [\pi(\Delta p)^2]^{-3/4} \exp[-\tfrac{1}{2}(p/\Delta p)^2] \times \exp(-i\mathbf{p}\cdot\mathbf{r}_0/\hbar), \quad b_{\alpha'}(\mathbf{p},0) = 0, \quad (28)$$

where $\Delta p = \hbar/\Delta r$. For $k_0 r_0 \gg 2\pi$, it is possible to choose $r_0 \gg \Delta r \gg \lambda$ such that $\Delta p \ll \hbar k_0$. With this choice of Δp, the momentum spread of the packet is less than the momentum with which the atom recoils when it absorbs or emits radiation.

To order $(k_0 r_0)^{-2}$, the emission spectrum arises from three channels. (i) Atom 1 decays by emitting radiation in an arbitrary direction. The amplitude for this channel is $(k_0 r_0)^0$. (ii) Atom 1 exchanges its excitation with atom 2, followed by emission from atom 2. Owing to recoil, the exchange process does not conserve energy, as is seen below. The amplitude for this channel is $(k_0 r_0)^{-1}$. (iii) It is also possible for atom 1 to exchange its excitation with atom 2 and atom 2 to reexchange the excitation with atom 1, followed by emission from atom 1. In principle, this channel can interfere with the first channel, giving rise to a contribution to the emission spectrum of order $(k_0 r_0)^{-2}$; however, the momentum of atom 1 after this double exchange differs from the initial momentum by $-2\hbar k_0 \hat{\mathbf{r}}_0$. As a consequence, the interference term vanishes since there is no overlap of the wave packets for the two channels when $\Delta p \ll \hbar k_0$. As a result, the contribution from this channel is neglected.

The emission spectrum can be defined as

$$I(\mathbf{k},\boldsymbol{\epsilon}) = \int d\mathbf{P} \int d\mathbf{p} |b_\beta(\mathbf{P},\mathbf{p},\infty)|^2, \quad (29)$$

that is, the probability to find both atoms in their ground states and emission into the mode $(\mathbf{k},\boldsymbol{\epsilon})$. The amplitude $b_\beta(\mathbf{P},\mathbf{p},t)$ can be found using Eqs. (8c) and (11) once we have expressions for $b_\alpha(\mathbf{p}-\frac{1}{2}\hbar\mathbf{k},t)$ and $b_{\alpha'}(\mathbf{p}+\frac{1}{2}\hbar\mathbf{k},\mathbf{t})$, which in turn can be obtained as solutions of Eq. (19) and the corresponding equation with α and α' interchanged. For $k_0 r_0 \gg 2\pi$ and $k_0 r_0 (\Delta r/r_0)^2 \ll 1$,

$$G(\mathbf{r}) \sim \tfrac{3}{2}\gamma \left(-\frac{i e^{i\mathbf{k}_0\cdot\mathbf{r}}}{k_0 r} \right) \sin^2 \alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}), \quad (30)$$

where

$$\mathbf{k}_0 = k_0 \hat{\mathbf{r}}_0. \quad (31)$$

Moreover, the energy E_p appearing in Eq. (19) can be written as

$$E_p = E_{p-\hbar\mathbf{k}_0} + 2\hbar\mathbf{k}_0\cdot\mathbf{p}/m + 2\hbar\omega^r_{k_0}, \quad (32)$$

where $\omega^r_{k_0}$ is given by Eq. (9). Substituting Eqs. (30) and (32) into Eq. (19), evaluating all terms in the integrand at $\mathbf{r}=\mathbf{r}_0$ *except* for the phase factors $e^{i\mathbf{k}_0\cdot\mathbf{r}}$ and $e^{-i\mathbf{p}_0\cdot\mathbf{r}/\hbar}$, and using Eq. (13), one finds that Eq. (19) reduces to

$$\dot{b}_{\alpha'}(\mathbf{p},t) = -\gamma b_{\alpha'}(\mathbf{p},t) + \tfrac{3}{2} i\gamma (k_0 r_0)^{-1} \sin^2 \alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}_0) \times \exp(2i\mathbf{k}_0\cdot\mathbf{p}t/m) \exp(-2i\omega^r_{k_0} t) \times b_\alpha(\mathbf{p}-\hbar\mathbf{k}_0, t-r_0/c), \quad (33)$$

with a similar equation for $\dot{b}_\alpha(\mathbf{p},t)$. These equations are now solved to zeroth order in $(k_0 r_0)^{-1}$ for b_α and first order in $(k_0 r_0)^{-1}$ for $b_{\alpha'}$ since this will give the emission spectrum correct to order $(k_0 r_0)^{-2}$.

To zeroth order in $(k_0 r_0)^{-1}$,

$$b_\alpha(\mathbf{p},t) = b_\alpha(\mathbf{p},0) \exp(-\gamma t)\Theta(t),$$

where $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$. When this solution is substituted into Eq. (33), one finds that, to order $(k_0 r_0)^{-1}$, $b_{\alpha'}$ is given as a solution to

$$\dot{b}_{\alpha'}(\mathbf{p},t) = -\gamma b_{\alpha'}(\mathbf{p},t) + \tfrac{3}{2} i\gamma (k_0 r_0)^{-1} \sin^2 \alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}_0) \times \exp(2i\mathbf{k}_0\cdot\mathbf{p}t/m) \exp(-2i\omega^r_{k_0} t) \times b_\alpha(\mathbf{p}-\hbar\mathbf{k}_0, 0) \exp(-\gamma t)\Theta(t-r_0/c), \quad (34)$$

where it has been assumed that $\gamma r_0/c \ll 1$. The initial wave function $b_\alpha(\mathbf{p},0)$ is sharply peaked about $\mathbf{p}=\mathbf{0}$, having a width $\Delta p \ll \hbar k_0$. This fact enables one to evaluate the momentum $\mathbf{p}$ appearing in the exponent in Eq. (34) at $\mathbf{p}=\hbar\mathbf{k}_0$ and to rewrite this equation as

$$\dot{b}_{\alpha'}(\mathbf{p},t) = -\gamma b_{\alpha'}(\mathbf{p},t) + \tfrac{3}{2} i\gamma (k_0 r_0)^{-1} \sin^2 \alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}_0) \times \exp(2i\omega^r_{k_0} t) b_\alpha(\mathbf{p}-\hbar\mathbf{k}_0, 0)\Theta(t-r_0/c). \quad (35)$$

This equation directly reflects the dynamics of the excitation exchange between the two atoms. Initially atom 1 is excited and the relative momentum of the two atoms is centered about $\mathbf{p}=\mathbf{0}$. When atom 2 gets excited, the relative momentum is shifted to $\mathbf{p}=\hbar\mathbf{k}_0$, which corresponds to atom 1 recoiling with momentum $-\hbar\mathbf{k}_0$ and atom 2 recoiling with momentum $\hbar\mathbf{k}_0$ as a result of emission and reabsorption of radiation emitted in the $\hat{\mathbf{r}}_0$ direction connecting the two atoms. Moreover, the exponential term in Eq. (35) indicates that there is an energy mismatch of $2\hbar\omega_{k_0}$ between states α and α'. It is easy to understand how this mismatch arises. The energy associated with the initial state of relative motion is zero, but that associated with the two atoms recoiling after the excitation exchange is $2(\hbar k_0)^2/2m = 2\hbar\omega^r_{k_0}$.

It is now possible to calculate $b_\beta(\mathbf{P},\mathbf{p},\infty)$ [recall that the emission spectrum is equal to $|b_\beta(\mathbf{P},\mathbf{p},\infty)|^2$] using Eqs. (8c), (35), and (9), along with the fact that $\Delta p < \hbar k_0$. After carrying out the integrations, one finds

$$b_\beta(\mathbf{P},\mathbf{p},\infty) = (i\hbar)^{-1} g_k^*(-\boldsymbol{\mu}^*\cdot\boldsymbol{\epsilon}_\mathbf{k}) \times\left[\frac{b_\alpha(\mathbf{P}+\hbar\mathbf{k},\mathbf{p}-\frac{1}{2}\hbar\mathbf{k},0)}{d_1}+\left(\frac{\frac{3}{2}i\gamma\,\sin^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}_0)}{k_0 r_0}\right)\times\frac{b_\alpha(\mathbf{P}+\hbar\mathbf{k},\mathbf{p}-\hbar\mathbf{k}_0+\frac{1}{2}\hbar\mathbf{k},0)}{d_2(d_2-2i\omega^r_{k_0})},\right] \tag{36}$$

where

$$d_1=\gamma+i\left((\omega_0-\Omega_k)+\frac{\mathbf{k}\cdot\mathbf{P}_0}{2m}-\omega_k^r\right), \qquad d_2=d_1+i\hbar\mathbf{k}_0\cdot\mathbf{k}/m, \tag{37}$$

and it has been assumed that $|d_i| r_0/c \ll 1$, $i=1,2$.

Using Eqs. (29), (36), (11), and (28), one finds that the emission spectrum is given by [7]

$$I(\mathbf{k},\boldsymbol{\epsilon})=|g_k\boldsymbol{\mu}\cdot\boldsymbol{\epsilon}_\mathbf{k}/\hbar|^2\left[\frac{1}{|d_1|^2}+\left|\frac{(3/2)\gamma\,\sin^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}_0)}{(k_0r_0)d_2(d_2-2i\omega^r_{k_0})}\right|^2 -2\,\mathrm{Re}\left\{\frac{\frac{3}{2}i\gamma\,\sin^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}_0)}{(k_0r_0)d_1d_2^*(d_2-2i\omega^r_{k_0})^*}\,e^{ikr_0(\cos\theta_{k_0}-1)}e^{-|\hbar(\mathbf{k}-\mathbf{k}_0)/(2\Delta p)|^2}\right\}\right], \tag{38}$$

where θ_{k_0} is the angle between $\mathbf{k}$ and $\mathbf{k}_0=k_0\hat{\mathbf{r}}_0$. The spectrum consists of three terms. The first term is associated with emission from atom 1 (channel 1) and is centered at

$$\Omega_k(1)=\omega_0+\frac{\mathbf{k}\cdot\mathbf{P}_0}{2m}-\omega_k^r. \tag{39}$$

The second term is associated with exchange of excitation between atoms 1 and 2 followed by emission from atom 2 (channel 2) and consists of a *doublet*, centered at frequencies

$$\Omega_k(2)=\Omega_k(1)+\frac{\hbar\mathbf{k}_0\cdot\mathbf{k}}{m}, \tag{40a}$$

$$\Omega_k(3)=\Omega_k(2)-2\omega^r_{k_0}. \tag{40b}$$

The third term consists of interference between channels 1 and 2 and contributes only in the forward direction $\mathbf{k}=\mathbf{k}_0$. The width (full width at half maximum) associated with each of the resonances is $\Gamma=2\gamma$. If $\omega^r_{k_0}>\Gamma$, the spectrum can be resolved into three components, with emission from atom 2 *spectrally* resolved from emission from atom 1. This is the spectral analog of the fact that the radiation pattern from a two-atom system can be used to distinguish from which atom the emission has occurred provided that $k_0 r_0 \gg 2\pi$.

The emission spectrum (38) is the principle result of this section. As a result of recoil, the emission spectrum is a triplet and the terms corresponding to emission from atom 2 can be distinguished from the term corresponding to emission from atom 1. If $\omega^r_{k_0}<\Gamma$, as is typically the case, it may not be possible to resolve the spectral components; however, this result is somewhat unimportant for the present discussion. What has been demonstrated is that, in principle, recoil leads to spectral components that can be resolved if $k_0 r_0 \gg 2\pi$.

The dynamics of the emission process is also interesting. Emission can be viewed in terms of sequential emission by the individual atoms rather than in terms of emission from the composite system of the atoms. The resonance positions (39) and (40) can be given a simple physical interpretation in terms of the emission process. The initial (average) momenta of the atoms are $\mathbf{p}_1=\mathbf{p}_2=\mathbf{P}_0/2$ and the initial energy of the system is

$$E_1=\frac{(P_0/2)^2}{2m}+\frac{(P_0/2)^2}{2m}+\hbar\omega_0. \tag{41}$$

Following emission of a photon by atom 1 into mode $\mathbf{k}$ of the radiation field, the momentum of atom 2 is unchanged, the momentum of atom 1 is $\mathbf{P}_0/2-\hbar\mathbf{k}$, the energy of the atoms is

$$E_2(\mathbf{k})=\frac{(\mathbf{P}_0/2-\hbar\mathbf{k})^2}{2m}+\frac{(P_0/2)^2}{2m}, \tag{42}$$

and the energy in the field is $\hbar\Omega_k$. If emission is in the $\hat{\mathbf{r}}_0$ direction, there can be an exchange of excitation between atoms 1 and 2. Following absorption by atom 2 of the photon emitted into mode $\mathbf{k}_0=k_0\hat{\mathbf{r}}_0$ by atom 1, the momentum of atom 1 is $\mathbf{P}_0/2-\hbar\mathbf{k}_0$, the momentum of atom 2 is $\mathbf{P}_0/2+\hbar\mathbf{k}_0$, and the total energy of the system is

$$E_3(\mathbf{k}_0)=\frac{(\mathbf{P}_0/2-\hbar\mathbf{k}_0)^2}{2m}+\frac{(\mathbf{P}_0/2+\hbar\mathbf{k}_0)^2}{2m}+\hbar\omega_0. \tag{43}$$

Finally, following emission by atom 2 into mode $\mathbf{k}$ of the vacuum field, the momentum of atom 2 changes to $\mathbf{P}_0/2+\hbar\mathbf{k}_0-\hbar\mathbf{k}$, the final energy of the atoms is

$$E_4(\mathbf{k}_0,\mathbf{k})=\frac{(\mathbf{P}_0/2-\hbar\mathbf{k}_0)^2}{2m}+\frac{(\mathbf{P}_0/2+\hbar\mathbf{k}_0-\hbar\mathbf{k})^2}{2m}, \tag{44}$$

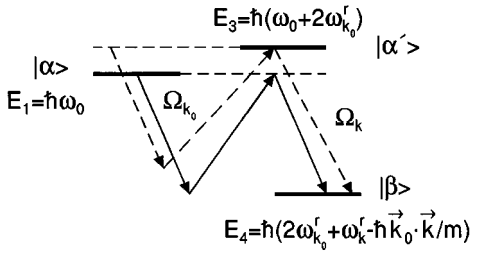

FIG. 2. Energy-level diagram for the two-atom system, including recoil, appropriate to the emission channel in which atom 1 exchanges its excitation energy with atom 2 via emission and reabsorption of a photon having frequency Ω_{k_0}, followed by emission from atom 2 of a photon having frequency Ω_k. Owing to recoil, state $|\alpha'\rangle$ differs in energy from state $|\alpha\rangle$ by $2\hbar\omega_{k_0}^r$. The solid arrows indicate a process that is resonant when $\Omega_k = E_1 - E_4 = \omega_0 - \omega_k^r - 2\omega_{k_0}^r + \hbar\mathbf{k}_0\cdot\mathbf{k}/m$, while the dashed arrows, starting in the Lorentzian tail of state $|\alpha\rangle$, indicate a process that is resonant when $\Omega_k = E_3 - E_4 = \omega_0 - \omega_k^r + \hbar\mathbf{k}_0\cdot\mathbf{k}/m$. The energies shown are for the case when the initial center-of-mass momentum equals zero.

and the energy in the field is $\hbar\Omega_k$.

Emission by atom 1 into mode $\mathbf{k}$ is a one-photon process that is resonant when $\hbar\Omega_k = E_1 - E_2(\mathbf{k})$, leading to Eq. (39). Emission by atom 2 into mode $\mathbf{k}$ is a *three-photon* process (see Fig. 2) in which a $\mathbf{k}_0$ photon is exchanged between atom 1 and 2, followed by emission of radiation into mode $\mathbf{k}$ by atom 2. As shown in Fig. 2, this process can be resonantly enhanced if the emission into mode $\mathbf{k}$ is such that (a) the three-photon process is resonant $[\hbar\Omega_k = E_1 - E_4(\mathbf{k}_0,\mathbf{k})]$, leading to Eq. (40b), or (b) the single-photon process from state 3 to state 4 is resonant $[\hbar\Omega_k = E_3(\mathbf{k}_0) - E_4(\mathbf{k}_0,\mathbf{k})]$, leading to Eq. (40a). In some sense this overall process can be viewed as a cascade emission in a three-level system. The initial state $|\alpha\rangle$ having energy E_1 is driven by the (emission and reabsorption) of the vacuum field to state $|\alpha'\rangle$ having energy E_3. The effective frequency of this two-photon driving field is *zero* since the same photon is emitted and absorbed in the excitation exchange. From state $|\alpha'\rangle$, the vacuum field mode having frequency Ω_k drives the system to state $|\beta\rangle$ having energy E_4.

Finally, it is of some interest to calculate the integrated spectrum

$$I = I_1 + I_2 + I_3 = \sum_{\boldsymbol{\epsilon}} \int d\mathbf{k}[I_1(\mathbf{k},\boldsymbol{\epsilon}) + I_2(\mathbf{k},\boldsymbol{\epsilon}) + I_3(\mathbf{k},\boldsymbol{\epsilon})], \tag{45}$$

where the three terms on the right-hand side of Eq. (45) correspond to the three terms in Eq. (38). The $\exp[ikr_0(\cos\theta_{k_0} - 1)]$ factor in $I_3(\mathbf{k},\boldsymbol{\epsilon})$ is rapidly varying in all but the $\theta_{k_0} = 0$ direction. Evaluating all the other factors in this term (which are slowly varying compared to this exponential factor) at $\theta_{k_0} = 0$, it is possible to carry out the integration over φ_k and θ_k to show explicitly that this term exactly cancels I_2. This cancellation is nothing more than a manifestation of the optical theorem. As a result, from Eqs. (45) and (38), one finds that $I = 1$, as it must since it represents the probability to find both atoms in their ground states and a photon present at time $t = \infty$. Using Eqs. (45) and (38) one finds that the ratio of emission into channel 2 to that in channel 1 is equal to

$$I_2/I_1 = \frac{\frac{9}{8}\sin^4\alpha_\mu(\hat{\mathbf{r}}_0)}{(k_0 r_0)^2}\,\frac{\Gamma^2}{\Gamma^2 + (2\omega_{k_0}^r)^2}.$$

This ratio clearly shows that the excitation exchange occurs with a frequency defect of $2\omega_{k_0}^r$. The width 2Γ represents the sum of the widths of the initial (state α) and final (state α') states involved in the excitation exchange.

IV. EMISSION SPECTRUM FOR $k_0 r \ll 1$

In calculating the emission spectrum for $k_0 r \ll 1$, it is convenient to express the amplitudes on the right-hand side of Eq. (8c) in terms of $b_\pm(\mathbf{r},t)$ defined in Eqs. (20) and (22). Using Eqs. (8c), (13), (22), and (11), one can rewrite Eq. (8c) as

$$\begin{aligned}\dot{b}_\beta(\mathbf{P},\mathbf{p},t) = {} & (i\hbar)^{-1}(2\pi\hbar)^{-3/2}\sqrt{2}\,g_k^*(-\boldsymbol{\mu}^*\cdot\boldsymbol{\epsilon}_{\mathbf{k}}) \\ & \times\exp[(-d_3+\gamma)t]\int d\mathbf{r}\,\exp(-i\mathbf{p}\cdot\mathbf{r}/\hbar) \\ & \times\exp(iE_p t/\hbar)]\left[\cos\left(\frac{\mathbf{k}\cdot\mathbf{r}}{2}\right)b_+(\mathbf{r},t)\right. \\ & \left. + i\,\sin\left(\frac{\mathbf{k}\cdot\mathbf{r}}{2}\right)b_-(\mathbf{r},t)\right] \\ & \times[(2\pi\hbar)^{3/2}V^{-1/2}\delta(\mathbf{P}+\hbar\mathbf{k}-\mathbf{P}_0)],\end{aligned} \tag{46}$$

where

$$d_3 = \gamma + i\left((\omega_0 - \Omega_k) + \frac{\mathbf{k}\cdot\mathbf{P}_0}{2m} - \frac{\omega_k^r}{2}\right). \tag{47}$$

This equation is valid for arbitrary $k_0 r$. In order to solve these equations, we must solve the Schrödinger equations for $b_\pm(\mathbf{r},t)$ in the presence of the complex potentials $\hbar[\pm s(r) - i\gamma_\pm(r)]$. In the region $k_0 r \ll 1$, it follows from Eqs. (18), (23), and (26) that

$$\begin{aligned} s(\mathbf{r}) &\sim \tfrac{3}{2}\gamma[\sin^2\alpha_\mu(\hat{\mathbf{r}}) - 2\cos^2\alpha_\mu(\hat{\mathbf{r}})]/(k_0 r)^3 \gg \gamma, \\ \gamma_+(r) &\sim 2\gamma, \\ \gamma_-(\mathbf{r}) &\sim \gamma[2\sin^2\alpha_\mu(\hat{\mathbf{r}}) + \cos^2\alpha_\mu(\hat{\mathbf{r}})](k_0 r)^2/10. \end{aligned} \tag{48}$$

Immediately, one runs into a problem that is not discussed often in considering two-atom superradiance. We are interested in emission when the atoms are separated by a distance $r \ll 1/k_0$, but the potential $\hbar s(r)$ is sufficiently strong to cause atoms to move a distance of order or greater than $1/k_0$ for time scales of order $\tau_\pm = 1/(2\gamma_\pm)$ relevant to the emission process. For example, for the symmetric $(+)$ state, the relevant time scale is $\tau_+ = 1/(2\gamma)$. On this time scale, the atom moves a distance of order $|\{\boldsymbol{\nabla}[s(r)]/m\}\tau_+^2| \sim (\omega_{k_0}^r/\Gamma) r_0 (k_0 r_0)^{-5}$. If we ask that the distance the atom moves be small compared to r_0, then we must require that

$k_0 r_0 > (\omega^r_{k_0}/\Gamma)^{1/5}$, which equals 0.25 for $(\omega^r_{k_0}/\Gamma)=0.001$. For the antisymmetric states, the lifetime is increased by a factor $10(k_0 r_0)^{-2}$, leading to the more stringent condition $k_0 r_0 > (100\omega^r_{k_0}/\Gamma)^{1/9}$, which equals 0.77 for $\omega^r_{k_0}/\Gamma=0.001$. Thus, for reasonable ratios of $\omega^r_{k_0}$ to Γ (typically $\omega^r_{k_0}/\Gamma \gtrsim 0.001$), it is not possible to restrict the atomic separation to distances $k_0 r \ll 1$ owing to the strength of the potential. To simplify the discussion, however, I assume that the ratio $\omega^r_{k_0}/\Gamma$ is sufficiently small to guarantee that the atomic separation does not change significantly during the emission process. Even if this corresponds to somewhat unphysical values of the parameters, it will afford us a qualitative picture of emission and the role of recoil in the region $k_0 r \ll 1$. Since $\omega^r_{k_0}/\Gamma \ll 1$, any recoil shifts are negligibly small compared to the natural linewidth.

The assumption that $\omega^r_{k_0}/\Gamma \ll 1$ allows one to neglect spreading of the wave packet on the time scale of emission [the wave packet spreads by a distance of order $(\Delta p/m)\tau_\pm \sim [\hbar/(m\Delta r)]\tau_\pm > [\hbar/(mr_0)]\tau_\pm \sim (\omega^r_{k_0}/\Gamma) r_0 (k_0 r_0)^{-2}$ for the symmetric state and $(10\omega^r_{k_0}/\Gamma) r_0 (k_0 r_0)^{-4}$ for the antisymmetric state]. Thus the symmetric and antisymmetric states can be thought to move on classical trajectories during the emission process. Motion along the classical trajectories leads to phase factors in the wave functions of the form $\exp\{\mp i \int_0^t s[\mathbf{r}(t')]dt'\}$. To further simplify matters, it is assumed that the integrands in these phase factors can be evaluated at $\mathbf{r}(t')=\mathbf{r}_0$ (this amounts to neglect of phase changes of order $m|\nabla[s(r)]/m|^2 \tau_\pm^3 \sim (\omega^r_{k_0}/\Gamma) r_0 (k_0 r_0)^{-8}$ for the symmetric state and $(1000\omega^r_{k_0}/\Gamma)(k_0 r_0)^{-14}$ for the antisymmetric state). This latter assumption is not essential to the calculation, but does allow one to arrive at analytical expressions for the spectrum. Had this assumption not been adopted, one would find that changes in the potential during the emission process result in a broadening of the emission lines.

In light of the above approximations, the solutions of Eq. (19) and the corresponding equation with α and α' interchanged are

$$b_\pm(\mathbf{p},t) = e^{-\gamma_\pm(\mathbf{r}_0) i \mp i s(\mathbf{r}_0) t} b_\pm(\mathbf{p},0), \tag{49}$$

where Eq. (27) has been used and retardation has been neglected. The spectrum is obtained by expanding the sin and cos terms in Eq. (46) to lowest order in $\mathbf{k}\cdot\mathbf{r}$ to obtain

$$\begin{aligned}\dot{b}_\beta(\mathbf{P},\mathbf{p},t) = {} & (i\hbar)^{-1}\sqrt{2} g_k^*(-\boldsymbol{\mu}^*\cdot\boldsymbol{\epsilon}_\mathbf{k})\exp[(-d_3+\gamma)t] \\ & \times [b_+(\mathbf{p},t) - \tfrac{1}{2}\hbar\mathbf{k}\cdot\nabla_\mathbf{p} b_-(\mathbf{p},t)] \\ & \times [(2\pi\hbar)^{3/2} V^{-1/2}\delta(\mathbf{P}+\hbar\mathbf{k}-\mathbf{P}_0)].\end{aligned} \tag{50}$$

Then, using Eqs. (29), (50), (49), and (28), along with the fact that the + and − components are spectrally distinct since $s(\mathbf{r}_0) \gg \gamma_\pm(\mathbf{r}_0)$ and the fact that $\Delta p \gg (\hbar/r_0)$ since $\Delta r \ll r_0$, one finds that the spectrum can be expressed as

$$I(\mathbf{k},\boldsymbol{\epsilon}) = |g_k \boldsymbol{\mu}\cdot\boldsymbol{\epsilon}_\mathbf{k}/\hbar|^2 \left[\frac{1}{|d_+|^2} + \frac{|\mathbf{k}\cdot\mathbf{r}_0|^2}{4|d_-|^2}\right], \tag{51}$$

where

$$d_\pm = \gamma_\pm(\mathbf{r}_0) + i\left((\omega_0-\Omega_k) \pm s(\mathbf{r}_0) + \frac{\mathbf{k}\cdot\mathbf{P}_0}{2m} - \frac{\omega^r_k}{2}\right). \tag{52}$$

The spectrum consists of a doublet, split in frequency by $2s(\mathbf{r}_0)$. The strength of the − component at line center is $25|\mathbf{k}\cdot\mathbf{r}_0|^2/\{[2\sin^2\alpha_\mu(\hat{\mathbf{r}}) + \cos^2\alpha_\mu(\hat{\mathbf{r}})](k_0 r)^2\}^2$ times *larger* than that of the + component at line center and the width of the − component is $[2\sin^2\alpha_\mu(\hat{\mathbf{r}}) + \cos^2\alpha_\mu(\hat{\mathbf{r}})](k_0 r)^2/20$ times smaller than that of the + component. The integrated intensities of the two components are equal. Although the recoil shift is much smaller than $\gamma_\pm$, it is important to note that the recoil shift of each spectral component $\omega^r_k/2 = \hbar k^2/[2(2m)]$ corresponds to recoil of the *two-atom system*. For $k_0 r \ll 1$, the atoms emit as a composite system, which is consistent with the idea that it is impossible to distinguish which atom of a system of two identical atoms has emitted radiation if the separation of the atoms is much smaller than a wavelength.

To gain additional insight into the emission process, one can redo the calculation starting directly from Eq. (8c). Using Eqs. (29), (8c), (22), (49), and (11), one finds that the contributions to the emission spectrum for the + and − components are given by

$$\begin{aligned} I_\pm(\mathbf{k},\boldsymbol{\epsilon}) = {} & \frac{1}{2}|g_k\boldsymbol{\mu}\cdot\boldsymbol{\epsilon}_\mathbf{k}/\hbar|^2 \\ & \times \int d\mathbf{p} \left| \frac{b_\pm\left(\mathbf{p}-\frac{\hbar\mathbf{k}}{2},0\right)}{\left|\delta_\pm - \frac{i\mathbf{p}\cdot\mathbf{k}}{m}\right|^2} \pm \frac{b_\pm\left(\mathbf{p}+\frac{\hbar\mathbf{k}}{2},0\right)}{\left|\delta_\pm + \frac{i\mathbf{p}\cdot\mathbf{k}}{m}\right|^2}\right|^2, \end{aligned} \tag{53}$$

where

$$\delta_\pm = \gamma_\pm(\mathbf{r}_0) + i\left((\omega_0-\Omega_k) \pm s(\mathbf{r}_0) + \frac{\mathbf{k}\cdot\mathbf{P}_0}{2m}\right). \tag{54}$$

When the initial wave function (28) is substituted into Eq. (53), one finds four terms in the integrand. Owing to the exponential factors in the wave function, the first term is sharply peaked at $\mathbf{p}=\hbar\mathbf{k}/2$, the second at $\mathbf{p}=-\hbar\mathbf{k}/2$, while the cross terms contain a factor $\exp[-(p^2+\hbar^2k^2/2)/\Delta p^2]\exp(\pm i\mathbf{k}\cdot\mathbf{r}_0) \simeq \exp[-p^2/\Delta p^2]\exp(\pm i\mathbf{k}\cdot\mathbf{r}_0)$, which is sharply peaked at $\mathbf{p}=\mathbf{0}$. Evaluating the slowly varying denominators of these terms at these values of $\mathbf{p}$ and carrying out the integration over $\mathbf{p}$, one finds that the spectral components can be written as

$$I_\pm(\mathbf{k},\boldsymbol{\epsilon}) = \frac{1}{2}|g_k\boldsymbol{\mu}\cdot\boldsymbol{\epsilon}_\mathbf{k}/\hbar|^2\left[\frac{1}{|\delta_\pm - i\omega^r_k|^2} \pm \frac{\cos(\mathbf{k}\cdot\mathbf{r}_0)}{|\delta_\pm|^2}\right]. \tag{55}$$

The recoil shift of the first component is $-\omega^r_k$, while the interference term has no recoil shift. These two terms combine to give an overall recoil shift of $-\omega^r_k/2$, characteristic of emission from a composite, two-atom system. The denominators in Eq. (55) can be rewritten $(d_\pm \mp i\omega^r_k/2)$; it is

then easy to show that the sum of $I_+ + I_-$ evaluated from Eq. (55) reduces to Eq. (51) in the limit that $kr_0 \ll 1$ and $(\omega_k^r/\gamma)^2 (kr_0)^{-4} \ll 1$, as has been assumed.

The spectrum written in the form (53) or (55) is particularly revealing. The terms corresponding to interference in the radiation emitted by atoms contain phase factors $\exp(\pm i\mathbf{k}\cdot\mathbf{r}_0)$. For $kr_0 \ll 1$, these phase factors are of order unity and the interference terms contribute for all directions of the emitted radiation; moreover, the interference terms are comparable in magnitude to the direct terms. Thus the atoms emit as a composite system. On the other hand, for $kr_0 \gg 1$, the interference terms contribute only for radiation emitted along the line connecting the two atoms and are smaller than the direct terms by a factor $(kr_0)^{-1}$.

It has already been noted that the use of the retarded times in the evolution equations is not strictly valid for $k_0 r_0 \ll 1$, although its use introduces negligible corrections. Just as the retarded time loses its significance in the near zone, the interpretation of recoil on excitation exchange between the atoms differs in the radiation and near zones, despite the explicit appearance of recoil momentum in Eq. (12). In the radiation zone, we have seen that an interpretation of excitation exchange involved atom 1 emitting and atom 2 absorbing radiation emitted in the direction $\hat{\mathbf{r}}_0$, with both atoms recoiling along this axis. In the near zone, the momentum spread of the wave packet $\Delta p \gg \hbar k_0$ is sufficiently large to allow for excitation exchange involving intermediate states in which the radiation field coupling the atoms can be emitted in any direction. The recoil in the near zone cannot be viewed in terms of individual recoil of the atoms; rather, the excitation exchange between the atoms can be interpreted as giving rise to the interatomic potential $\hbar s(\mathbf{r})$.

V. SUMMARY

The emission spectrum of a system of two identical atoms has been calculated, including effects of atomic recoil. For separations of the atoms much larger than an optical wavelength, the emission process can be viewed as involving two channels: direct emission from one of the atoms and excitation exchange between the two atoms followed by emission from the second atom. Interference between the channels occurs only for emission along the line connecting the atoms. In principle, the recoil shift can be used to distinguish the two channels. For separations much less than an optical wavelength, the spectrum can be viewed as arising from a coherent emission from the composite, two-atom system. The recoil shift in the spectrum is that associated with emission from an ''atom'' of mass $2m$. The spectrum is split into a doublet, owing to the strength of the interatomic potential. For realistic values of the parameters, it is impossible for atoms to remain in the region $kr_0 \ll 1$ for times of order of the emission time, owing to the strength of the potential.

It might prove interesting to carry out numerical calculation for atomic separations $kr_0 \sim 1$, where neither of the above limits and interpretations remains valid. In this context, there may be some different effects related to the polarization of the emitted radiation [8]. During the excitation exchange, the atoms can acquire some relative orbital angular momentum as a result of recoil, which will cause the internal magnetic state polarization to be modified as a result of the exchange process. In this manner, recoil can lead to changes in the polarization of the emitted radiation. To study this effect in detail, the calculations must be generalized to states of arbitrary angular momentum [9].

Although experimental verification of some of these predictions may be all but impossible at the current time, one could imagine creating an atomic lattice, turning off the trapping fields, exciting the atoms by electron collisions, and observing the resultant spectra.

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APPENDIX

The integral that must be evaluated is

$$H(\mathbf{r},t) = \int_0^t dt' \int_0^{\omega_c} \Omega_k^3 d\Omega_k \exp[i(\omega_0 - \Omega_k)(t-t')] b_\alpha(t') \times \left[\left(\frac{\sin kr}{kr} + \frac{\cos kr}{(kr)^2} - \frac{\sin kr}{(kr)^3} \right) \sin^2 \alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}) + 2 \left(-\frac{\cos kr}{(kr)^2} + \frac{\sin kr}{(kr)^3} \right) \cos^2 \alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}}) \right], \tag{A1}$$

where $\Omega_k = kc$ and ω_c is a cutoff frequency that will eventually go to infinity. It is necessary to consider only the integral

$$F(r,t) = -\int_0^t dt' \int_0^{\omega_c} d\Omega_k \sin kr \times \exp[i(\omega_0 - \Omega_k)(t-t')] b_\alpha(t') \tag{A2}$$

since the other integrals can be obtained from it by differentiation with respect to r. Introducing dimensionless variables

$$\begin{aligned} s &= c(t-t')/r - 1, \quad & s_f &= ct/r - 1, \\ s' &= c(t-t')/r + 1, \quad & s_f^* &= ct/r + 1, \\ y &= k_0 r = \omega_0 r/c, \quad & z &= \omega_c r/c, \end{aligned} \tag{A3}$$

one can rewrite Eq. (A2) as

$$F(r,t) = \frac{1}{2} \left[e^{iy} \int_{-1}^{s_f} ds \left(\frac{1 - e^{-isz}}{s} \right) e^{isy} b_\alpha \left(\frac{r}{c} (s_f - s) \right) \right.$$

$$-e^{-iy}\int_1^{s_f'} ds'\left(\frac{1-e^{-is'z}}{s'}\right)e^{is'y}b_\alpha\left(\frac{r}{c}\,(s_f'-s')\right). \tag{A4}$$

When $ct/r \gg 1$, the first integral is dominant since the range of integration includes the pole at $s=0$. Keeping only this term for the moment and evaluating the amplitude b_α at $s=0$, one finds

$$F(r,t)=\frac{1}{2}\,e^{iy}\begin{bmatrix}-\mathrm{Ci}(|y|)+\mathrm{Ci}(|s_f y|)+\mathrm{Ci}(z-y)-\mathrm{Ci}[|s_f(z-y)|]\\ +i\ \mathrm{Si}(y)+i\ \mathrm{Si}(z-y)+i\ \mathrm{Si}(s_f y)+i\ \mathrm{Si}[s_f(z-y)]\end{bmatrix}b_\alpha\left(t-\frac{r}{c}\right), \tag{A5}$$

where $\mathrm{Ci}(x)=-\int_x^\infty ds\ \cos(s)/s$ and $\mathrm{Si}=\int_0^x ds\ \sin(s)/s$. Letting the cutoff ω_c go to infinity is equivalent to letting $z\sim\infty$. In this limit, Eq. (A5) reduces to

$$F(r,t)=\frac{1}{2}\,e^{iy}\begin{bmatrix}-\mathrm{Ci}(|y|)+\mathrm{Ci}(|s_f y|)\\ +i\ \mathrm{Si}(y)+i\pi\Theta(s_f)+i\ \mathrm{Si}(s_f y)\end{bmatrix}b_\alpha\left(t-\frac{r}{c}\right), \tag{A6}$$

where $\Theta(x)=1$ for $x>0$ and 0 for $x<0$. If, in addition, we take the limit that $s_f=ct/r-1\gg 1$ and $y\gg 1$, Eq. (A6) reduces to

$$F(r,t)=i\pi e^{iy}b_\alpha\left(t-\frac{r}{c}\right), \tag{A7}$$

which coincides with the corresponding term in Eq. (18). The remaining terms are obtained by differentiation with respect to r. The terms involving derivatives of b_α can be neglected since they are smaller than the other terms by a factor of order γ/ω_0. In this way, one arrives at Eq. (18).

Thus the result of the main text involving retarded times is strictly valid only in the limits $|s_f|=|ct/r-1|\gg y^{-1}$ and $y\gg 1$. When these inequalities are not satisfied, the second integral in Eq. (A4) must be considered as well. There is no justification for evaluating b_α at the retarded time in this integral.

It is an interesting exercise to set b_α equal to unity in Eq. (A4), which will lead to the perturbation theory result for $\gamma t\ll 1$. In this way we can see if $b_{\alpha'}$ is nonvanishing for $t<r/c$, a result that would appear to violate causality. Setting $b_\alpha=1$ in Eq. (A4) and using the facts that $z\gg 1$ and $s_f' z\gg 1$, one finds

$$\begin{aligned}F(r,t)=&\tfrac{1}{2}e^{iy}[-\mathrm{Ci}(|y|)+\mathrm{Ci}(|s_f y|)+i\pi\Theta(s_f)+i\ \mathrm{Si}(y)\\ &+i\ \mathrm{Si}(s_f y)]-\tfrac{1}{2}e^{-iy}[-\mathrm{Ci}(|y|)+\mathrm{Ci}(|s_f' y|)\\ &-i\ \mathrm{Si}(y)+i\ \mathrm{Si}(s_f' y)].\end{aligned} \tag{A8}$$

It is easy to verify that this expression is nonvanishing for $0<ct/r<1$, regardless of the value of y. It would appear that causality is violated.

An objection may be raised that ''counterrotating'' terms associated with the state $|e_1,e_2;\mathbf{k}\rangle$ have been omitted. Such terms are incorporated easily into Eq. (A8) by the addition of terms having $y\to -y$, which is equivalent to replacing nearly resonant terms varying as $\exp[i(\Omega_k-\omega_0)t]$ with counterrotating terms varying as $\exp[i(\Omega_k+\omega_0)t]$ in Eq. (A2). When this is carried out one finds

$$\begin{aligned}F(r,t)=&\,i\pi\Theta(s_f)e^{iy}+\cos y[-\mathrm{Ci}(s_f' y)+\mathrm{Ci}(|s_f y|)]\\ &-\sin y[\mathrm{Si}(s_f' y)+\mathrm{Si}(s_f y)-\pi\Theta(s_f)].\end{aligned} \tag{A9}$$

This expression is also nonvanishing for $0<ct/r<1$.

The question of causality is planned to be addressed in a future paper. The fact that the joint probability to find atom 1 in its ground state, atom 2 excited, and no photons in the field is nonvanishing for $0<ct/r<1$ does not violate causality, in the sense that it is impossible to convey information with a speed faster than the speed of light based on this joint probability [10]. If the total probability to find atom 2 excited was nonvanishing for $0<ct/r<1$, this *would* constitute a violation of causality; however, it can be shown that this probability vanishes identically for $ct/r<1$ using the basis states $|e_1,g_2;0\rangle$, $|g_1,g_2;\mathbf{k}\rangle$, $|g_1,e_2;0\rangle$, $|e_1,e_2;\mathbf{k}\rangle$, $|g_1,e_2;\mathbf{k}\rangle$, $|e_1,g_2;\mathbf{k}\rangle$, and $|g_1,e_2;\mathbf{k},\mathbf{k}'\rangle$ [10,11].

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[9] To generalize the calculation to states having an arbitrary angular momentum, one replaces the states $|\alpha\rangle$ and $|\alpha'\rangle$ by $|H,m_H;G,m_G\rangle$ and $|G,m_G;H,m_H\rangle$, respectively, where H and G are the total angular momentum quantum numbers for the excited and ground states, m_H and m_G are the corresponding quantum numbers for the z projection of angular momentum, and the first symbols in the ket refer to atom 1 and the second to atom 2. The starting point for the calculation is

obtained by replacing Eq. (14) with

$$\dot{b}_{G,m_G;H,m_H}(\mathbf{p},t) = -\gamma b_{G,m_G;H,m_H}(\mathbf{p},t) - (2\pi\hbar)^{-3/2} \times \sum_{m'_G,m'_H} \int_0^t dt' \int d\mathbf{r}\, \exp(-i\mathbf{p}\cdot\mathbf{r}/\hbar) \times \exp(iE_p t'/\hbar) G_{m_G,m_H;m'_H,m'_G}(\mathbf{r},t-t') \times b_{H,m'_H;G,m'_G}(\mathbf{r},t'),$$

where $G_{m_G,m_H;m'_H,m'_G}$ is obtained from G in Eq. (16) by replacing the factors $\mu^2\sin^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}})$ and $\mu^2\cos^2\alpha_{\boldsymbol{\mu}}(\hat{\mathbf{r}})$ by

$$\langle H,m_H|\mu_x|G,m'_G\rangle\langle H,m'_H|\mu_x|G,m_G\rangle^* + \langle H,m_H|\mu_y|G,m'_G\rangle\langle H,m'_H|\mu_y|G,m_G\rangle^*$$

and

$$\langle H,m_H|\mu_z|G,m'_G\rangle\langle H,m'_H|\mu_z|G,m_G\rangle^*,$$

respectively, where all matrix elements are evaluated in a coordinate system quantized along $\mathbf{r}$.

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