Quantum dynamics of an atomic Bose-Einstein condensate in a double-well potential

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We consider the quantum dynamics of a neutral atom Bose-Einstein condensate in a double-well potential, including many-body hard-sphere interactions. Using a mean-field factorization we show that the coherent oscillations due to tunneling are suppressed when the number of atoms exceeds a critical value. An exact quantum solution, in a two-mode approximation, shows that the mean-field solution is modulated by a quantum collapse and revival sequence. [S1050-2947(97)01406-6]

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I. INTRODUCTION

The recent experimental observation of Bose-Einstein condensation (BEC) in dilute systems of trapped neutral atoms [1–5] opens a new context for studying the quantum mechanics of mesoscopic systems. In particular, atomic BECs can be expected to display a variety of quantum interference phenomena, and Javanainen [6] and Grossmann and Holthaus [7] have previously suggested the possibility of condensate tunneling between two adjacent atomic traps. This tunneling, resulting in oscillatory exchange of the atoms between the traps, is analogous to the Josephson effect [8] for neutral atoms, in which the exchange arises from the relative phase between the macroscopic wave function in the two traps. These authors did not, however, take account of the many-body hard-sphere interactions between the atoms.

We consider the case of an atomic BEC formed in a double-well potential with well separated minima, where each potential well represents an atomic trap. Using the mean-field factorization assumption, together with a two-mode approximation, we find an analytic solution to the Gross-Pitaevskii equation [9] including many-body interactions. If the condensate is initially localized in one well, it can oscillate between the wells by quantum tunneling [6,7]. However, due to the nonlinearity arising from particle interactions, this oscillation is suppressed when the number of atoms in the condensate exceeds a critical value. This suppression of the tunneling corresponds to the self-trapping transition previously studied in the discrete self-trapping equation [10–12], and also corresponds to switching action in the nonlinear directional coupler of nonlinear optics [13]. We also calculate the full quantum dynamics and show that the oscillations arising in the mean-field approximation are modulated by a collapse and revival sequence. Collapse and revivals have also been studied in the context of the quantum

theory of the nonlinear directional coupler [14], and also in the relative phase between two superfluids or superconductors [15]. The time for a complete collapse and revival depends very strongly on the number of particles in the condensate, becoming longer as the particle number is increased. Observation of quantum tunneling in this system may be easier to observe than other condensed systems [16], or the nonlinear directional coupler [14], due to the small dissipation in atom optical contexts.

The remainder of this paper is organized as follows. In Sec. II we present our basic model, in particular, the two-mode approximation for the quantum dynamics of coupled condensates. Here we also discuss the limits of validity of the model. Section III gives a discussion of the mean-field solution of the quantum problem, showing that tunneling is suppressed beyond a critical atom number, and the full quantum problem is addressed in Sec. IV where we show that the mean-field solution is modulated by a series of collapses and revivals. Finally, our summary and conclusions are given in Sec. V.

II. BASIC MODEL

A. Two-mode approximation

Our model system is a symmetric double-well single-particle potential $V(\mathbf{r})$ with minima at $\mathbf{r}_1$ and $\mathbf{r}_2$, and with no loss of generality we set $V(\mathbf{r}_{1,2}) = 0$. We assume the potential is such that the two lowest states are closely spaced and well separated from higher levels of the potential, and that many-particle interactions do not significantly change this situation. This assumption permits a two-mode approximation to the many-body description of the system. To proceed we expand the potential around each minimum as

$$V(\mathbf{r}) = \widetilde{V}^{(2)}(\mathbf{r} - \mathbf{r}_j) + \cdots, \quad j = 1,2 \tag{1}$$

where $\widetilde{V}^{(2)}(\mathbf{r} - \mathbf{r}_j)$ is the parabolic approximation to the potential in the vicinity of each minimum. We now define the state $u_0(\mathbf{r})$ as the normalized single-particle ground-state

mode of the local potential $\widetilde{V}^{(2)}(\mathbf{r})$, with energy E_0, and define the local mode solutions of the individual wells $u_{1,2}(\mathbf{r}) = u_0(\mathbf{r} - \mathbf{r}_{1,2})$. These local modes are not exactly orthogonal, but we may write

$$\int d^3\mathbf{r}\, u_j^*(\mathbf{r}) u_k(\mathbf{r}) = \delta_{jk} + \epsilon(1 - \delta_{jk}), \quad j,k = 1,2. \quad (2)$$

Here ϵ is the overlap between the modes of opposite wells. If the position uncertainty in the state $u_0(\mathbf{r})$ is much less than the separation of the minima of the global potential $V(\mathbf{r})$, then $\epsilon \ll 1$, and first-order perturbation theory, with ϵ as an expansion parameter, will suffice. In first-order perturbation theory the modes are determined to order ϵ^0, which ignores inter-well coupling, in which case the local modes may be treated as orthogonal. The energy eigenstates of the global double-well potential may then be approximated as the symmetric $(+)$ and asymmetric $(-)$ combinations

$$u_\pm(\mathbf{r}) \approx \frac{1}{\sqrt{2}} [u_1(\mathbf{r}) \pm u_2(\mathbf{r})], \quad (3)$$

with corresponding eigenvalues $E_\pm = E_0 \pm \mathcal{R}$, and

$$\mathcal{R} = \int d^3\mathbf{r}\, u_1^*(\mathbf{r}) [V(\mathbf{r}) - \widetilde{V}^{(2)}(\mathbf{r} - \mathbf{r}_1)] u_2(\mathbf{r}). \quad (4)$$

The tunneling frequency Ω between the two minima is then given by the energy level splitting of these two lowest states, $\Omega = 2\mathcal{R}/\hbar$. The matrix element $\mathcal{R}$, which is of order ϵ^1, describes the coupling between the local modes.

The many-body Hamiltonian describing atomic BEC in a potential is [17]

$$\hat{H}(t) = \int d^3\mathbf{r} \left[\frac{\hbar^2}{2m} \nabla \hat{\psi}^\dagger \cdot \nabla \hat{\psi} + V + \frac{U_0}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right], \quad (5)$$

where m is the atomic mass, $U_0 = 4\pi\hbar^2 a/m$ measures the strength of the two-body interaction, and a is the s-wave scattering length, $\hat{\psi}(\mathbf{r},t)$ and $\hat{\psi}^\dagger(\mathbf{r},t)$ are the Heisenberg picture field operators which annihilate and create atoms at position $\mathbf{r}$, and normal ordering has been used. In the two-mode approximation we expand the field operators in terms of the local modes and introduce the Heisenberg picture annihilation and creation operators

$$c_j(t) = \int d^3\mathbf{r}\, u_j^*(\mathbf{r}) \hat{\psi}(\mathbf{r},t) \quad (6)$$

so that $[c_j, c_k^\dagger] = \delta_{jk}$ to order ϵ^0. Then retaining terms up to order ϵ, the many-body Hamiltonian reduces to the following two-mode approximation:

$$\hat{H}_2(t) = E_0(c_1^\dagger c_1 + c_2^\dagger c_2) + \frac{\hbar\Omega}{2}(c_1 c_2^\dagger + c_1^\dagger c_2)$$

$$+ \hbar\kappa[(c_1^\dagger)^2 c_1^2 + (c_2^\dagger)^2 c_2^2], \quad (7)$$

where $\kappa = U_0/2\hbar V_{\text{eff}}$, and $V_{\text{eff}}^{-1} = \int d^3\mathbf{r}|u_0(\mathbf{r})|^4$ is the effective mode volume of each well. Here we have retained only self-phase modulation arising from self-interaction within each well since the cross-interaction terms involve matrix

elements such as $(U_0/2)\int d^3\mathbf{r}|u_1(\mathbf{r})|^2|u_2(\mathbf{r})|^2$, which are of order ϵ^2 compared to the self-phase modulation matrix element $\hbar\kappa$, and should therefore be consistently neglected to first order.

The Hamiltonian (7) has the form of that for the discrete self-trapping equation [10,11], and has previously been studied in the context of the quantum dimer [12], as a model for anharmonic oscillations in small molecules, and also in the context of the nonlinear optical directional coupler [13,14]. Here we explore the consequences of this model for atomic BEC in a double-well potential. In the limit of negligible many-body interactions, $\hbar\kappa \to 0$, the Hamiltonian (7) reduces to that previously employed to study condensate tunneling [6,7].

B. Model double-well system

We can illustrate the general features of the double-well system by considering a potential of the form

$$V(\mathbf{r}) = b\left(x^2 - \frac{d}{2b}\right)^2 + \frac{1}{2}m\omega_t^2(y^2 + z^2), \quad (8)$$

where the interwell coupling occurs along x, and ω_t is the trap frequency in the y-z plane. This potential has elliptic fixed points at $\mathbf{r}_1 = +q_0\mathbf{x}$, $\mathbf{r}_2 = -q_0\mathbf{x}$, where $q_0^2 = d/2b$, at which the linearized motion is harmonic with frequency $\omega_0 = (4d/m)^{1/2}$. Thus, setting $\omega_t = \omega_0$ for simplicity, we choose

$$\widetilde{V}^{(2)}(\mathbf{r}) = \frac{1}{2}m\omega_0^2(x^2 + y^2 + z^2). \quad (9)$$

We will fix ω_0 by fixing d and consider variations of q_0 only. This is equivalent to varying the height of the barrier D separating the two wells as $D = dq_0^2$. It is convenient to scale the length in units of the position uncertainty in a harmonic oscillator ground state $r_0 = \sqrt{\Delta}$, where $\Delta = \hbar/2m\omega_0$. The barrier height is then given by $D = (\hbar\omega_0/8)(q_0^2/\Delta)$. For a suitable choice of D, only two energy eigenstates lie beneath the barrier. The local mode of each well is then given by

$$u_0(\mathbf{r}) = \left(\frac{1}{2\pi\Delta}\right)^{3/4} e^{-(x^2 + y^2 + z^2)/4\Delta}. \quad (10)$$

These states are simply Gaussian, which enables all integrals to be performed explicitly. For example, the tunneling frequency Ω may be evaluated as

$$\Omega = \frac{q_0^2\omega_0}{2\Delta} e^{-q_0^2/2\Delta}, \quad (11)$$

and the effective mode volume is given by $V_{\text{eff}} = 8(\pi\Delta)^{3/2}$.

C. Limits of validity

The two-mode approximation is valid when many-body interactions produce only small modifications of the ground-state properties of the individual potentials. This is true when

$$\hbar\omega_0 = \frac{\hbar^2}{2mr_0^2} \gg \frac{N|U_0|}{V_{\text{eff}}}. \qquad (12)$$

Using $V_{\text{eff}} \approx 8\pi^{3/2}r_0^3$ for this case, we obtain the following condition on the number of atoms:

$$N \ll \frac{r_0}{|a|}. \qquad (13)$$

Taking typical numbers of $r_0 = 1$ μm, and $a = 5$ nm, yields $N \ll 200$. Thus the two-mode approximation is valid for small number of atoms compared to current experiments with $N = 10^3 - 10^6$ [1–5]. If we consider a larger trap, say $r_0 = 10$ μm, then the theory is valid for a few hundred atoms, and this is the case we consider here. In the following sections we shall show that the condensate tunneling discussed in Refs. [6,7] is strongly modified by many-body interactions even for such low numbers of atoms.

We remark that the approximations employed here are in contrast to the Thomas-Fermi approximation [18,19] which is appropriate to current experiments with $N = 10^3 - 10^6$. In the Thomas-Fermi approximation the many-body interactions dominate over the kinetic energy, and as a result the properties of the ground state are strongly modified with respect to the linear ones. In this case the two-mode aproximation employed here is not applicable.

III. MEAN-FIELD APPROXIMATION

Before proceeding to the full quantum analysis of the Hamiltonian (7), we first consider the mean-field approximation. For this we employ the Hartree approximation [20] for a fixed number of atoms N, and write the atomic state vector as

$$|\Psi_N(t)\rangle = \frac{1}{\sqrt{N!}} \left[\int d^3\mathbf{r} \, \phi_N(\mathbf{r},t)\hat{\psi}^\dagger(\mathbf{r},0) \right]^N |0\rangle, \qquad (14)$$

where $|0\rangle$ is the vacuum. The self-consistent nonlinear Schrödinger equation or Gross-Pitaevskii equation for the condensate wave function $\phi_N(\mathbf{r},t)$ follows from the Schrödinger equation $i\hbar|\dot{\Psi}_N(t)\rangle = \hat{H}(0)|\Psi_N(t)\rangle$, and is given by [9,20,21]

$$i\hbar\frac{\partial\phi_N}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + NU_0|\phi_N|^2 \right]\phi_N. \qquad (15)$$

For a particular choice of the global potential $V(\mathbf{r})$, Eq. (15) can be solved numerically for a given initial condition. In particular, this equation allows simulations of condensate tunneling to be performed without the limitations imposed by the two-mode approximation.

In the two-mode approximation we use the local modes described above and write

$$\phi_N(\mathbf{r},t) = e^{-iE_0t/\hbar}[b_1(t)u_1(\mathbf{r}) + b_2(t)u_2(\mathbf{r})]. \qquad (16)$$

Then, to first order in ϵ we obtain the coupled-mode equations

$$\frac{db_j}{dt} = -\frac{i\Omega}{2}b_{3-j} - 2i\kappa N|b_j|^2 b_j. \qquad (17)$$

The number of atoms in the jth well is given by

$$N_j(t) = \langle\Psi_N(t)|\hat{c}_j^\dagger\hat{c}_j|\Psi_N(t)\rangle = N|b_j(t)|^2, \qquad (18)$$

and this provides the link between the coupled-mode amplitudes and the expectation values of the quantum problem.

The coupled-mode equations (17) have an exact solution [10]. For the case that all N atoms are initially localized in well 1, $N_1(0) = N|b_1(0)|^2 = N$, the number of atoms in well 1 varies in time as

$$N_1(t) = \frac{N}{2}[1 + \text{cn}(\Omega t|N^2/N_c^2)], \qquad (19)$$

with $N_1(t) + N_2(t) = N$. Here $\text{cn}(\phi|m)$ is a Jacobi elliptic function, and N_c is the critical number of atoms given by

$$N_c = \frac{\Omega}{\kappa}. \qquad (20)$$

For $N < N_c$ this solution exhibits complete and periodic oscillations between the two condensates with a period $K(N^2/N_c^2)$ which depends on the number of atoms, where $K(m)$ is a complete elliptic integral of the first kind. For $N \ll N_c$, cn becomes cos, and the oscillations are precisely like those in the Josephson effect [6–8]. As the number of atoms is increased the oscillation period increases, until at $N = N_c$ the period is infinite. This marks a bifurcation in the nonlinear system and at this point the system asymptotically evolves to equal number of atoms $N/2$ in each well. For $N > N_c$ the period of oscillation reduces again but the exchange between the wells is no longer complete. That is, the coherent tunneling oscillations are inhibited at high numbers of atoms, and this is the analog of the self-trapping transition [10] for the double-well BEC. Note that this result arises even for a fixed number of atoms N, and does not therefore rely on coherence between different number states. It does, however, require there to be a well defined relative phase between the amplitudes $b_{1,2}$ of the two potential wells.

The choice of initial conditions depends on the condensate state. In a typical case one might expect that there would be equal numbers of atoms in each of the wells, and thus the many-body ground state would reflect the fundamental symmetry of the potential. This would mean that the quantity $(|b_2|^2 - |b_1|^2)$ would initially be zero. However, as the total number of atoms is conserved $(|b_2|^2 + |b_1|^2) = 1$, we must have

$$b_1^*b_2 = \frac{1}{2}e^{-i\theta}, \qquad (21)$$

where θ is the relative phase between the amplitudes b_1 and b_2, respectively. The condensate may thus be regarded as having a well defined phase between the two potential wells. According to the usual notion of spontaneous symmetry breaking [22–24] this phase is randomly selected for a given

realization of the system, and if averaged over many realizations would yield a zero contribution from the phase dependent term in the ensemble.

To investigate the consequences of spontaneous symmetry breaking for the semiclassical dynamics, it will be convenient to define the three real variables

$$S_x = \frac{1}{2}(|b_2|^2 - |b_1|^2),$$ (22)

$$S_y = -\frac{i}{2}(b_1^* b_2 - \text{c.c.}),$$ (23)

$$S_z = \frac{1}{2}(b_1^* b_2 + \text{c.c.}).$$ (24)

In Sec. IV we will show that S_y is the mean momentum of the condensate, while S_z is the atomic number difference between the two single-particle energy eigenstates of the double-well system. If $S_x = 0$ we must have $S_y = \frac{1}{2}\sin\theta$ and $S_z = \frac{1}{2}\cos\theta$. If the mean momentum of the condensate is initially zero, $\theta = 0$ and $S_z(0) = 1/2$. Such an initial condition is a stationary point of the dynamics, as is easily seen if we write the equations of motion in terms of the real variables defined above:

$$\dot{S}_x = -\Omega S_y,$$ (25)

$$\dot{S}_y = \Omega S_x - 4\kappa N S_x S_z,$$ (26)

$$\dot{S}_z = 4\kappa N S_x S_y.$$ (27)

These equations indicate a linear precession around the S_z axis at rate Ω, and a nonlinear precession around the S_x axis at a rate $4\kappa N S_x$. It is easily seen that $S_x^2 + S_y^2 + S_z^2 = 1/4$ is a constant of the motion, which corresponds to conservation of particle number.

In Fig. 1 we show the mean-field solutions for the quantity $\langle S_x \rangle = \frac{1}{2}(|b_2|^2 - |b_1|^2)$ which represents the occupation difference of the two wells: The solid line is for $\kappa N/\Omega = 0.9$, and complete oscillations between the wells is observed (we scale time in units of the tunneling period so that $\Omega = 1$ in all the figures). In contrast, the dashed curve is for $\kappa N/\Omega = 2$, and the coherent oscillations are no longer complete. This corresponds to the discrete self-trapping identified in Ref. [10].

IV. QUANTUM DYNAMICS

A. Quantum model

Within the two-mode approximation we can obtain an exact solution to the full quantum problem in order to assess the effect of quantum fluctuations on the predictions of the Gross-Pitaevskii equation. The total number operator $\hat{N} = c_1^\dagger c_1 + c_2^\dagger c_2$ is a constant of motion and we thus set it equal to the total number of atoms N. We now define three operators, which obey SU(2) commutation relations, by

$$\hat{J}_z = \frac{1}{2}(c_1^\dagger c_2 + c_2^\dagger c_1),$$ (28)
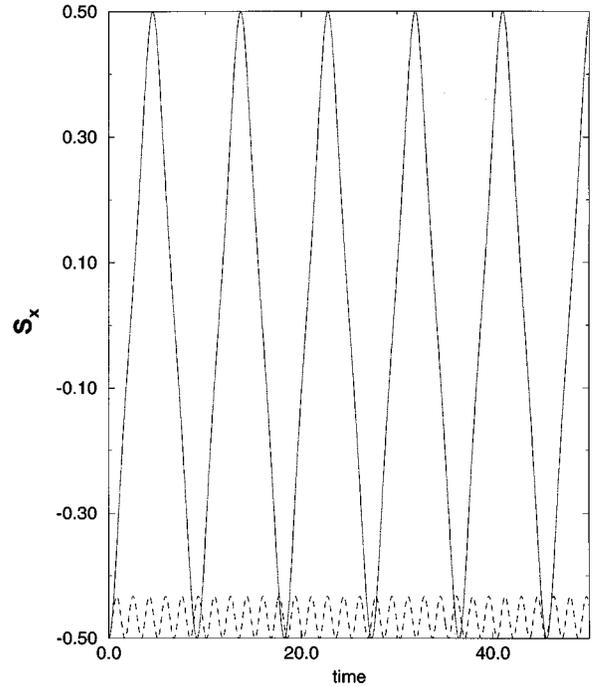

FIG. 1. Mean-field solutions for the occupation difference $\langle S_x \rangle = (|b_2|^2 - |b_1|^2)/2$ versus time in units of the inverse tunneling period. The solid line is for $\kappa N/\Omega = 0.9$, and the dashed line is for $\kappa N/\Omega = 2$, the critical value being for $\kappa N/\Omega = 1$.

$$\hat{J}_x = \frac{1}{2}(c_2^\dagger c_2 - c_1^\dagger c_1),$$ (29)

$$\hat{J}_y = \frac{i}{2}(c_2^\dagger c_1 - c_1^\dagger c_2).$$ (30)

The Casimir invariant is easily seen to be

$$\hat{J}^2 = \frac{\hat{N}}{2}\left(\frac{\hat{N}}{2} + 1\right).$$ (31)

This is analogous to an angular momentum model with total angular momentum given by $j = N/2$.

The operator $\hat{J}_z$ corresponds to the particle occupation number difference between the single-particle energy eigenstates. For example, the maximal weight eigenstate $|j,j\rangle_z$ corresponds to all the particles occupying the highest single-particle energy eigenstate, $\psi_2(x)$. The operator $\hat{J}_x$ gives the particle number difference between the localized states (u_1, u_2) of each well. In fact, for the one dimensional case, the position operator in the field representation is

$$\hat{x} \rightarrow \frac{2q_0}{N}\hat{J}_x.$$ (32)

Thus the maximal and minimal weight eigenstates of $\hat{J}_x$ correspond to the localization of all the particles in one well or the other. The interpretation of $\hat{J}_y$ is crucial for an understanding of tunneling. In one dimension, the field representation of the single-particle momentum operator $i\hbar(d/dx)$ is

$$\hat{p} \rightarrow -\frac{\hbar \Omega}{q_0 \omega_0} \hat{J}_y. \tag{33}$$

Thus the operator $\hat{J}_y$ represents the condensate momentum.

The two-mode Hamiltonian (7) may be written

$$\hat{H}_2 = \hbar \Omega \hat{J}_z + 2 \hbar \kappa \hat{J}_x^2, \tag{34}$$

where we neglected constant energy shifts which depend on the total number N. This Hamiltonian describes linear precession around the z axis at the tunneling frequency and a nonlinear precession around the x axis at a rate determined by the value of x component of angular momentum. It is interesting to note that Eq. (34) looks similar to the nonlinear top models considered by Haake [25]. This Hamiltonian is symmetric under rotations of π about the z axis. Such a transformation corresponds to $\hat{J}_x \rightarrow -\hat{J}_x$ which in view of the interpretation of $\hat{J}_x$ discussed above corresponds to the parity symmetry of the double-well potential. Thus all eigenstates belong to one of two parity classes corresponding to the two eigenvalues of this transformation.

B. Energy eigenstates

The semiclassical solution suggests that for N small the first term in Eq. (34) dominates, in which case the energy eigenstates are close to the $N+1$ eigenstates of $\hat{J}_z$. The condensate state will be near the minimum weight state $|j,-j\rangle_z$. This state is of course just the single-particle ground state of the double-well potential, and thus the density function of the condensate will be symmetric as expected. In this case the dynamics is dominated by a precession around the z axis. If the system then starts with broken symmetry so that

$$\langle \hat{J}_y \rangle = N S_y \neq 0 \tag{35}$$

(which corresponds to a nonzero momentum state), precession around the z axis will cause $\langle \hat{J}_x \rangle$ to oscillate at frequency Ω. This means that the condensate accumulates first in one well then the other at a frequency determined by the single-particle tunneling frequency Ω. This is analogous to the general case for superfluidity when spontaneous symmetry breaking gives the condensate a phase and a nonzero momentum [22].

On the other hand, for large N we expect the system to be dominated by the second nonlinear term in the Hamiltonian. This suggests that the eigenstates of the Hamiltonian are close to the eigenstates of $\hat{J}_x^2$. The ground state, and thus the condensate state, is close to the zero weight state $|j,0\rangle_x$, with all other states being doubly degenerate. Note that this state corresponds to an equal number of particles in each of the localized states in each well and thus will also have a symmetric density function. Some results on the spectrum of this model were presented in Ref. [12].

In Figs. 2 and 3 we calculate the eigenvalues of the Hamiltonian for different values of the ratio Ω/κ, and total particle number N. In Fig. 2 the ratio is large and the low lying part of the spectrum is dominated by the eigenstates of $\hat{J}_z$, with a characteristic linear increase of the energy with
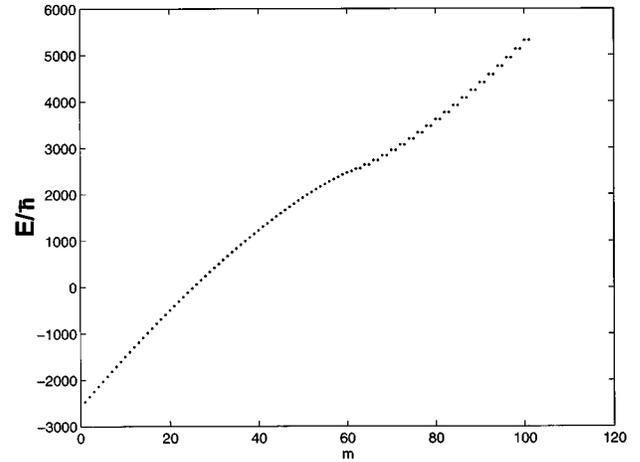

FIG. 2. Plot of the energy spectrum in the two-mode approximation with $\Omega/\kappa = 50$ and $N=100$.

the integer m labeling the sequence of eigenstates. As the ratio of Ω/κ increases, Fig. 3, the doubly degenerate eigenstates of $\hat{J}_x^2$ begin to dominate and the energies increase quadratically with the integer m labeling the eigenstates.

The most natural set of states which exhibit spontaneous broken symmetry for this system are the angular momentum coherent states [26] defined in terms of the $\hat{J}_z$ eigenstates by

$$|\alpha\rangle = \sum_{m=-j}^{j} \binom{2j}{m+j}^{1/2} \frac{\alpha^{m+j}}{(1+|\alpha|^2)^j} |j,m\rangle, \tag{36}$$

with $\alpha = e^{-i\phi} \tan(\theta/2)$. For these states we have that $\langle \hat{J}_x \rangle = (N/2)\sin\theta\cos\phi$, $\langle \hat{J}_y \rangle = (N/2)\sin\theta\sin\phi$, and $\langle \hat{J}_z \rangle = (N/2)\cos\theta$. These states have a binomial, rather than Poisson, distribution of particle number over the two single-particle energy eigenstates of the potential. These states were recently used by Wong et al. [27] to test aspects of broken symmetry. As previously for a condensate of zero momentum we would have $\theta = 0$.
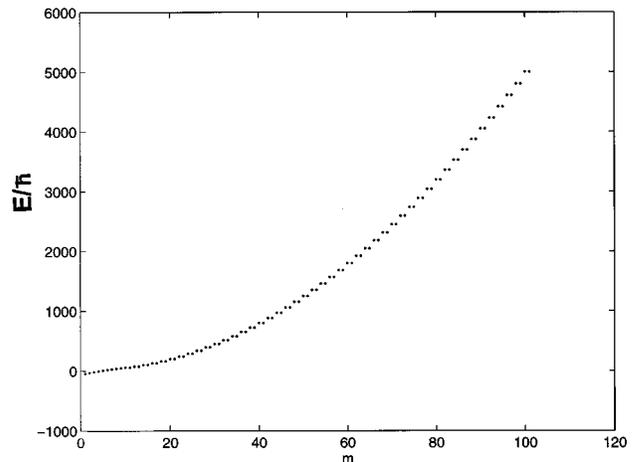

FIG. 3. Plot of the energy spectrum in the two-mode approximation with $\Omega/\kappa = 1$ and $N=100$.
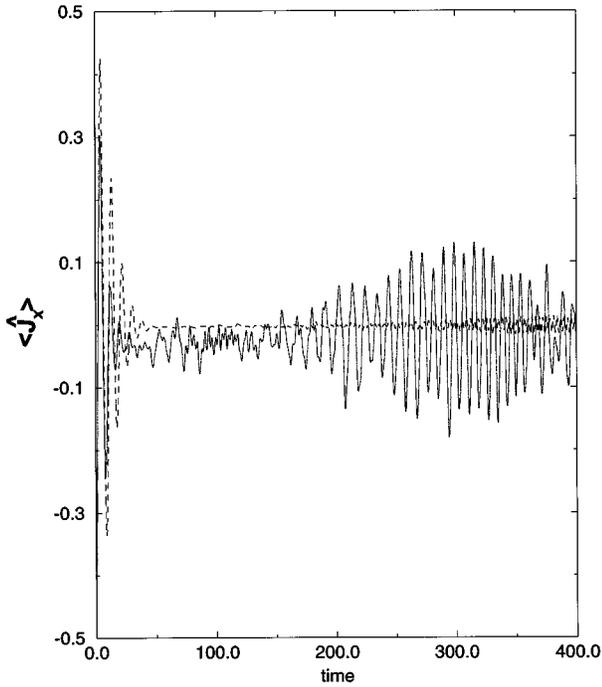
FIG. 4. Plot of the quantum mean value dynamics of $\hat{S}_x$ for an intial state $|j,-j\rangle_x$, below threshold. The solid curve is for $N=100$, the dashed curve is for $N=400$ with $\Omega=1.0$, $\kappa N=0.9$.

C. Quantum results

We now determine the quantum dynamics of this model and contrast the results with the semiclassical results. The Heisenberg equations of motion are

$$\frac{d\hat{S}_x}{dt} = -\Omega\hat{S}_y, \tag{37}$$

$$\frac{d\hat{S}_y}{dt} = \Omega\hat{S}_x - 2\kappa N(\hat{S}_z\hat{S}_x + \hat{S}_x\hat{S}_z), \tag{38}$$

$$\frac{d\hat{S}_z}{dt} = 2\kappa N(\hat{S}_y\hat{S}_x + \hat{S}_x\hat{S}_y), \tag{39}$$

where we have defined the scaled, or intensive, many-body operators by $\hat{S}_\alpha = \hat{J}_\alpha/N$. If we now consider the equations of motion for the mean values and factorize all product averages, we can define an equivalent mean-field model. The resulting equations are equivalent to Eqs. (17) with the identification $\langle S_x\rangle = \frac{1}{2}(|b_2|^2 - |b_1|^2)$, $\langle S_y\rangle = -(i/2)(b_1^* b_2 - b_2^* b_1)$, $\langle S_z\rangle = \frac{1}{2}(b_1^* b_2 + b_1 b_2^*)$.

To obtain the quantum dynamics, we represent the two-mode Hamiltonian Eq. (34) in the eigenbasis of J_z, and expand the states in the same basis. The time evolution can then be found by integrating the Schrödinger equation in this basis. In Figs. 4 and 5 we plot the mean value $\langle \hat{J}_x(t)/N\rangle = S_x$ for the initial state $|j,-j\rangle_x$, corresponding to a state localized in one well, and equivalent to that used for the mean-field solutions shown in Fig. 1.

We take two cases, $N=100$, $N=400$, with κN above and below the critical or threshold case with $\kappa N=\Omega$. For convenience we normalize the time in units of the single-
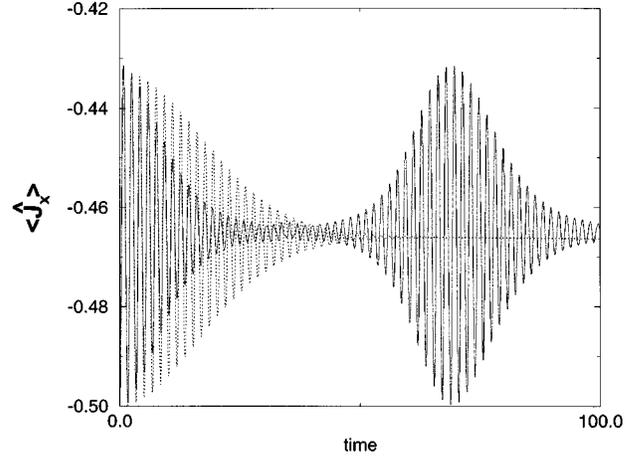

FIG. 5. Plot of the quantum mean value dynamics of $\hat{S}_x$ for an intial state $|j,-j\rangle_x$, below threshold. The solid curve is for $N=100$, the dashed curve is for $N=400$ with $\Omega=1.0$, $\kappa N=2.0$.

particle tunneling period so that in these units $\Omega=1$. In Fig. 4 we plot the mean value of S_x versus time for an initial state $|j,-j\rangle_x$ for the case $\kappa=0.9$. For short times the quantum and mean-field dynamics are similar, with the same oscillation frequency. However, the oscillations of the quantum mean decay due to the intrinsic quantum fluctuations in the initial condition. That is, although the total particle number is fixed the number of atoms in each individual well are not and must be considered fluctuating quantities [23]. More interesting, however, is the revival of the oscillation that occurs at later times. This is entirely due to the discrete spectrum of the many-body Hamiltonian [12,14]. The revival is rather irregular in the below threshold case in Fig. 4 when compared with the above threshold case, Fig. 5. In both cases increasing the number of atoms N while keeping κN fixed increases the collapse and revival time. Thus it is clear that the mean-field factorization approximation will be valid for sufficiently long time scales if N is large enough.

To observe this result it would be necessary to prepare the condensate in a maximal eigenstate of $\hat{J}_x$, that is, entirely localized in one well or the other. To observe the collapse and revival one would need to monitor the initially unoccupied well. This could be done using off-resonant light scattering [28–30], which is dependent on the particle density, so long as the probe laser could be focused down to distinguish a single well.

V. SUMMARY AND CONCLUSIONS

In summary, we have investigated the tunneling of a neutral atom Bose-Einstein condensate in a double-well potential, and have shown that many-body interactions lead to significant effects even for small numbers of atoms. In particular, using mean-field theory we found that beyond a critical atom number the quantum tunneling is suppressed, analogous to the self-trapping transition. Using a full quantum theory in the two-mode approximation we showed that the mean-field solution is modulated by a quantum collapse and revival sequence.

The single-particle tunneling frequency will depend on the details of how the double well is constructed. In fact in

the experiment of Davies *et al.* [3] something like a double well was formed by using an off-resonant optical dipole force to perturb a magnetic-rf trap. Suppose that the harmonic frequency at the bottom of the trap were of the order of 1 kHz. In the case of sodium this results in $\Delta = 1.4 \times 10^{-12} \text{ m}^2$. In the experiment of Davies *et al.*, U_0 is approximately $1.8 \times 10^{-50} \text{ J m}^3$. This gives a value for $\kappa = 53 \text{ s}^{-1}$. If we use the expression for the tunneling frequency given in Eq. (11), then the maximum value of Ω is 37% of the harmonic frequency at the bottom of the wells. If this harmonic frequency is 1 kHz, then the critical number of atoms is $N_c \approx 7$, a rather small number. Thus in a realistic experiment it is likely that the single-particle tunneling will be strongly suppressed by the atomic interactions. Furthermore it is known that quantum tunneling is very sensitive to noise, being rapidly suppressed for even small noise sources. For example, small fluctuations in the potential can cause the

bottoms of the double wells to fluctuate in energy. This will tend to cause localization of the condensate in one well or the other. However, due to the considerable isolation of atomic condensates from their environments we expect that this problem will be not as serious for these systems as it has been for other many-particle tunneling systems such as Josephson tunneling.

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