Quantum-classical correspondence via Liouville dynamics. II. Correspondence for chaotic Hamiltonian systems

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We prove quantum-classical correspondence for bound conservative classically chaotic Hamiltonian systems. In particular, quantum Liouville spectral projection operators and spectral densities, and hence classical dynamics, are shown to approach their classical analogs in the $h \to 0$ limit. Correspondence is shown to occur via the elimination of essential singularities. In addition, applications to matrix elements of observables in chaotic systems are discussed. [S1050-2947(96)05212-2]

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I. INTRODUCTION

The validity of quantum mechanics as a description of the macroscopic world is contingent upon the reduction of the laws of quantum mechanics to Newton's laws in the limit where the characteristic actions of a system are large with respect to Planck's constant [1]. Thus, diagonal and off-diagonal matrix elements must reduce to their classical analogs and quantum dynamics must reproduce the predictions of classical mechanics as $h \to 0$. Despite the fundamental importance of quantum-classical correspondence it has only been satisfactorily demonstrated [2–5] in the very restrictive case of regular systems, i.e., systems that classically possess as many constants of the motion as degrees of freedom. Indeed some authors have suggested that bound quantum systems with a discrete quantum spectrum and a chaotic classical analog may violate the correspondence principle [6]. These doubts about the validity of the correspondence principle for chaotic systems stem from the difficulty of reconciling the quasiperiodic nature of bound state quantum dynamics with the chaotic nature of classical dynamics for the same Hamiltonian. The issue of correspondence for quantum systems whose classical analogs exhibit chaos (irregular systems) is thus of great interest.

Verification of correspondence should be distinguished from the development of semiclassical approximation methods. While semiclassical theories provide a natural starting point for an exploration of the classical limit their existence does not guarantee correspondence. For example, semiclassical theories for regular systems preceded the development of modern quantum mechanics [7], but an understanding of correspondence for regular systems has only recently been achieved [2,3,5]. By comparison, attempts to develop semiclassical quantization rules for chaotic systems have had some success [8], whereas the correspondence limit remains largely unexplored [9]. In this paper we demonstrate that the existing semiclassical theories of quantum dynamics for classically chaotic systems are sufficiently well developed to allow us to show that such systems do in fact approach their proper correspondence limits as Planck's constant approaches zero. This completes the Liouville correspondence program outlined in the preceding paper [5], and significantly extends the results of our study of quantum maps [10,11], where we rigorously demonstrated that a nonchaotic quantum map dynamics can completely recover a fully chaotic classical dynamics in the limit $h \to 0$.

The Liouville picture affords a means of gaining insight into the connections between quantum and classical mechanics [3,12,13], and is a natural framework for studies of correspondence. As outlined in the preceding paper (henceforth referred to as paper I) [5] the essential ingredients for Liouville dynamics are eigenstates and eigenvalues of the Liouville operators in both mechanics. In particular, the dynamics is completely characterized by the Liouville eigenfunctions and eigenvalues or the spectral projectors once the class of allowed initial distributions is specified. Here we consider correspondence in chaotic systems from this Liouville perspective.

Quantum Liouville eigenfunctions for conservative Hamiltonian systems whose classical analogs are chaotic take the form $|n\rangle\langle m|$ where $|n\rangle$ are eigenstates of the Hamiltonian, i.e., $\hat{H}|n\rangle = E_n|n\rangle$. These distributions are eigenfunctions of the complete set of operators $\hat{L},\hat{\mathcal{H}}$ where $\hat{L} = \frac{1}{2}[\hat{H},\ ]$ is the quantum Liouville operator and where $\hat{\mathcal{H}} = \frac{1}{2}[\hat{H},\ ]_+$ is the Hermitian energy operator in the Liouville picture [5]. That is, they are solutions of both the time independent Liouville equation

$$\hat{L}|n\rangle\langle m| = \lambda_{n,m}|n\rangle\langle m|, \tag{1}$$

where $\lambda_{n,m} = (E_n - E_m)/\hbar$, and of the energy eigenequation

$$\hat{\mathcal{H}}|n\rangle\langle m| = E_{n,m}|n\rangle\langle m|, \tag{2}$$

with $E_{n,m} = (E_n + E_m)/2$.

Consistent with von Neumann's criteria for quantum ergodicity [14], we deal with quantum systems with a chaotic classical analog [15,16] for which the spectrum of energies E_n is nondegenerate. For such systems the states $|n\rangle\langle m|$ are specified by the integers n and m, or equivalently by the frequency $\lambda_{n,m}$ and energy $E_{n,m}$. Since the distributions $|n\rangle\langle m|$ govern the quantum dynamics [5] an understanding of their $h \to 0$ limit, or of their Wigner representation $\rho^w_{n,m}(\mathbf{x})$,

$$\rho^w_{n,m}(\mathbf{x}) \equiv h^{-s/2} \int d\mathbf{v}\, e^{i\mathbf{p}\cdot\mathbf{v}/\hbar} \langle \mathbf{q} - \mathbf{v}/2|n\rangle\langle m|\mathbf{q} + \mathbf{v}/2\rangle, \tag{3}$$

would seem essential for verification of correspondence. [Here $\mathbf{x}=(\mathbf{p},\mathbf{q})$ where $\mathbf{p}$ are the momenta and $\mathbf{q}$ are the coordinates.] However, as shown below, the relevant objects for the study of correspondence in chaotic systems are the quantum spectral projection operators [17], which are of the form $\rho^{w*}_{n,m}(\mathbf{x}_0)\rho^{w}_{n,m}(\mathbf{x})$ in the Wigner representation. That is, we demonstrate that for irregular systems these quantum Liouville spectral projection operators approach classical spectral projection operators Υ of the same frequency and energy as $h\to 0$, i.e., that

$$\rho^{w*}_{n,n}(\mathbf{x}_0)\rho^{w}_{n,n}(\mathbf{x})\to dE\ \Upsilon_{E_n}(\mathbf{x},\mathbf{x}_0) \tag{4}$$

and

$$\rho^{w*}_{n,m}(\mathbf{x}_0)\rho^{w}_{n,m}(\mathbf{x})\to dE\ d\lambda\ \Upsilon_{E_{n,m},\lambda_{n,m}}(\mathbf{x};\mathbf{x}_0) \quad (n\neq m). \tag{5}$$

Here the distributions Υ_E and $\Upsilon_{E,\lambda}$ are the stationary and nonstationary Liouville spectral projection operators of classical dynamics [5], discussed in paper I. We also show that the spectrum of the quantum Liouville operator goes to that of the classical operator as $h\to 0$ and that the correspondence emerges smoothly via the elimination of essential singularities. Proof of Eqs. (4) and (5), plus proof of the correspondence of the Liouville spectra, suffices to prove quantum-classical correspondence in chaotic systems.

Note that, unlike their quantum analogs $\rho^{w*}_{n,m}(\mathbf{x}_0)\rho^{w}_{n,m}(\mathbf{x})$, the nonstationary chaotic classical spectral projection operators $\Upsilon_{E_{n,m},\lambda_{n,m}}$ cannot be written as a product of Liouville eigenfunctions but rather consist of a sum of products of Liouville eigenfunctions (see paper I). Hence Eq. (5) suggests, as discussed below, that the individual nonstationary quantum Liouville eigenfunctions $\rho^{w}_{n,m}(\mathbf{x})$ for quantum systems with chaotic classical analogs do not have well-defined correspondence limits. This situation is quite different from that of the integrable case discussed in paper I and necessitates the introduction of new tools to prove correspondence.

This paper is organized as follows: Section II introduces a useful Dirac notation to simplify our formal manipulations, and the proof of Eqs. (4) and (5) is expressed in this Dirac form. Section III proves correspondence for both the Liouville spectral projectors and the Liouville eigenvalues. This treatment ignores higher-order corrections relating to scars, which are treated in Sec. IV. The proof of correspondence allows us to consider the classical limit of operator matrix elements, which is discussed in Sec. V. Section VI provides a summary.

II. A DIRAC FORMULATION OF LIOUVILLE DYNAMICS

The effectiveness of the Liouville picture is limited by the clumsiness of the associated density matrix notation. In this section we introduce a useful Dirac notation that simplifies manipulations considerably [18]. We will also employ a Dirac notation for the classical Liouville dynamics in order to maintain symmetry between the quantum and classical formulations.

Let $|n\rangle$ be a complete, orthonormal set of basis states for the quantum Hilbert space associated with the solutions of the Schrödinger equation. That is, $\langle n|m\rangle=\delta_{n,m}$ and $\sum_n|n\rangle\langle n|=1$. From these states we construct distributions $\hat{\rho}_{n,m}=|n\rangle\langle m|$, which are a basis in the Hilbert space associated with solutions of the von Neumann equation. It is natural to assign a Dirac notation to these basis states, i.e.,

$$|n,m)\equiv\hat{\rho}_{n,m}. \tag{6}$$

A complete orthonormal basis $|n\rangle$ of Schrödinger states then yields a complete set of Liouville states $|n,m)$. One can now easily deduce that the dual space is spanned by the linear functionals

$$(n,m|=\mathrm{Tr}\{\hat{\rho}^{\dagger}_{n,m}\cdot\} \tag{7}$$

by requiring that $(n,m|k,l)=\delta_{n,m}\delta_{k,l}$. Note that the normalization of the states $|n,m)$ has been chosen so that $|n,m)(n,m|$ is a projection operator. Completeness implies that

$$\sum_{n,m}|n,m)(n,m|=1. \tag{8}$$

The spectral decomposition of the Liouville operator then takes the form

$$\hat{L}=\sum_{n,m}\lambda_{n,m}|n,m)(n,m|. \tag{9}$$

The Hermitian operators [19] $|n,m)(n,m|=\hat{\rho}_{n,m}\mathrm{Tr}\{\hat{\rho}^{\dagger}_{n,m}\cdot\}$ are obviously the spectral projection operators in the Liouville picture, in the same way that $|n\rangle\langle n|$ are the spectral projection operators in the traditional Hamiltonian picture. Arbitrary superoperators of the form $[\hat{O},\ ]_{\pm}$ (i.e., $[\hat{O},\ ]_{\pm}\hat{\rho}=\hat{O}\hat{\rho}\pm\hat{\rho}\hat{O}$), of interest below, can be expanded on the $|n,m)$ states as

$$[\hat{O},\ ]_{\pm}=\sum_{n,m,k,l}|n,m)(n,m|[\hat{O},\ ]_{\pm}|k,l)(k,l|, \tag{10}$$

where the superoperator ''matrix elements'' are $(n,m|[\hat{O},\ ]_{\pm}|k,l)=O_{n,k}\delta_{l,m}\pm O_{l,m}\delta_{n,k}$.

Physical states $|\rho)$ are defined as

$$|\rho)=h^{-s/2}\hat{\rho}, \tag{11}$$

with corresponding kets

$$(\rho|=h^{-s/2}\mathrm{Tr}\{\hat{\rho}^{\dagger}\cdot\}=h^{-s/2}\mathrm{Tr}\{\hat{\rho}\cdot\}, \tag{12}$$

with the latter equality due to the fact that $\hat{\rho}^{\dagger}=\hat{\rho}$. Equations (11) and (12), which define the physical states, differ from Eqs. (6) and (7), which define the basis states by a factor of $h^{-s/2}$, which is introduced so that the quasiprobability distributions associated with the physical states have the correct dimensions in a phase-space representation, i.e., inverse action to a power equal to the number of degrees of freedom. We also assign states $|A)$ to operators $\hat{A}$ (i.e., operators operating on the Hilbert space spanned by $|n\rangle$) via $|A)=h^{s/2}\hat{A}$ and $(A|=h^{s/2}\mathrm{Tr}\{\hat{A}^{\dagger}\cdot\}$. The expectation of $\hat{A}$ is then given by $(\rho|A)=(A|\rho)^{*}=\mathrm{Tr}\{\hat{\rho}\hat{A}\}$.

In this notation the von Neumann (quantum Liouville) equation [20] is

$$\frac{\partial}{\partial t}|\rho(t))=-i\hat{L}|\rho(t)), \tag{13}$$

and the Wigner-Weyl representation [21,22] of a state ρ takes the form

$$(\mathbf{x}|\rho)=h^{-s/2}\mathrm{Tr}\{h^{-s/2}\hat{\Delta}(\mathbf{x})\hat{\rho}\}, \tag{14}$$

where

$$\hat{\Delta}(\mathbf{x})=h^{-s}\int d\mathbf{u}\, d\mathbf{v}\, e^{i[\mathbf{v}\cdot(\mathbf{p}-\hat{\mathbf{p}})+\mathbf{u}\cdot(\mathbf{q}-\hat{\mathbf{q}})]/\hbar}. \tag{15}$$

Thus, employing Eq. (11), we identify

$$(\mathbf{x}|=\mathrm{Tr}\{h^{-s/2}\hat{\Delta}(\mathbf{x})\cdot\}=h^{-s/2}\mathrm{Tr}\{\hat{\Delta}^{\dagger}(\mathbf{x})\cdot\}, \tag{16}$$

where the second equality is due to the fact that $\hat{\Delta}(\mathbf{x})$ is Hermitian. The particular form of the corresponding ket [23,24] is determined by demanding that $(\mathbf{x}|\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}')$. Thus, here

$$|\mathbf{x})=h^{-s/2}\hat{\Delta}(\mathbf{x}), \tag{17}$$

where $(\mathbf{x}|\mathbf{x}')=h^{-s}\mathrm{Tr}\{\hat{\Delta}(\mathbf{x})\hat{\Delta}(\mathbf{x}')\}=\delta(\mathbf{x}-\mathbf{x}')$. Since $|\mathbf{x})$ and $(\mathbf{x}|$ span the Hilbert space and its dual space, they satisfy the closure relation

$$\int d\mathbf{x}\ |\mathbf{x})(\mathbf{x}|=1. \tag{18}$$

Definitions (16) and (17) in conjunction with Eqs. (11) and (12) guarantee that the probability densities $(\mathbf{x}|\rho)$ have the correct dimensions. Other phase-space representations [in which $(\mathbf{x}|$ and $|\mathbf{x})$ may be quite dissimilar], and a general transformation theory between them is provided elsewhere [26].

Consider then the Liouville spectral decomposition [i.e., Eqs. (1) and (2)] for a chaotic quantum system in the Dirac notation. As eigenfunctions of $\hat{L}$ and $\hat{\mathcal{H}}$ the $|n,m)$ satisfy

$$\hat{L}|n,m)=\lambda_{n,m}|n,m), \tag{19}$$

and

$$\hat{\mathcal{H}}|n,m)=E_{n,m}|n,m). \tag{20}$$

In the Wigner-Weyl representation Eqs. (19) and (20) become

$$(\mathbf{x}|\hat{L}|n,m)=L(\mathbf{x})(\mathbf{x}|n,m)=\lambda_{n,m}(\mathbf{x}|n,m) \tag{21}$$

and

$$(\mathbf{x}|\hat{\mathcal{H}}|n,m)=\mathcal{H}(\mathbf{x})(\mathbf{x}|n,m)=E_{n,m}(\mathbf{x}|n,m), \tag{22}$$

where $L(\mathbf{x})=(2i/\hbar)H(\mathbf{x})\sin(\hbar\sigma/2)$ is the quantum Liouville operator, and $\mathcal{H}(\mathbf{x})=H(\mathbf{x})\cos\{\hbar\sigma/2\}$ is the energy operator. Here $\sigma=\overleftarrow{\partial}/\partial\mathbf{x}\, J\overrightarrow{\partial}/\partial\mathbf{x}$ is the Poisson bracket, i.e., $A(\mathbf{x})\sigma B(\mathbf{x})=\{A,B\}$, and $J=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$ is the $(2s\times 2s)$-dimensional symplectic matrix [27]. Expanding $\mathcal{H}(\mathbf{x})$ in powers of h shows that the classical analog of $\mathcal{H}(\mathbf{x})$ is the energy function $H(\mathbf{x})$, and that $L_c(\mathbf{x})$ is the correspondence limit of $L(\mathbf{x})$.

Consider now the classical case. The classical analog of a phase-space representation is a choice of canonical variables for a classical distribution ρ_c. Thus we denote the phase-space representation of ρ_c by $\rho_c(\mathbf{x})=(\mathbf{x}|\rho_c)$.

The classical Liouville spectral decomposition, and the properties of the eigendistributions discussed in paper I [5] are readily restated using the Dirac notation. Associating states $|E)$ with the classical distributions $\rho_E(\mathbf{x})$, which span the point spectrum, and states $|E,\lambda,l)$ with the classical distributions $\rho^l_{E,\lambda}(\mathbf{x})$, which span the continuous spectrum, the full set of equations for the spectral decomposition becomes

$$(E'|E)=\delta(E'-E), \tag{23}$$

$$(E'|E,\lambda,l)=0, \tag{24}$$

$$(E'\lambda',l'|E,\lambda,l)=\delta_{l',l}\delta(E'-E)\delta(\lambda'-\lambda), \tag{25}$$

$$\int_0^\infty dE\ \ |E)(E|+\int_0^\infty dE ⨍ d\lambda\sum_l\ |E,\lambda,l)(E,\lambda,l|=1, \tag{26}$$

$$e^{-iL_ct}|E)=|E), \tag{27}$$

and

$$e^{-iL_ct}|E,\lambda,l)=e^{-i\lambda t}|E,\lambda,l). \tag{28}$$

Here the line through the integral in Eq. (26) indicates that the point spectrum eigenvalue $\lambda=0$ has been removed (see paper I). Two further equations relate to the second constant of the motion, a classical energy operator $\mathcal{H}_c$:

$$\mathcal{H}_c|E)=E|E) \tag{29}$$

and

$$\mathcal{H}_c|E,\lambda,l)=E|E,\lambda,l). \tag{30}$$

In the phase-space representation parametrized by $\mathbf{x}$ these equations become

$$(\mathbf{x}|\mathcal{H}_c|E)=H(\mathbf{x})(\mathbf{x}|E)=E(\mathbf{x}|E) \tag{31}$$

and

$$(\mathbf{x}|\mathcal{H}_c|E,\lambda,l)=H(\mathbf{x})(\mathbf{x}|E,\lambda,l)=E(\mathbf{x}|E,\lambda,l). \tag{32}$$

A complete set of stationary and nonstationary classical Liouville eigenfunctions $\rho_E(\mathbf{x})=(\mathbf{x}|E)$ and $\rho^l_{E,\lambda}(\mathbf{x})=(\mathbf{x}|E,\lambda,l)$, were introduced in paper I where the integer l labels the infinite degeneracy of the continuous spectrum [16]. In addition, spectral projection operators $\Upsilon_E(\mathbf{x};\mathbf{x}_0)$ and $\Upsilon_{E,\lambda}(\mathbf{x};\mathbf{x}_0)$ were introduced; these are the phase-space representations of the classical operators $\delta(E-\mathcal{H}_c)=|E)(E|$ and $\delta(E-\mathcal{H}_c)\delta(\lambda-L_c)=\Sigma_l|E,\lambda,l)(E,\lambda,l|$. Specifically,

$$(\mathbf{x}|E)(E|\mathbf{x}_0)=\Upsilon_E(\mathbf{x};\mathbf{x}_0)=\frac{\delta(E-H(\mathbf{x}_0))\delta(E-H(\mathbf{x}))}{\int d\mathbf{x}'\,\delta(E-H(\mathbf{x}'))}=\rho_E(\mathbf{x})\rho_E^*(\mathbf{x}_0) \tag{33}$$

and

$$\sum_l (\mathbf{x}|E,\lambda,l)(E,\lambda,l|\mathbf{x}_0)=\Upsilon_{E,\lambda}(\mathbf{x};\mathbf{x}_0)=\frac{1}{2\pi}\delta(E-H(\mathbf{x}_0))\int_{-\infty}^{\infty}dt'\,e^{i\lambda t'}\,\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t'))=\sum_l \rho_{E,\lambda}^{l*}(\mathbf{x}_0)\rho_{E,\lambda}^{l}(\mathbf{x}), \tag{34}$$

where $\mathbf{X}(\mathbf{x},-t')$ is the phase-space point from which $\mathbf{x}$ emerges over a time t'.

In terms of these eigenfunctions the spectral decomposition [Eq. (26)] takes the form

$$\int_0^{\infty}dE\ \ \rho_E^*(\mathbf{x}_0)\rho_E(\mathbf{x})+\int_0^{\infty}dE⨍d\lambda\ \sum_l \rho_{E,\lambda}^{l*}(\mathbf{x}_0)\rho_{E,\lambda}^{l}(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_0) \tag{35}$$

or

$$\int_0^{\infty}dE\ \ \Upsilon_E(\mathbf{x};\mathbf{x}_0)+\int_0^{\infty}dE⨍d\lambda\ \ \Upsilon_{E,\lambda}(\mathbf{x};\mathbf{x}_0)=\delta(\mathbf{x}-\mathbf{x}_0). \tag{36}$$

Thus, the evolution of any initial distribution $\rho(\mathbf{x},0)$ can be written, in quantum mechanics, as an expansion:

$$\rho(\mathbf{x},t)=\sum_n c_{n,n}\ \rho_{n,n}^{w}(\mathbf{x})+\sum_{n\neq m} c_{n,m}\ \rho_{n,m}^{w}(\mathbf{x})e^{-i\lambda_{n,m}t}=\int d\mathbf{x}_0\rho(\mathbf{x}_0,0)\left[\sum_n \rho_{n,n}^{w*}(\mathbf{x}_0)\rho_{n,n}^{w}(\mathbf{x})+\sum_{n\neq m}\rho_{n,m}^{w*}(\mathbf{x}_0)\rho_{n,m}^{w}(\mathbf{x})e^{-i\lambda_{n,m}t}\right] \tag{37}$$

and in classical mechanics as

$$\begin{aligned}\rho^c(\mathbf{x},t)&=\int dE\ \ c_E\rho_E(\mathbf{x})+\int dE⨍d\lambda\sum_l c_{E,\lambda,l}\ \rho_{E,\lambda}^{l}(\mathbf{x})e^{-i\lambda t}\\&=\int d\mathbf{x}_0\rho(\mathbf{x}_0,0)\left[\int dE\ \ \rho_E^*(\mathbf{x}_0)\rho_E(\mathbf{x})+\int dE⨍d\lambda\sum_l \rho_{E,\lambda}^{l*}(\mathbf{x}_0)\rho_{E,\lambda}^{l}(\mathbf{x})e^{-i\lambda t}\right]\\&=\int d\mathbf{x}_0\rho(\mathbf{x}_0,0)\left[\int dE\ \ \Upsilon_E(\mathbf{x};\mathbf{x}_0)+\int dE⨍d\lambda\Upsilon_{E,\lambda}(\mathbf{x};\mathbf{x}_0)e^{-i\lambda t}\right].\end{aligned} \tag{38}$$

Equations (37) and (38) make clear that a demonstration of correspondence for the spectral projection operators and their eigenvalues is sufficient to establish correspondence for the dynamics, i.e., that $\rho(\mathbf{x},t)\to\rho^c(\mathbf{x},t)$ as $h\to 0$. That is, formally establishing correspondence requires demonstrating

$$|n,n)(n,n|\to dE\ \ |E_n)(E_n| \tag{39}$$

and

$$|n,m)(n,m|\to dE\ d\lambda\ \sum_l |E_{n,m},\lambda_{n,m},l)(E_{n,m},\lambda_{n,m},l|, \tag{40}$$

or

$$(\mathbf{x}|n,n)(n,n|\mathbf{x}_0)=\rho_{n,n}^{w*}(\mathbf{x}_0)\rho_{n,n}^{w}(\mathbf{x})\to dE\ \ \Upsilon_{E_n}(\mathbf{x};\mathbf{x}_0)=dE(\mathbf{x}|E_n)(E_n|\mathbf{x}_0) \tag{41}$$

and

$$(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)=\rho_{n,m}^{w*}(\mathbf{x}_0)\rho_{n,m}^{w}(\mathbf{x})\to dE\ d\lambda\ \ \Upsilon_{E_{n,m},\lambda_{n,m}}(\mathbf{x};\mathbf{x}_0)=dE\ d\lambda\ \sum_l (\mathbf{x}|E_{n,m},\lambda_{n,m},l)(E_{n,m},\lambda_{n,m},l|\mathbf{x}_0), \tag{42}$$

with the infinitesimals dE and $d\lambda$ to be determined. These limits are proven in Sec. III A.

III. CORRESPONDENCE

Consider then the correspondence limit, i.e., the limit of the quantum Liouville dynamics as $h\to 0$, with the $h\to 0$ limit taken before the $T\to\infty$ limit [9,28]. This order, $h\to 0$ first, is consistent with the actual physics in which one first chooses a particular system and then propagates it for long times. Technically, this is achieved by first broadening the system energy by some amount $\epsilon \gg h/T_{\min}$ (thus restricting the dynamics to finite time) and taking the $h\to 0$ limit with ϵ fixed. The broadening ϵ can then be chosen infinitesimal, $\epsilon\to\langle\Delta E\rangle$ (where $\langle\Delta E\rangle$ is the average spacing between neighboring energy levels) allowing for long time dynamics. Here $T_{\min}$ is the period of the shortest periodic orbit.

The physical significance of correspondence under these limits is clear. A transition from quantum to classical behavior will be observed in the dynamics of a physical system as $h\to 0$ provided that (a) the apparatus with which we observe its dynamics has a fixed, classically small but quantum mechanically large, energy resolution, and that (b) we do not observe its dynamics beyond the recurrence time given approximately by $h/\langle\Delta E\rangle$.

A. Correspondence for spectral projection operators

Here we examine the correspondence limits of the spectral projection operators $|n,n)(n,n|$ and $|n,m)(n,m|$. We focus attention on the nonstationary case. The stationary case [Eq. (41)] has already been obtained by Berry and Voros [4,29,30], but we work through this case to demonstrate the consistency of our approach. In the latter case consider Berry's formula [4,30,31]

$$\lim_{\epsilon\to 0}\pi\epsilon W(\mathbf{x};E_n,\epsilon)=\int d\mathbf{q}'\,e^{i\mathbf{p}\cdot\mathbf{q}'/\hbar}\langle\mathbf{q}-\mathbf{q}'/2|n\rangle\langle n|\mathbf{q}+\mathbf{q}'/2\rangle$$
$$=h^{s/2}\;(\mathbf{x}|n,n) \tag{43}$$

for finite ϵ, in order to investigate the semiclassical form of the stationary Liouville eigendistribution $(\mathbf{x}|n,n)$ in the Wigner representation. Here $W(\mathbf{x};E,\epsilon)$ is the Lorentzian weighted sum of Wigner functions over a width ϵ about an energy E, i.e.,

$$W(\mathbf{x};E,\epsilon)\equiv\int d\mathbf{q}'\,e^{-i\mathbf{p}\cdot\mathbf{q}'/\hbar}\langle\mathbf{q}+\mathbf{q}'/2|\,\delta_\epsilon(E-\hat{H})|\mathbf{q}-\mathbf{q}'/2\rangle, \tag{44}$$

where

$$\delta_\epsilon(E-\hat{H})=-\frac{1}{\pi}\mathrm{Im}\frac{1}{E-\hat{H}+i\epsilon}$$
$$=\frac{1}{h}\int_{-\infty}^{\infty}dt\;\;e^{i(E-\hat{H})t/\hbar}e^{-\epsilon|t|/\hbar}. \tag{45}$$

For quantum systems with chaotic classical analogs, Berry [30] has shown, for small h and small ϵ, that (where $\sim$ denotes the form in the limit)

$$W(\mathbf{x};E,\epsilon)\sim\delta_\epsilon(E-H(\mathbf{x}))+\sum_j W^j_{\mathrm{scar}}(\mathbf{x};E,\epsilon), \tag{46}$$

and thus, by employing Eq. (43), that

$$(\mathbf{x}|n,n)\sim\pi\epsilon h^{-s/2}\delta_\epsilon(E_n-H(\mathbf{x}))$$
$$+\pi\epsilon h^{-s/2}\sum_j W^j_{\mathrm{scar}}(\mathbf{x};E_n,\epsilon). \tag{47}$$

Here W^j_{scar} is of the order h^{s-1} smaller than the $\delta_\epsilon(E_n-H(\mathbf{x}))$ term, and hence vanishes rapidly as $h\to 0$. These scar terms, neglected in this section, are considered in Sec. IV.

Neglecting the scar terms gives

$$(\mathbf{x}|n,n)\sim h^{-s/2}\pi\epsilon\delta(E_n-H(\mathbf{x}))\sim\pi\epsilon(\mathbf{x}|E_n)/\langle\Delta E\rangle^{1/2}, \tag{48}$$

where $(\mathbf{x}|E_n)=\rho_{E_n}(\mathbf{x})=\delta(E_n-H(\mathbf{x}))/[\int d\mathbf{x}'\,\delta(E_n-H(\mathbf{x}'))]^{1/2}$ (see paper I) and where $\langle\Delta E\rangle=h^s/\int d\mathbf{x}'\,\delta(E-H(\mathbf{x}'))$ is the average adjacent energy-level spacing. Therefore

$$(\mathbf{x}|n,n)(n,n|\mathbf{x}_0)\sim(\pi\epsilon)^2(\mathbf{x}|E_n)(E_n|\mathbf{x}_0)/\langle\Delta E\rangle. \tag{49}$$

Correspondence for the stationary eigenstates [Eq. (41)] then results if we take the limit $\pi\epsilon\to\langle\Delta E\rangle$ and note that $\langle\Delta E\rangle\to dE$.

The proof of Eq. (42) follows in a similar fashion from the following important relationship (proven in Appendix A) between stationary and nonstationary quantum Liouville eigenfunctions [32]:

$$(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)$$
$$=h^{-s}\int d\mathbf{u}\;d\mathbf{v}\;e^{-i(\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{v}/\hbar}e^{i(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{u}/\hbar}$$
$$\times((\mathbf{p}+\mathbf{u}/2,\mathbf{q}+\mathbf{v}/2)|n,n)(m,m|(\mathbf{p}_0-\mathbf{u}/2,\mathbf{q}_0-\mathbf{v}/2))$$
$$=h^{-s}\int d\mathbf{y}\;e^{i(\mathbf{x}-\mathbf{x}_0)\cdot J\cdot\tilde{\mathbf{y}}/\hbar}\cdot(\mathbf{x}+\mathbf{y}/2|n,n)$$
$$\times(m,m|\mathbf{x}_0-\mathbf{y}/2). \tag{50}$$

Here the tilde denotes the transpose (a column vector) of the row vector $\mathbf{y}$. Substituting Eq. (43) into Eq. (50) gives

$$(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)$$
$$=\lim_{\epsilon_1,\epsilon_2\to 0}h^{-2s}\pi^2\epsilon_1\epsilon_2\int d\mathbf{y}\;\;e^{i(\mathbf{x}-\mathbf{x}_0)\cdot J\cdot\tilde{\mathbf{y}}/\hbar}$$
$$\times W(\mathbf{x}+\mathbf{y}/2;E_n,\epsilon_1)W^*(\mathbf{x}_0-\mathbf{y}/2;E_m,\epsilon_2) \tag{51}$$

if the energy eigenvalues E_n and E_m are both nondegenerate, the case for a chaotic system. In the limit of small ϵ_1,ϵ_2 we obtain

$$(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)\sim h^{-2s}\pi^2\epsilon_1\epsilon_2\int d\mathbf{y}\;\; e^{i(\mathbf{x}-\mathbf{x}_0)\cdot J\cdot\tilde{\mathbf{y}}/\hbar}\times W(\mathbf{x}+\mathbf{y}/2;E_n,\epsilon_1)\times W^*(\mathbf{x}_0-\mathbf{y}/2;E_m,\epsilon_2), \tag{52}$$

which is amenable to semiclassical analysis.

As a first approximation we neglect the scar corrections and employ

$$W(\mathbf{x};E,\epsilon)\sim\delta_\epsilon(E-H(\mathbf{x}))=h^{-1}\int_{-\infty}^{\infty}dt\;\; e^{i[E-H(\mathbf{x})]t/\hbar}e^{-\epsilon|t|/\hbar} \tag{53}$$

as we did in the stationary case. Substituting Eq. (53) into Eq. (52), followed by a simple change of variables $\mathbf{y}\rightarrow h\mathbf{y}$, yields

$$\begin{aligned}(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)&\sim\pi^2\epsilon_1\epsilon_2\int d\mathbf{y}\;\; e^{2\pi i(\mathbf{x}-\mathbf{x}_0)\cdot J\cdot\tilde{\mathbf{y}}}\delta_{\epsilon_1}(E_n-H(\mathbf{x}+h\mathbf{y}/2))\delta_{\epsilon_2}(E_m-H(\mathbf{x}_0-h\mathbf{y}/2))\\&=\frac{\pi^2\epsilon_1\epsilon_2}{h^2}\int d\mathbf{y}\; dt_1dt_2e^{-\epsilon_1|t_1|/\hbar}e^{-\epsilon_2|t_2|/\hbar}e^{2\pi i(\mathbf{x}-\mathbf{x}_0)\cdot J\cdot\tilde{\mathbf{y}}}\exp\{i[E_n-H(\mathbf{x}+h\mathbf{y}/2)]t_1/\hbar\}\\&\quad\times\exp\{i[E_m-H(\mathbf{x}_0-h\mathbf{y}/2)]t_2/\hbar\}.\end{aligned} \tag{54}$$

Note, at this stage, the presence of essential singularities in each of the highly oscillatory phase-factors $\exp\{i[E_n-H(\mathbf{x}+h\mathbf{y}/2)]t_1/\hbar\}$ and $\exp\{i[E_m-H(\mathbf{x}_0-h\mathbf{y}/2)]t_2/\hbar\}$.

We now let $h\rightarrow 0$ with ϵ_1,ϵ_2 fixed. Expanding the displaced Hamiltonian functions in Eq. (54) in powers of h:

$$H(\mathbf{x}+h\mathbf{y}/2)\sim H(\mathbf{x})+\frac{h}{2}\frac{\partial H(\mathbf{x})}{\partial\mathbf{x}}\cdot\tilde{\mathbf{y}}\sim H(\mathbf{x})-\frac{h}{2}\left[\frac{\partial H(\mathbf{x})}{\partial\mathbf{x}}J\right]\cdot J\cdot\tilde{\mathbf{y}}, \tag{55}$$

$$H(\mathbf{x}_0-h\mathbf{y}/2)\sim H(\mathbf{x}_0)-\frac{h}{2}\frac{\partial H(\mathbf{x}_0)}{\partial\mathbf{x}_0}\cdot\tilde{\mathbf{y}}\sim H(\mathbf{x}_0)+\frac{h}{2}\left[\frac{\partial H(\mathbf{x}_0)}{\partial\mathbf{x}_0}J\right]\cdot J\cdot\tilde{\mathbf{y}} \tag{56}$$

and substituting these expressions into Eq. (54) gives

$$\begin{aligned}(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)&\sim\frac{\pi^2\epsilon_1\epsilon_2}{h^2}\int d\mathbf{y}\; dt_1dt_2e^{-\epsilon_1|t_1|/\hbar}e^{-\epsilon_2|t_2|/\hbar}e^{i[E_n-H(\mathbf{x})]t_1/\hbar}e^{i[E_m-H(\mathbf{x}_0)]t_2/\hbar}\\&\quad\times\exp\left\{2\pi i\left[\left(\mathbf{x}-\frac{\partial H(\mathbf{x})}{\partial\mathbf{x}}Jt_1/2\right)-\left(\mathbf{x}_0-\frac{\partial H(\mathbf{x}_0)}{\partial\mathbf{x}_0}Jt_2/2\right)\right]\cdot J\cdot\tilde{\mathbf{y}}\right\}.\end{aligned} \tag{57}$$

The factor $e^{-\epsilon_1|t_1|/\hbar}e^{-\epsilon_2|t_2|/\hbar}$ guarantees that the integrand is zero for all but short times since $\epsilon_1|t_1|/\hbar\gg 2\pi|t_1|/T_{\min}$ and $\epsilon_2|t_2|/\hbar\gg 2\pi|t_2|/T_{\min}$, so that we can use the short time approximation:

$$\mathbf{X}(\mathbf{x},-t_1/2)\sim\mathbf{x}-\frac{\partial H(\mathbf{x})}{\partial\mathbf{x}}\cdot Jt_1/2, \tag{58}$$

$$\mathbf{X}'(\mathbf{x}_0,-t_2/2)\sim\mathbf{x}_0-\frac{\partial H(\mathbf{x}_0)}{\partial\mathbf{x}_0}\cdot Jt_2/2. \tag{59}$$

Using these results in Eq. (57) gives

$$\begin{aligned}(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)&\sim\frac{\pi^2\epsilon_1\epsilon_2}{h^2}\int d\mathbf{y}\; dt_1dt_2e^{-\epsilon_1|t_1|/\hbar}e^{-\epsilon_2|t_2|/\hbar}e^{i[E_n-H(\mathbf{x})]t_1/\hbar}e^{i[E_m-H(\mathbf{x}_0)]t_2/\hbar}\\&\quad\times\exp\{2\pi i[\mathbf{X}(\mathbf{x},-t_1/2)-\mathbf{X}'(\mathbf{x}_0,-t_2/2)]\cdot J\cdot\tilde{\mathbf{y}}\}\\&=\frac{\pi^2\epsilon_1\epsilon_2}{h^2}\int dt_1dt_2e^{-\epsilon_1|t_1|/\hbar}e^{-\epsilon_2|t_2|/\hbar}e^{i[E_n-H(\mathbf{x})]t_1/\hbar}e^{i[E_m-H(\mathbf{x}_0)]t_2/\hbar}\delta(\mathbf{X}(\mathbf{x},-t_1/2)-\mathbf{X}'(\mathbf{x}_0,-t_2/2)).\end{aligned} \tag{60}$$

Noting that Eq. (60) is identically zero unless $\mathbf{x}$ and $\mathbf{x}_0$ are on the same trajectory and using the fact that the Hamiltonian is time independent allows us to replace $H(\mathbf{x})$ by $H(\mathbf{x}_0)$ in the exponential. Next we perform a canonical transformation (i.e., time translation) to put Eq. (60) in the form

$$(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)\sim\frac{\pi^2\epsilon_1\epsilon_2}{h^2}\int dt_1dt_2e^{-\epsilon_1|t_1|/\hbar}e^{-\epsilon_2|t_2|/\hbar}e^{i[E_n-H(\mathbf{x}_0)]t_1/\hbar}e^{i[E_m-H(\mathbf{x}_0)]t_2/\hbar}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-(t_1-t_2)/2))$$

$$=\frac{\pi^2 2\epsilon_1\epsilon_2}{h^2}\int dt'dt_0e^{-\epsilon_1|t_0|/\hbar}e^{-\epsilon_2|t_0-2t'|/\hbar}e^{i[E_n-H(\mathbf{x}_0)]t_0/\hbar}e^{i[E_m-H(\mathbf{x}_0)](t_0-2t')/\hbar}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t')),\quad(61)$$

where we have changed variables to $t'=(t_1-t_2)/2$ and $t_0=t_1$. Note that $e^{-\epsilon_1|t_0|/\hbar}\sim 0$ unless $t_0\sim 0$ so that we can replace $e^{-\epsilon_2|t_0-2t'|/\hbar}$ by $e^{-2\epsilon_2|t'|/\hbar}$ in Eq. (61). Defining $\epsilon_0=\epsilon_1/2$ and $\epsilon'=2\epsilon_2$ we obtain

$$(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)\sim\frac{\pi^2 2\epsilon_0\epsilon'}{h^2}\int dt'dt_0e^{-2\epsilon_0|t_0|/\hbar}e^{-\epsilon'|t'|/\hbar}e^{i[E_n-H(\mathbf{x}_0)]t_0/\hbar}e^{i[E_m-H(\mathbf{x}_0)](t_0-2t')/\hbar}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t')).\quad(62)$$

Using identity (45) in reverse then yields two equivalent forms:

$$(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)\sim\frac{\pi^2\epsilon_0\epsilon'}{h}\delta_{\epsilon_0}(E_{n,m}-H(\mathbf{x}_0))\int dt'e^{-\epsilon'|t'|/\hbar}e^{-2i[E_m-H(\mathbf{x}_0)]t'/\hbar}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t'))\quad(63)$$

and

$$(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)\sim\frac{\pi^2\epsilon_0\epsilon'}{h}\delta_{\epsilon_0}(E_{n,m}-H(\mathbf{x}_0))\int dt'e^{-\epsilon'|t'|/\hbar}e^{i\lambda_{n,m}t'}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t')).\quad(64)$$

Note that Eq. (64) no longer exhibits the essential singularities present in Eq. (54). This is due to the expansions in Eqs. (55) and (56) through which the essential singularities are eliminated. We found that the same mechanism, i.e., elimination of essential singularities, was responsible for correspondence in chaotic mappings of the torus [10].

We now take the limit ''$T\to\infty$,'' that is, we let $\epsilon_0/\hbar,\epsilon'/\hbar\to 0$. Note that ϵ_0 and ϵ' essentially define a cutoff in time beyond which the semiclassical approximations break down. The commonly adopted cutoff time is the density of states time $T_{ds}\sim h/\langle\Delta E\rangle$. Since our t' is symmetric about zero, propagation to T_{ds} implies that $-T_{ds}/2\leqslant t'\leqslant T_{ds}/2$. To achieve this we let $\epsilon' T_{ds}/2\hbar\to 1$ or, substituting $T_{ds}\sim h/\langle\Delta E\rangle$, $\pi\epsilon'\to\langle\Delta E\rangle$. Thus the precise limits we must take to achieve the ''$T\to\infty$'' limit $\epsilon_0/\hbar,\epsilon'/\hbar\to 0$ are $\pi\epsilon_0,\pi\epsilon'\to\langle\Delta E\rangle$. The relation between the $\epsilon_0/\hbar,\epsilon'/\hbar\to 0$ limit and the $T_{ds}\to\infty$ limit is explicit in a formula proven by Kay [33]:

$$\lim_{\epsilon'/\hbar\to 0}\epsilon'/\hbar\int dt'e^{-\epsilon'|t'|/\hbar}\cdot=\lim_{T\to\infty}\frac{2}{T}\int_{-T/2}^{T/2}dt'\cdot.\quad(65)$$

We consider the correspondence limit of Eq. (64) for the case of $n\neq m$, as well as for $n=m$. Consider first $n\neq m$. Performing the limits as outlined above and making use of Eq. (65) we obtain

$$(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)=\lim_{\epsilon_0/\hbar\to 0}\frac{\pi\epsilon_0}{2}\delta_{\epsilon_0}(E_{n,m}-H(\mathbf{x}_0))\lim_{\epsilon'/\hbar\to 0}\frac{\epsilon'}{\hbar}\int dt'e^{-\epsilon'|t'|/\hbar}e^{i\lambda_{n,m}t'}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t'))$$

$$=\lim_{\epsilon_0/\hbar\to 0}\frac{\pi\epsilon_0}{2}\delta_{\epsilon_0}(E_{n,m}-H(\mathbf{x}_0))\lim_{T\to\infty}\frac{2}{T}\int_{-T/2}^{T/2}dt'e^{i\lambda_{n,m}t'}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t'))$$

$$=\lim_{\epsilon_0/\hbar\to 0}\frac{\pi\epsilon_0}{2}\delta_{\epsilon_0}(E_{n,m}-H(\mathbf{x}_0))2\frac{d\lambda}{2\pi}\int_{-\infty}^{\infty}dt'e^{i\lambda_{n,m}t'}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t')),$$

$$=dE\ d\lambda\ \delta(E_{n,m}-H(\mathbf{x}_0))\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'e^{i\lambda_{n,m}t'}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t')).\quad(66)$$

Here we have interpreted the limit of $1/T$ to be $d\lambda/2\pi$ and $\pi\epsilon_0=\langle\Delta E\rangle\approx dE$. To see that this is correct recall that we have taken the limit as $\pi\epsilon'\to\langle\Delta E\rangle$, which corresponds to letting $T\to T_{\rm ds}$. The inverse of the density of states time can be roughly interpreted as the average nearest-neighbor Liouville frequency divided by 2π. The average nearest-neighbor Liouville frequency $\langle\Delta E\rangle/\hbar$ can evidently be interpreted as $d\lambda$. Given Eq. (34) we have

$$\lim_{h\to 0}\ (\mathbf{x}|n,m)(n,m|\mathbf{x}_0)=dE\ d\lambda\ \Upsilon_{E_{n,m},\lambda_{n,m}}(\mathbf{x};\mathbf{x}_0), \tag{67}$$

hence proving correspondence.

Note that in establishing Eq. (67) we have also shown that individual quantum Liouville eigenstates $(\mathbf{x}|n,m)$, with $n\neq m$, do not have correspondence limits when the classical system is chaotic. We may also infer the reason: individual states $(\mathbf{x}|n,m)$, $n\neq m$, possess essential singularities that cancel in the product $(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)$ to give a well-defined correspondence limit.

Consider now the case where $n=m$ in Eq. (64). Using Eq. (65) and the fact that the classical dynamics is ergodic, we can replace the time average by a phase average and rewrite the integral in Eq. (64) as

$$\begin{aligned}\lim_{\epsilon'/\hbar\to 0}\ \epsilon'/\hbar\int dt'\, e^{-\epsilon'|t'|/\hbar}\,\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t'))\\ =\lim_{T\to\infty}\frac{2}{T}\int_{-T/2}^{T/2}dt'\ \delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t'))\\ =2\int d\mathbf{x}_0\ \frac{\delta(E-H(\mathbf{x}_0))}{\int d\mathbf{x}'\ \delta(E-H(\mathbf{x}'))}\,\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t'))\\ =2\frac{\delta(E-H(\mathbf{x}))}{\int d\mathbf{x}'\ \delta(E-H(\mathbf{x}'))}.\end{aligned} \tag{68}$$

Substituting this expression into Eq. (64) and again interpreting $\pi\epsilon_0\sim\langle\Delta E\rangle\sim dE$ we obtain [Eq. (41)] the desired correspondence:

$$\lim_{h\to 0}\ (\mathbf{x}|n,n)(n,n|\mathbf{x}_0)=dE\ \ \Upsilon_{E_n}(\mathbf{x};\mathbf{x}_0). \tag{69}$$

Since

$$\Upsilon_E(\mathbf{x};\mathbf{x}_0)=\rho_E^*(\mathbf{x}_0)\rho_E(\mathbf{x})=(\mathbf{x}|E)(E|\mathbf{x}_0), \tag{70}$$

Eq. (70) implies that a product of stationary quantum Liouville eigenfunctions goes to a product of stationary classical Liouville eigenfunctions. However, this is not the case for the nonstationary projectors. That is,

$$\begin{aligned}\Upsilon_{E,\lambda}(\mathbf{x};\mathbf{x}_0)=\sum_l\ \rho_{E,\lambda}^{l*}(\mathbf{x}_0)\rho_{E,\lambda}^{l}(\mathbf{x})\\ =\sum_l\ (\mathbf{x}|E,\lambda,l)(E,\lambda,l|\mathbf{x}_0)\end{aligned} \tag{71}$$

is not a simple product of Liouville eigenfunctions, but rather a sum of products. Thus Eq. (34) implies that the correspondence limit of a product of nonstationary quantum Liouville eigenfunctions is a sum of products of nonstationary classical Liouville eigenfunctions, due to the degeneracy of the classical states.

B. Correspondence for the Liouville spectrum

In addition to the limit relations for the spectral projection operators discussed in Sec. III A, correspondence requires that the quantum spectrum reduces to its classical analog in the $h\to 0$ limit. Since the classical Liouville spectrum is continuous we examine spectral densities, rather than individual Liouville eigenvalues. The quantum Liouville operator $\delta(\lambda-\hat{L})$ can be expanded on the Liouville eigenbasis as

$$\delta(\lambda-\hat{L})=\sum_{n,m}\ |n,m)(n,m|\,\delta(\lambda-\hat{L})|n,m)(n,m|. \tag{72}$$

The trace $D(\lambda)$ of $\delta(\lambda-\hat{L})$, is the quantum Liouville spectral density, i.e.,

$$\begin{aligned}D(\lambda)=\mathrm{Tr}\,[\,\delta(\lambda-\hat{L})]=\sum_{n,m}\ \ (n,m|\,\delta(\lambda-\hat{L})|n,m)\\ =\sum_{n,m}\ \ (n,m|\,\delta(\lambda-\lambda_{n,m})|n,m)=\sum_{n,m}\ \delta(\lambda-\lambda_{n,m}).\end{aligned} \tag{73}$$

With a view toward investigating the classical limit we note that we can rewrite the first equality of Eq. (73) by inserting the identity $\delta(\lambda-\hat{L})=(2\pi)^{-1}\int dt\ \ e^{i(\lambda-\hat{L})t}$ as

$$D(\lambda)=\sum_{n,m}\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\ e^{i\lambda t}(n,m|e^{-i\hat{L}t}|n,m). \tag{74}$$

But, inserting the closure relation (18), and noting that $\hat{L}|\mathbf{x})=L(\mathbf{x})|\mathbf{x})$, gives

$$(n,m|e^{-i\hat{L}t}|n,m)=\int\ d\mathbf{x}\ \ (n,m|\mathbf{x})e^{-iL(\mathbf{x})t}(\mathbf{x}|n,m). \tag{75}$$

It follows that Eq. (74) can be rewritten in the form

$$\begin{aligned}D(\lambda)=\sum_{n,m}\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\ e^{i\lambda t}\int\ d\mathbf{x}\ (n,m|\mathbf{x})e^{-iL(\mathbf{x})t}(\mathbf{x}|n,m)\\ =\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\ e^{i\lambda t}\int\ d\mathbf{x}\ d\mathbf{x}_0\left[\sum_{n,m}\ (n,m|\mathbf{x})(\mathbf{x}_0|n,m)\right]\\ \times e^{-iL(\mathbf{x})t}\delta(\mathbf{x}-\mathbf{x}_0)\\ =\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\ e^{i\lambda t}\int\ d\mathbf{x}\ d\mathbf{x}_0\delta(\mathbf{x}-\mathbf{x}_0)e^{-iL(\mathbf{x})t}\delta(\mathbf{x}-\mathbf{x}_0),\end{aligned} \tag{76}$$

where we have used $\Sigma_{m,n}(n,m|\mathbf{x})(\mathbf{x}_0|n,m)=\delta(\mathbf{x}-\mathbf{x}_0)$. Formally expanding $L(\mathbf{x})$ in powers of Planck's constant and taking the $h\to 0$ limit, gives $L(\mathbf{x})\to L_c(\mathbf{x})$, where $L_c(\mathbf{x})=iH(\mathbf{x})\sigma$ is the classical Liouville operator. It follows that as $h\to 0$

$$D(\lambda)\to \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{i\lambda t}\int d\mathbf{x}\ d\mathbf{x}_0\ \delta(\mathbf{x}-\mathbf{x}_0)e^{-iL_c(\mathbf{x})t}\times\delta(\mathbf{x}-\mathbf{x}_0)$$
$$\to \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{i\lambda t}\int d\mathbf{x}\ d\mathbf{x}_0\ \ \delta(\mathbf{x}-\mathbf{x}_0)\times\delta(\mathbf{X}(\mathbf{x},-t)-\mathbf{x}_0)$$
$$\to \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{i\lambda t}\int d\mathbf{x}\ \delta(\mathbf{X}(\mathbf{x},-t)-\mathbf{x}). \tag{77}$$

But

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{i\lambda t}\int d\mathbf{x}\ \delta(\mathbf{X}(\mathbf{x},-t)-\mathbf{x}) = \int_0^{\infty} dE\int_{-\infty}^{\infty} d\lambda_0\ \delta(\lambda-\lambda_0)\int d\mathbf{x}\ \Upsilon_{E,\lambda}(\mathbf{x};\mathbf{x})\equiv D_c(\lambda), \tag{78}$$

where $D_c(\lambda)$ is the classical Liouville spectral density [25].

Thus, in a formal sense we have correspondence, i.e., $D(\lambda)\to D_c(\lambda)$, in the $h\to 0$ limit. However, the proof is unsatisfactory because the classical spectrum is highly degenerate and this limit is not well defined, i.e., $D_c(\lambda)=\infty$. This arises from the fact that the classical Liouville spectrum is infinitely degenerate due to its stability with respect to variations with energy [25,34]. In addition, this formal proof provides little insight into the way that the spectra approach one another.

This problem can be bypassed by considering the Liouville spectral density for energies in a classically small, but quantum mechanically large, energy interval $E_0-\epsilon/2\leqslant E\leqslant E_0+\epsilon/2$. That is, we define

$$D_\epsilon(E_0;\lambda)=\sum_{\substack{n,m\\ E_0-\epsilon/2\leqslant E_{n,m}\leqslant E_0+\epsilon/2}}\delta(\lambda-\lambda_{n,m}) = \hbar\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d(E+\hbar\lambda/2)d(E-\hbar\lambda/2), \tag{79}$$

where $d(E)=\Sigma_n\delta(E-E_n)$. Then, as shown below, expression (79) has a well-defined classical limit, i.e.,

$$\lim_{\epsilon\to 0}\lim_{h\to 0} D_\epsilon(E_0;\lambda)=D_c(E_0;\lambda) \tag{80}$$

with

$$D_c(E_0;\lambda)=\delta(\lambda)+\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ e^{i\lambda t}\int_{H(\mathbf{x})=E_0} d\mathbf{x}\times\delta(\mathbf{X}(\mathbf{x},-t)-\mathbf{x}). \tag{81}$$

Here $D_c(E_0;\lambda)$ is the classical Liouville density of states [34,35] on the energy surface $H(\mathbf{x})=E_0$. Note that the factor $\delta(\mathbf{X}(\mathbf{x},-t)-\mathbf{x})$ in Eq. (81) is nonzero only for points $\mathbf{x}$ that lie on periodic orbits of period t. Thus the integral $\int_{H(\mathbf{x})=E_0} d\mathbf{x}\ \delta(\mathbf{X}(\mathbf{x},-t)-\mathbf{x})$ can be written as a sum over periodic orbits [34], giving [25,34]

$$D_c(E_0;\lambda)=\delta(\lambda)+\frac{1}{\pi}\sum_j \frac{T_j(E_0)\cos[\lambda T_j(E_0)]}{k_j|\det[M_j(E_0)-I]|}, \tag{82}$$

where T_j is the period of periodic orbit j, k_j is its winding number, M_j is its $(2s-2)\times(2s-2)$-dimensional stability matrix, and the sum is over positive traversals of the periodic orbits.

To show Eq. (80) we employ Gutzwiller's formula $d(E)=\underline{d}(E)+d_{\rm osc}(E)$ for the density of states, in Eq. (79). Here $\underline{d}(E)=\langle\Delta E\rangle^{-1}$ is the average density of states, and $d_{\rm osc}(E)$ is an oscillatory correction given by the formula

$$d_{\rm osc}(E)\sim\sum_j d_{\rm osc}^j(E), \tag{83}$$

where

$$d_{\rm osc}^j(E)\sim\frac{T_j(E)}{k_j\pi\hbar}\frac{\cos[S_j(E)/\hbar+\gamma_j]}{\sqrt{|\det(M_j(E)-I)|}}. \tag{84}$$

Here $S_j(E)$ is the action of the periodic orbit j and $\gamma_j=\sigma_j\pi/2$, where σ_j is the Maslov index of the orbit [36–38]. The sum in Eq. (83) is over positive traversals of the periodic orbits.

The Gutzwiller formula for the density of states is not generally convergent, but can be made so by broadening over energy [39]. That is, we replace $d(E)$ by the energy broadened density $d_\mu(E)=\Sigma_n\Omega_\mu(E-E_n)$ where $\Omega_\mu(x)=1/\mu$ for $-\mu/2\leqslant x\leqslant\mu/2$ and is zero otherwise. Note that $\lim_{\mu\to 0}\Omega_\mu(x)=\delta(x)$. The energy broadening modifies the standard Gutzwiller expansion by damping out contributions from very long periodic orbits.

We rewrite Eq. (79) in the form

$$D_\epsilon(E_0;\lambda)=\lim_{\mu\to\langle\Delta E\rangle}\hbar\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d_\mu(E+\hbar\lambda/2)\times d_\mu(E-\hbar\lambda/2) \tag{85}$$

in order to employ the energy broadened (and hence convergent [40]) form of Gutzwiller's formula. The correspondence limit is now $h\to 0$, followed by $\epsilon,\mu\to\langle\Delta E\rangle$ [41].

We now separate the Liouville density $D_\epsilon(E_0;\lambda)$ into its diagonal and off-diagonal parts. Since the system is chaotic we assume that it exhibits level repulsion, i.e., that the probability of two neighboring energy levels exhibiting an accidental degeneracy is zero. It therefore follows that the only contribution to the Liouville spectrum at $\lambda=0$ is from the diagonal $(n=m)$ terms,

$$\lim_{\lambda\to 0} D_{\epsilon}(E_0;\lambda)\sim \lim_{\mu\to\langle\Delta E\rangle} \hbar\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE \sum_n \Omega_{\mu}(E+\hbar\lambda/2-E_n)\Omega_{\mu}(E-\hbar\lambda/2-E_n), \tag{86}$$

so that we can rewrite Eq. (85) in the limit $h\to 0$ (i.e., $h\lambda\to 0$ for all λ) in the form

$$D_{\epsilon}(E_0;\lambda)\sim \lim_{\mu\to\langle\Delta E\rangle} \hbar\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE \sum_n \Omega_{\mu}(E+\hbar\lambda/2-E_n)\Omega_{\mu}(E-\hbar\lambda/2-E_n)$$
$$+\lim_{\mu\to\langle\Delta E\rangle} \hbar\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE[d_{\mu}(E+\hbar\lambda/2)d_{\mu}(E-\hbar\lambda/2)-d_{\mu}^2(E)]. \tag{87}$$

Note that as $\mu\to\langle\Delta E\rangle$

$$\sum_n \Omega_{\mu}(E+\hbar\lambda/2-E_n)\Omega_{\mu}(E-\hbar\lambda/2-E_n)\to\delta(\hbar\lambda)d(E) \tag{88}$$

and so the first term in Eq. (87) becomes

$$\lim_{\mu\to\langle\Delta E\rangle} \hbar\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE \sum_n \Omega_{\mu}(E+\hbar\lambda/2-E_n)\Omega_{\mu}(E-\hbar\lambda/2-E_n)\sim\hbar\epsilon\delta(\hbar\lambda)\frac{1}{\epsilon}\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d(E). \tag{89}$$

If we choose $\epsilon\sim h/T_{\min}$, where $T_{\min}$ is the period of the shortest periodic orbit of energy E_0, then

$$\overline{d(E_0)}\sim\frac{1}{\epsilon}\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d(E) \tag{90}$$

and so we may write Eq. (87) in the form

$$D_{\epsilon}(E_0;\lambda)=\hbar\epsilon\delta(\hbar\lambda)\overline{d(E_0)}+\lim_{\mu\to\langle\Delta E\rangle} \hbar\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ [d_{\mu}(E+\hbar\lambda/2)d_{\mu}(E-\hbar\lambda/2)-d_{\mu}^2(E)]. \tag{91}$$

Now we assume that $d_{\mu}(E)\sim\overline{d(E)}+d_{\mathrm{osc},\mu}(E)$, with $(1/\epsilon)\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d_{\mathrm{osc},\mu}(E)\sim 0$, and substituting this expression into Eq. (91) we obtain

$$D_{\epsilon}(E_0;\lambda)\sim\hbar\epsilon\delta(\hbar\lambda)\overline{d(E_0)}+\hbar\epsilon\overline{d}^{\,2}(E_0)-\hbar\lim_{\mu\to\langle\Delta E\rangle}\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d_{\mu}^2(E)$$
$$+\lim_{\mu\to\langle\Delta E\rangle} \hbar\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d_{\mathrm{osc},\mu}(E+\hbar\lambda/2)d_{\mathrm{osc},\mu}(E-\hbar\lambda/2). \tag{92}$$

The form of the energy broadened density is now introduced. In particular [42],

$$d_{\mu}(E)\sim\overline{d(E)}+\sum_j \frac{T_j(E)\ \ \mathrm{sinc}[\mu T_j(E)/2\hbar]}{k_j\pi\hbar\sqrt{|\det[M_j(E)-I]|}}\cos[S_j(E)/\hbar+\gamma_j]\sim\overline{d(E)}+\sum_j A_{\mu,j}(E)\cos[S_j(E)/\hbar+\gamma_j], \tag{93}$$

where

$$A_{\mu,j}(E)\equiv\frac{T_j(E)\ \ \mathrm{sinc}[\mu T_j(E)/2\hbar]}{k_j\pi\hbar\sqrt{|\det[M_j(E)-I]|}}, \tag{94}$$

and $\mathrm{sinc}(x)=\sin(x)/x$ is the damping function.

With $d_{\mathrm{osc},\mu}(E)\sim\Sigma_j A_{\mu,j}(E)\cos[S_j(E)/\hbar+\gamma_j]$ the last term of Eq. (92) can be written as

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d_{\mathrm{osc},\mu}(E+\hbar\lambda/2)d_{\mathrm{osc},\mu}(E-\hbar\lambda/2) \sim \sum_{j,j'} \int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ A_{\mu,j}(E+\hbar\lambda/2)A_{\mu,j'}(E-\hbar\lambda/2) \times \cos[S_j(E+\hbar\lambda/2)/\hbar+\gamma_j]\cos[S_{j'}(E-\hbar\lambda/2)/\hbar+\gamma_{j'}]$$
$$\sim \sum_{j,j'} \int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ A_{\mu,j}(E+\hbar\lambda/2)A_{\mu,j'}(E-\hbar\lambda/2) \times (\cos\{[S_j(E+\hbar\lambda/2)-S_{j'}(E-\hbar\lambda/2)]/\hbar+\gamma_j-\gamma_{j'}\} + \cos\{[S_j(E+\hbar\lambda/2)+S_{j'}(E-\hbar\lambda/2)]/\hbar+\gamma_j+\gamma_{j'}\}). \tag{95}$$

Now let $h\to 0$, followed by $\epsilon,\mu\to\langle\Delta E\rangle$. Observing that $\Omega_\mu(E-E_n)\Omega_\mu(E-E_m)\sim\Omega_\mu(E-E_n)\delta_{n,m}/\mu$ if μ is sufficiently small (a consequence of level repulsion), if follows that $d_\mu(E)^2\sim d_\mu(E)/\mu$. Taking $\mu\to\langle\Delta E\rangle$ we see that

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d_\mu^2(E) \sim \int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d(E)/\langle\Delta E\rangle \sim \epsilon\overline{d}^{\,2}(E_0). \tag{96}$$

Substituting this result back into Eq. (92) we obtain the result that

$$D_\epsilon(E_0;\lambda) \sim \hbar\epsilon\delta(\hbar\lambda)\overline{d(E_0)} + \lim_{\mu\to\langle\Delta E\rangle} \hbar \int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ d_{\mathrm{osc},\mu}(E+\hbar\lambda/2)d_{\mathrm{osc},\mu}(E-\hbar\lambda/2). \tag{97}$$

It remains to evaluate Eq. (95), a complicated procedure outlined in Appendix B. Then taking the limit as $\epsilon\to\langle\Delta E\rangle$, and substituting the contributions from Eqs. (B10), (B12), (B16), and (B22) into Eq. (95) and substituting Eq. (95) back into Eq. (97) gives the result

$$D_\epsilon(E_0;\lambda) \sim \delta(\lambda) + \frac{1}{\pi}\sum_j \frac{T_j(E_0)\cos\left(\dfrac{S_j(E_0+\lambda\hbar/2)-S_j(E_0-\hbar\lambda/2)}{\hbar}\right)}{k_j|\det(M_j(E_0)-I)|} + O(h^{s-1}e^{iz/\hbar}). \tag{98}$$

For $s\geqslant 2$, we thus see that

$$\lim_{\epsilon\to 0}\lim_{h\to 0} D_\epsilon(E_0;\lambda) = \delta(\lambda) + \frac{1}{\pi}\sum_j \frac{T_j(E_0)\cos[\lambda T_j(E_0)]}{k_j|\det[M_j(E_0)-I]|} = D_c(E_0;\lambda). \tag{99}$$

That is, the quantum Liouville spectrum properly approaches the classical Liouville spectrum as $h\to 0$. Note that Eq. (99) emerges from Eq. (98) via elimination of essential singularities.

IV. SCAR CORRECTIONS

In the last section we began with Eq. (52) and utilized Berry's formula [Eq. (46)] for $W(\mathbf{x};E,\epsilon)$, neglected scar corrections, and arrived at a proof of Eqs. (4) and (5), i.e., the correspondence rules discussed above. We now consider the corrections to these limits, which arise due to the scars from the periodic orbits.

Consider first the following formula for $W_{\mathrm{scar}}^j(\mathbf{x};E,\epsilon)$, the scar contribution to $W(\mathbf{x};E,\epsilon)$ from periodic orbit j:

$$W_{\mathrm{scar}}^j(\mathbf{x};E,\epsilon) \sim \frac{2^s}{\sqrt{|\det(M_j+I)|}} e^{-\epsilon T_j/\hbar} \times \cos\{[S_j-\boldsymbol{\xi}\cdot[J(M_j-I)/(M_j+I)]\cdot\tilde{\boldsymbol{\xi}}]/\hbar + \gamma_j\} h^{-1}\int_{-\infty}^{\infty} dt\ e^{-\epsilon|t|/\hbar} e^{i\{(E-H)t-1/24\,\dot{\mathbf{x}}\wedge\ddot{\mathbf{x}}t^3\}/\hbar}. \tag{100}$$

Here the variables $\boldsymbol{\xi}$ are the $2(s-1)$ coordinates of the surface of section transverse to the periodic orbit, and $\dot{\mathbf{x}}\wedge\ddot{\mathbf{x}} = |\nabla V(\mathbf{q})|^2/m + (\mathbf{p}\cdot\nabla)^2 V(\mathbf{q})/m^2$. The derivation of Eq. (100) is given by Berry [30] although he does not write it out explicitly.

Let $h\to 0$ with $\epsilon\gg h/T_{\min}$. (For convenience we drop the j subscripts on M_j, S_j, γ_j, and T_j.) Note that the t^3 term in the time integral of Eq. (100) can be neglected because only short times count for small h. Thus

$$W^j_{\rm scar}(\mathbf{x};E,\epsilon)\sim\frac{2^s}{\sqrt{|\det(M+I)|}}e^{-\epsilon T/\hbar}\cos\{[S(E)-\boldsymbol{\xi}\cdot[J\times(M-I)/(M+I)]\cdot\tilde{\boldsymbol{\xi}}]/\hbar+\gamma\}\delta_\epsilon(E-H(\mathbf{x}))$$
$$=\frac{2^s}{\sqrt{|\det(M+I)|}}e^{-\epsilon T/\hbar}\delta_\epsilon(E-H(\mathbf{x}))\times[e^{i\{[S(E)-\boldsymbol{\xi}\cdot[J(M-I)/(M+I)]\cdot\tilde{\boldsymbol{\xi}}]/\hbar+\gamma\}}+e^{-i\{[S(E)-\boldsymbol{\xi}\cdot[J(M-I)/(M+I)]\cdot\tilde{\boldsymbol{\xi}}]/\hbar+\gamma\}}/2. \tag{101}$$

Consider now factors such as $e^{\pm i\boldsymbol{\xi}\cdot[J(M-I)/(M+I)]\cdot\tilde{\boldsymbol{\xi}}/\hbar}$ in the $h\to 0$ limit. Note that the integral

$$\int d\mathbf{z}\; e^{\pm i\mathbf{z}\cdot\Omega\cdot\tilde{\mathbf{z}}/\alpha}f(\mathbf{z}) \tag{102}$$

[where $\mathbf{z}=(z_1,\ldots,z_N)$ and Ω is an $(N\times N)$-dimensional matrix independent of $\mathbf{z}$] in the limit $\alpha\to 0$ can be evaluated by the stationary phase to give

$$\int d\mathbf{z}\; e^{\pm i\mathbf{z}\cdot\Omega\cdot\tilde{\mathbf{z}}/\alpha}f(\mathbf{z})\sim\frac{[\alpha\pi]^{N/2}}{\sqrt{|\det(\Omega)|}}f(0)e^{\pm i\pi{\rm sgn}\Omega/4}. \tag{103}$$

This suggests the existence of a distributional identity

$$e^{\mp i\pi{\rm sgn}\Omega/4}\frac{\sqrt{|\det(\Omega)|}}{[\alpha\pi]^{N/2}}e^{\pm i\mathbf{z}\cdot\Omega\cdot\tilde{\mathbf{z}}/\alpha}\to\delta(\mathbf{z}). \tag{104}$$

This formula, when applied to the exponent in Eq. (101), gives

$$e^{\pm i\boldsymbol{\xi}\cdot[J(M-I)/(M+I)]\cdot\tilde{\boldsymbol{\xi}}/\hbar}\to e^{\pm i\sigma}(h/2)^{s-1}\sqrt{\left|\frac{\det(M+I)}{\det(M-I)}\right|}\delta(\boldsymbol{\xi}) \tag{105}$$

(here $\sigma=\pi{\rm sgn}[J(M-I)/(M+I)]/4$). Substituting into Eq. (101) gives

$$W^j_{\rm scar}(\mathbf{x};E,\epsilon)\sim\frac{2h^{s-1}}{\sqrt{|\det(M-I)|}}e^{-\epsilon T/\hbar}\cos\{S(E)/\hbar+\gamma\}\times\delta_\epsilon(E-H(\mathbf{x}))\delta(\boldsymbol{\xi}(\mathbf{x})), \tag{106}$$

which is the classical limit obtained by Berry [30].

To obtain the scar corrections to the Liouville spectral projectors we insert Eq. (46) into Eq. (52) and use Eq. (106) for the scar term. There are two types of corrections of the forms: "$\delta_\epsilon(E-H)\times$ scar" and "scar $\times$ scar." The latter are of much higher order in h and are neglected. The correction terms to Eq. (54) due to periodic orbits of period T and energy E_n are then

$$\mathcal{S}_{n,m}(\mathbf{x};\mathbf{x}_0)=\frac{2\pi^2\epsilon_1\epsilon_2h^{s-2}}{\sqrt{|\det(M-I)|}}e^{-\epsilon_1T/\hbar}\cos\{S(E_n)/\hbar+\gamma\}\int d\mathbf{y}\;dt_1dt_2e^{-\epsilon_1|t_1|/\hbar}e^{-\epsilon_2|t_2|/\hbar}e^{2\pi i(\mathbf{x}-\mathbf{x}_0)\cdot J\cdot\tilde{\mathbf{y}}}\delta(\boldsymbol{\xi}(\mathbf{x}+h\mathbf{y}/2))$$
$$\times\exp\{i[E_n-H(\mathbf{x}+h\mathbf{y}/2)]t_1/\hbar\}\exp\{i[E_m-H(\mathbf{x}_0-h\mathbf{y}/2)]t_2/\hbar\}. \tag{107}$$

We neglect h corrections to $\boldsymbol{\xi}$; i.e., we assume that $\boldsymbol{\xi}(\mathbf{x}+h\mathbf{y}/2)\sim\boldsymbol{\xi}(\mathbf{x})$. This can be justified as follows: (a) expanding $\boldsymbol{\xi}(\mathbf{x}+h\mathbf{y}/2)$ to first order in h and using the fact that $J(M-I)/(M+I)$ is symmetric [30], and Eq. (105), allows us to show that

$$\delta(\boldsymbol{\xi}(\mathbf{x}+h\mathbf{y}/2))\sim\delta(\boldsymbol{\xi}(\mathbf{x}))\cos\left\{2\pi\left[\mathbf{y}\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{x}}\right]\cdot[J(M-I)/(M+I)]\cdot\tilde{\boldsymbol{\xi}}\right\}, \tag{108}$$

and (b) noting that the right-hand side of Eq. (108) is zero unless $\boldsymbol{\xi}(\mathbf{x})\sim 0$, and that the argument of the cosine factor is proportional to $\boldsymbol{\xi}(\mathbf{x})$ allows us to replace the cosine factor by unity.

Using the short-time expansions of Eqs. (58) and (59) and doing the integrals over $\mathbf{u}$ and $\mathbf{v}$ gives

$$\mathcal{S}_{n,m}(\mathbf{x};\mathbf{x}_0)=\frac{2\pi^2\epsilon_1\epsilon_2h^{s-2}}{\sqrt{|\det(M-I)|}}e^{-\epsilon_1T/\hbar}\cos\{S(E_n)/\hbar+\gamma\}\delta(\boldsymbol{\xi}(\mathbf{x}))\int dt_1dt_2e^{-\epsilon_1|t_1|/\hbar}e^{-\epsilon_2|t_2|/\hbar}\exp\{i[E_n-H(\mathbf{x})]t_1/\hbar\}$$
$$\times\exp\{i[E_m-H(\mathbf{x}_0)]t_2/\hbar\}\delta(\mathbf{X}(\mathbf{x},-t_1/2)-\mathbf{X}'(\mathbf{x}_0,-t_2/2)). \tag{109}$$

Changing variables to $t'=(t_1-t_2)/2$ and $t_0=t_1$ and defining $\epsilon_0=\epsilon_1/2$ and $\epsilon'=2\epsilon_2$ as in Sec. III A we obtain

$$\mathcal{S}_{n,m}(\mathbf{x};\mathbf{x}_0)\sim\frac{\pi\epsilon_0h^{s-1}}{2\sqrt{|\det(M-I)|}}e^{-\epsilon_0T/\hbar}\cos\{S(E_n)/\hbar+\gamma\}\delta_{\epsilon_0}(E_{n,m}-H(\mathbf{x}))\delta(\boldsymbol{\xi}(\mathbf{x}))\lim_{\epsilon'/\hbar\to 0}\epsilon'/\hbar\int dt'\,e^{-\epsilon'|t'|/\hbar}$$
$$\times\exp\{i\lambda_{n,m}t'\}\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t')). \tag{110}$$

For the stationary case $n=m$ we interchange limits via Eq. (65) to obtain

$$\mathcal{S}_{n,n}(\mathbf{x};\mathbf{x}_0)=\frac{\langle\Delta E\rangle^2}{h\sqrt{|\det(M-I)|}}e^{-\epsilon_0 T/\hbar}\cos\{S(E_n)/\hbar+\gamma\}\,\delta(E_n-H(\mathbf{x}))\,\delta(E_n-H(\mathbf{x}_0))\,\delta(\boldsymbol{\xi}(\mathbf{x})). \tag{111}$$

The other cross term, due to periodic orbits of energy E_m, gives a similar contribution $\mathcal{S}_{n,n}(\mathbf{x}_0;\mathbf{x})$, and so the overall correction to Eq. (54) with $n=m$, due to the periodic orbits, is (see also Berry [30])

$$\mathcal{S}_{n,n}(\mathbf{x};\mathbf{x}_0)+\mathcal{S}_{n,n}(\mathbf{x}_0;\mathbf{x})\sim\sum_j\frac{\langle\Delta E\rangle^2}{h\sqrt{|\det(M_j-I)|}}e^{-\epsilon_0 T_j/\hbar}\cos\{S_j(E_n)/\hbar+\gamma_j\}\,\delta(E_n-H(\mathbf{x}_0))\,\delta(E_n-H(\mathbf{x}))[\,\delta(\boldsymbol{\xi}_j(\mathbf{x}))+\delta(\boldsymbol{\xi}_j(\mathbf{x}_0))]. \tag{112}$$

This result may be rewritten in terms of the distributions

$$\Upsilon^j_{E,0}(\mathbf{x};\mathbf{x}_0)=\frac{k_j}{T_j}\,\delta(E-H(\mathbf{x}_0))\,\delta(E-H(\mathbf{x}))\,\delta(\boldsymbol{\xi}_j(\mathbf{x})), \tag{113}$$

which we defined in paper I [5] and which are stationary spectral projectors with uniform density on the periodic orbits. Using Eq. (113), Eq. (112) becomes

$$\mathcal{S}_{n,n}(\mathbf{x};\mathbf{x}_0)+\mathcal{S}_{n,n}(\mathbf{x}_0;\mathbf{x})=\sum_j\frac{T_j\langle\Delta E\rangle^2}{k_j h\sqrt{|\det(M_j-I)|}}e^{-\epsilon_0 T_j/\hbar}\cos\{S_j(E_n)/\hbar+\gamma_j\}[\Upsilon^j_{E_n,0}(\mathbf{x};\mathbf{x}_0)+\Upsilon^j_{E_n,0}(\mathbf{x}_0;\mathbf{x})]. \tag{114}$$

This result shows that the scar corrections to the limit as $h\to 0$ of the spectral projectors corresponding to stationary states are comprised of weighted sums over the stationary classical projectors, which have uniform density on the classical periodic orbits. For the nonstationary case $n\neq m$, again interchanging limits via Eq. (65) and considering only points on the periodic orbit, it follows that the integral in Eq. (110) can be written as

$$\lim_{\epsilon'/\hbar\to 0}\epsilon'/\hbar\int dt'\,e^{-\epsilon'|t'|/\hbar}e^{i\lambda_{n,m}t'}\,\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t'))=\lim_{k\to\infty}\frac{1}{2k+1}\frac{\sin\{(2k+1)\lambda_{n,m}\tau/2\}}{\sin(\lambda_{n,m}\tau/2)}\frac{2}{\tau}\int_{-\tau/2}^{\tau/2}dt'\,e^{i\lambda_{n,m}t'}\,\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t')). \tag{115}$$

(Here $\tau=T_j/k_j$.) Since it can be readily shown that

$$\lim_{k\to\infty}\frac{1}{2k+1}\frac{\sin\{(2k+1)\lambda_{n,m}\tau/2\}}{\sin(\lambda_{n,m}\tau/2)}=\begin{cases}1 & \text{if } \lambda_{n,m}=2\pi l/\tau \text{ for } l\in\mathbf{Z}\\ 0 & \text{otherwise}\end{cases}$$

it follows that there are scar corrections only for the nonstationary distributions whose frequency matches an integer multiple of the frequency of one of the periodic orbits. When this condition is met the scar correction to the nonstationary spectral projector $(n,m|\mathbf{x})(\mathbf{x}_0|n,m)$ is obtained from Eqs. (110) and (115) as

$$\mathcal{S}_{n,m}(\mathbf{x};\mathbf{x}_0)\sim\frac{\langle\Delta E\rangle h^{s-1}}{\sqrt{|\det(M-I)|}}e^{-\epsilon_0 T/\hbar}\cos\{S(E_n)/\hbar+\gamma\}\,\delta_{\epsilon_0}(E_{n,m}-H(\mathbf{x}))\,\delta(\boldsymbol{\xi}(\mathbf{x}))\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}dt'\,e^{i\lambda_{n,m}t'}\,\delta(\mathbf{x}_0-\mathbf{X}(\mathbf{x},-t')). \tag{116}$$

Similar corrections $\mathcal{S}_{m,n}(\mathbf{x}_0;\mathbf{x})$ arise from periodic orbits of energy E_m, i.e., $\mathcal{S}_{n,m}(\mathbf{x};\mathbf{x}_0)$ is given by Eq. (116) with E_n replaced by E_m.

Consider now these nonstationary corrections in more detail. The scar contribution [Eq. (116)] involves the following product of factors

$$\delta_{\epsilon_0}(E_{n,m}-H(\mathbf{x}))\,\delta(\boldsymbol{\xi}(\mathbf{x})). \tag{117}$$

The variables $\boldsymbol{\xi}(\mathbf{x})$ are effectively zero on a local family of periodic orbits with energies close to E_n and periods close to T. The distribution $\delta(\boldsymbol{\xi}(\mathbf{x}))$ is thus zero except on this local family. If the energy $E_{n,m}$ lies outside of the neighborhood in energy of the local family then the product of delta functions [Eq. (117)] will be everywhere zero. As a consequence the product is generally zero for $n\neq m$. The same considerations hold for the scar term $S_{n,m}(\mathbf{x};\mathbf{x}_0)$ with periodic orbits of energy E_m. Thus, scar corrections to the nonstationary Liouville eigenfunctions are typically negligible in the semiclassical limit. Only periodic orbits of period $T_j/k_j=2\pi l/\lambda_{n,m}$, $l\in\mathbf{Z}$, contribute and of these only the ones with energy E_n or E_m close to $E_{n,m}$ make a nonzero contribution.

Thus, we see that the stationary and nonstationary contributions of periodic orbits [Eqs. (114) and (116)] at most make corrections of order $h^{2s-1}e^{-\epsilon T/\hbar}$, which vanish in the correspondence limit ($h\to 0$ followed by $\pi\epsilon\to\langle\Delta E\rangle$). Furthermore, in the classical limit these corrections are only supported on the measure zero set of periodic orbits.

V. CORRESPONDENCE: APPLICATIONS TO MATRIX ELEMENTS

The results obtained above allow us to systematize and extend previous results on the classical limiting forms of matrix elements of quantum observables. With our normalization of the quantum Liouville eigenfunctions, matrix elements satisfy the following relationship:

$$\langle n|\hat{A}|m\rangle = h^{-s/2}\int d\mathbf{x}\,\rho^{w}_{n,m}(\mathbf{x})A^{w}(\mathbf{x}). \tag{118}$$

As a consequence we may relate matrix elements of observables to the spectral projection operators via the expression

$$|\langle n|\hat{A}|m\rangle|^2 = \int d\mathbf{x}\,d\mathbf{x}_0\;A^{w*}(\mathbf{x})(\mathbf{x}|n,m)(n,m|\mathbf{x}_0)A^{w}(\mathbf{x}_0). \tag{119}$$

We focus on the correspondence limit of matrix elements for observables $\hat{A}$ with a well-defined classical analog, i.e., for $A^{w}(\mathbf{x})\to A(\mathbf{x})$ where $A(\mathbf{x})$ is the classical analog.

Consider first that Eqs. (119), (34), and (42) imply that $(n\neq m)$

$$\lim_{h\to 0}|\langle n|\hat{A}|m\rangle|^2 = \frac{\langle\Delta E\rangle}{h}\int_{-\infty}^{\infty}dt'\,e^{i\lambda_{n,m}t'}\;\langle A(0)A(t')\rangle_{E_{n,m}}, \tag{120}$$

where $\langle A(0)A(t')\rangle_{E_{n,m}}$ denotes the microcanonical average of $A(\mathbf{x},0)A(\mathbf{x},t')$ at an energy $E_{n,m}$. Noting that

$$\int_{-\infty}^{\infty}dt'\,e^{i\lambda_{n,m}t'}\;\langle A(0)A(t')\rangle_{E_{n,m}} = 2\pi\langle A\rangle^2_{E_{n,m}}\delta(\lambda_{n,m}) + \int_{-\infty}^{\infty}dt'\,e^{i\lambda_{n,m}t'}\;\langle[A(0)-\langle A\rangle_{E_{n,m}}][A(t')-\langle A\rangle_{E_{n,m}}]\rangle_{E_{n,m}} \tag{121}$$

$$= \int_{-\infty}^{\infty}dt'\,e^{i\lambda_{n,m}t'}\;\langle[A(0)-\langle A\rangle_{E_{n,m}}][A(t')-\langle A\rangle_{E_{n,m}}]\rangle_{E_{n,m}}, \tag{122}$$

if $\lambda_{n,m}\neq 0$, it follows that

$$\lim_{h\to 0}|\langle n|\hat{A}|m\rangle|^2 = \frac{\langle\Delta E\rangle}{h}\int_{-\infty}^{\infty}dt'\,e^{i\lambda_{n,m}t'}\;\langle[A(0)-\langle A\rangle_{E_{n,m}}][A(t')-\langle A\rangle_{E_{n,m}}]\rangle_{E_{n,m}} \tag{123}$$

for the nonstationary case. This result is essentially in agreement with the predictions of Feingold and Peres [43]. Their result differs from Eq. (123) insofaras they are missing a factor of $\hbar^{-1}$, and they left the energy $E_{n,m}$ unspecified.

The arguments leading to Eq. (123) can be readily generalized to the case of mixed operators with the result that

$$\lim_{h\to 0}\langle m|\hat{A}|n\rangle\langle n|\hat{B}|m\rangle = \frac{\langle\Delta E\rangle}{h}\int_{-\infty}^{\infty}dt'\,e^{i\lambda_{n,m}t'}\;\langle[A(0)-\langle A\rangle_{E_{n,m}}][B(t')-\langle B\rangle_{E_{n,m}}]\rangle_{E_{n,m}}. \tag{124}$$

This important result establishes a connection between the Liouville spectrum of classical time correlation functions and quantum matrix elements for chaotic systems.

The case of $n=m$ also results from Eq. (119) and Eq. (41) to give the well-known results:

$$\lim_{h\to 0}|\langle n|\hat{A}|n\rangle|^2 = \langle A\rangle^2_{E_n},$$
$$\lim_{h\to 0}\langle n|\hat{A}|n\rangle\langle n|\hat{B}|n\rangle = \langle A\rangle_{E_n}\langle B\rangle_{E_n}. \tag{125}$$

Scar corrections to the off-diagonal matrix elements are negligible. Scar corrections for the stationary case $|\langle n|\hat{A}|n\rangle|^2$ are of the form

$$\sum_j \frac{2T_j\langle\Delta E\rangle}{k_j h\sqrt{|\det(M_j-I)|}}e^{-\epsilon_0 T_j/\hbar}\times\cos\{S_j(E_n)/\hbar+\gamma_j\}\;\langle A\rangle_{E_n}\;\langle A\rangle^{j}_{E_n,0}, \tag{126}$$

where

$$\langle A\rangle^{j}_{E,0} \equiv \frac{1}{\tau_j}\int_{-\tau_j/2}^{\tau_j/2}dt'\;A(\mathbf{X}(\mathbf{x}^j,t')) \tag{127}$$

is the average of A over the periodic orbit j and $\mathbf{x}^j$ is a point on the periodic orbit, while corrections to Eq. (125) are of the form

$$\sum_j \frac{T_j\langle\Delta E\rangle}{k_j h\sqrt{|\det(M_j-I)|}}e^{-\epsilon_0 T_j/\hbar}\cos\{S_j(E_n)/\hbar+\gamma_j\}\times[\langle A\rangle_{E_n}\langle B\rangle^{j}_{E_n,0}+\langle B\rangle_{E_n}\langle A\rangle^{j}_{E_n,0}]. \tag{128}$$

Comparing Eq. (123) to Eq. (126) it is readily apparent that squared off-diagonal matrix elements for a chaotic system are of the same order of magnitude as the fluctuations in the squared diagonal matrix elements, i.e., $O(h^{s-1})$. This confirms an early conjecture by Pechukas [44].

Finally, we compare the chaotic case to the well-known [3,45] classical limit for matrix elements of individual matrix elements for integrable systems, i.e.,

$$\lim_{h\to 0}\langle \mathbf{n}|\hat{A}|\mathbf{m}\rangle = \frac{1}{(2\pi)^s}\int d\boldsymbol{\theta}\; A(\mathbf{I}_{\mathbf{n},\mathbf{m}},\boldsymbol{\theta})e^{i(\mathbf{n}-\mathbf{m})\cdot\boldsymbol{\theta}}. \tag{129}$$

The most significant difference is that Eq. (129), which holds for both the stationary and nonstationary integrable cases, gives a classical limit for individual matrix elements whereas the chaotic results allow classical limits only for products of matrix elements [Eqs. (123), (124), and (125)]. This is a direct consequence of the fact that $\rho_{n,m}$ has a classical limit for regular systems but not for chaotic systems.

VI. SUMMARY

Correspondence for chaotic quantum systems has been considered from the viewpoint of distribution dynamics in the Wigner-Weyl representation of quantum mechanics. The connections between quantum and classical dynamics have been clarified through the formulation of the correspondence problem in terms of Liouville spectral projection operators. Our demonstration of correspondence for these objects and for the Liouville spectrum shows that quantum dynamics is capable of reproducing chaotic classical dynamics in the $h\to 0$ limit. The mechanism of correspondence here, as in our studies of chaotic mappings, appears to be the elimination of essential singularities. Corrections arising from periodic orbits were also considered with the result that stationary quantum spectral projection operators have corrections in the form of weighted sums of stationary classical projectors on periodic orbits. Scar corrections for the nonstationary Liouville spectral projection operators were found to be negligible. Applications of our correspondence results to matrix elements revealed connections between the matrix elements of quantum observables and the spectrum of classical time correlation functions. The corrections to the diagonal matrix elements due to periodic orbits were shown to be of the same order of magnitude in h as the square of the off-diagonal matrix elements.

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APPENDIX A

We prove identity (50) by showing that the right-hand side can be reduced to the left. To begin we substitute the definitions of $(\mathbf{x}|n,n)$ into the right-hand side to obtain

$$\begin{aligned} h^{-2s}\int d\mathbf{u}\, d\mathbf{v}\, d\mathbf{q}'d\mathbf{q}''\;\; & e^{-i(\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{v}/\hbar}e^{i(\mathbf{q}-\mathbf{q}_0)\cdot\mathbf{u}/\hbar} \\ & \times e^{i(\mathbf{p}+\mathbf{u}/2)\cdot\mathbf{q}'/\hbar}e^{i(\mathbf{p}_0-\mathbf{u}/2)\cdot\mathbf{q}''/\hbar}\langle\mathbf{q}+\mathbf{v}/2-\mathbf{q}'/2|n\rangle \\ & \times\langle n|\mathbf{q}+\mathbf{v}/2+\mathbf{q}'/2\rangle\langle\mathbf{q}_0-\mathbf{v}/2-\mathbf{q}''/2|m\rangle \\ & \times\langle m|\mathbf{q}_0-\mathbf{v}/2+\mathbf{q}''/2\rangle. \end{aligned} \tag{A1}$$

Performing the integral over $\mathbf{u}$ gives

$$\begin{aligned} h^{-s}\int & d\mathbf{v}\; d\mathbf{q}'d\mathbf{q}''\,\delta(\mathbf{q}-\mathbf{q}_0+\mathbf{q}'/2-\mathbf{q}''/2) \\ & \times e^{-i(\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{v}/\hbar}e^{i\mathbf{p}\cdot\mathbf{q}'/\hbar}e^{i\mathbf{p}_0\cdot\mathbf{q}''/\hbar} \\ & \times\langle\mathbf{q}+\mathbf{v}/2-\mathbf{q}'/2|n\rangle\langle n|\mathbf{q}+\mathbf{v}/2+\mathbf{q}'/2\rangle \\ & \times\langle\mathbf{q}_0-\mathbf{v}/2-\mathbf{q}''/2|m\rangle\langle m|\mathbf{q}_0-\mathbf{v}/2+\mathbf{q}''/2\rangle. \end{aligned} \tag{A2}$$

Now doing the integral over $\mathbf{q}''$ gives

$$\begin{aligned} h^{-s}2^s\int & d\mathbf{v}\; d\mathbf{q}'\,e^{i\mathbf{p}\cdot(-\mathbf{v}+\mathbf{q}')/\hbar}e^{-i\mathbf{p}_0\cdot(2\mathbf{q}_0-2\mathbf{q}-\mathbf{v}-\mathbf{q}')/\hbar} \\ & \times\langle\mathbf{q}-(-\mathbf{v}+\mathbf{q}')/2|n\rangle\langle m|\mathbf{q}+(-\mathbf{v}+\mathbf{q}')/2\rangle \\ & \times\langle\mathbf{q}_0+(2\mathbf{q}_0-2\mathbf{q}-\mathbf{v}-\mathbf{q}')/2|m\rangle \\ & \times\langle n|\mathbf{q}_0-(2\mathbf{q}_0-2\mathbf{q}-\mathbf{v}-\mathbf{q}')/2\rangle. \end{aligned} \tag{A3}$$

Changing variables to $\mathbf{v}'=-\mathbf{v}+\mathbf{q}'$, and $\mathbf{v}''=2\mathbf{q}_0-2\mathbf{q}-\mathbf{v}-\mathbf{q}'$ gives a Jacobian determinant of absolute value $1/2^s$. Inserting the definitions of $(\mathbf{x}|n,m)$ we immediately obtain the right-hand side of Eq. (50).

APPENDIX B:

Here we systematically evaluate each of the contributions to Eq. (95), in accord with the correspondence limit $h\to 0$, $\epsilon \gg h/T_{\min}$, followed by $\epsilon\to\langle\Delta E\rangle$. We assume that $T_j(E)$ and $M_j(E)$ vary slowly with energy, i.e., $T_j(E\pm\hbar\lambda/2)\sim T_j(E)$ and $M_j(E\pm\hbar\lambda/2)\sim M_j(E)$ for small h, and note, as a consequence, that the amplitudes $A_{\mu,j}$ are also slowly varying, i.e.,

$$A_{\mu,j}(E+\hbar\lambda/2)\sim A_{\mu,j}(E), \tag{B1}$$

and

$$A_{\mu,j'}(E+\hbar\lambda/2)\sim A_{\mu,j'}(E). \tag{B2}$$

We thus define the slowly varying amplitudes

$$B_{j,j'}(E)\equiv A_{\mu,j}(E)A_{\mu,j'}(E). \tag{B3}$$

The integrals to be evaluated [Eq. (95)] therefore take the form

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\; B_{j,j'}(E)\cos\left[\frac{S_j(E+\hbar\lambda/2)\pm S_{j'}(E-\hbar\lambda/2)}{\hbar}+\gamma_j\pm\gamma_{j'}\right], \tag{B4}$$

and we must consider three separate cases.

Case 1: We begin by considering the integrals for which the actions add. Here, because we are summing over positive traversals,

$$\frac{d}{dE}[S_j(E+\hbar\lambda/2)+S_{j'}(E-\hbar\lambda/2)]=T_j(E+\hbar\lambda/2)+T_{j'}(E-\hbar\lambda/2)>0 \tag{B5}$$

and so there are no stationary phase points in the interval $E_0-\epsilon/2\leqslant E\leqslant E_0+\epsilon/2$. It can be shown [46] that integrals of the form $\int_a^b dz\ f(z)e^{i\phi(z)/\hbar}$, with $f(z)$ slowly varying and $\phi'(z)\neq 0$ for $z\in[a,b]$, can be approximated by

$$\int_a^b dz\ f(z)e^{i\phi(z)/\hbar}\sim\frac{\hbar}{i}\left[\frac{f(b)}{\phi'(b)}e^{i\phi(b)/\hbar}-\frac{f(a)}{\phi'(a)}e^{i\phi(a)/\hbar}\right] \tag{B6}$$

in the limit that $h\rightarrow 0$. Application of Eq. (B6) to the integral

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2}dE\ B_{j,j'}(E)\cos\left[\frac{S_j(E+\hbar\lambda/2)+S_{j'}(E-\hbar\lambda/2)}{\hbar}+\gamma_j+\gamma_{j'}\right] \tag{B7}$$

gives

$$\begin{aligned}\int_{E_0-\epsilon/2}^{E_0+\epsilon/2}&dE\ B_{j,j'}(E)\cos\left[\frac{S_j(E+\hbar\lambda/2)+S_{j'}(E-\hbar\lambda/2)}{\hbar}+\gamma_j+\gamma_{j'}\right]\\ &\sim\hbar\left\{\frac{B_{j,j'}(E_0+\epsilon/2)}{T_j(E_0+\epsilon/2)+T_{j'}(E_0+\epsilon/2)}\sin\left[\frac{S_j(E_0+\epsilon/2+\hbar\lambda/2)+S_{j'}(E_0+\epsilon/2-\hbar\lambda/2)}{\hbar}+\gamma_j+\gamma_{j'}\right]\right.\\ &\quad\left.-\frac{B_{j,j'}(E_0-\epsilon/2)}{T_j(E_0-\epsilon/2)+T_{j'}(E_0-\epsilon/2)}\sin\left[\frac{S_j(E_0-\epsilon/2+\hbar\lambda/2)+S_{j'}(E_0-\epsilon/2-\hbar\lambda/2)}{\hbar}+\gamma_j+\gamma_{j'}\right]\right\}.\end{aligned} \tag{B8}$$

Taking the limit as $\epsilon\rightarrow\langle\Delta E\rangle$, $B_{j,j'}(E_0\pm\epsilon/2)\sim B_{j,j'}(E_0)$ and $T_j(E_0\pm\epsilon/2)\sim T_j(E_0)$, so that we can rewrite Eq. (B8) in the form

$$\begin{aligned}\int_{E_0-\epsilon/2}^{E_0+\epsilon/2}&dE\ B_{j,j'}(E)\cos\left[\frac{S_j(E+\hbar\lambda/2)+S_{j'}(E-\hbar\lambda/2)}{\hbar}+\gamma_j+\gamma_{j'}\right]\\ &\sim\frac{2\hbar B_{j,j'}(E_0)}{T_j(E_0)+T_{j'}(E_0)}\cos\left[\frac{S_j(E_0+\hbar\lambda/2)+S_{j'}(E_0-\hbar\lambda/2)}{\hbar}+\gamma_j+\gamma_{j'}\right]\sin\{[T_j(E_0)+T_{j'}(E_0)]\epsilon/\hbar\}\sim 2\epsilon B_{j,j'}(E_0)\\ &\quad\times\cos\left[\frac{S_j(E_0+\hbar\lambda/2)+S_{j'}(E_0-\hbar\lambda/2)}{\hbar}+\gamma_j+\gamma_{j'}\right]\mathrm{sinc}\{[T_j(E_0)+T_{j'}(E_0)]\epsilon/\hbar\}.\end{aligned} \tag{B9}$$

Further simplifications show that these contributions take the form

$$\begin{aligned}\int_{E_0-\epsilon/2}^{E_0+\epsilon/2}&dEB_{j,j'}(E)\cos\left[\frac{S_j(E+\hbar\lambda/2)+S_{j'}(E-\hbar\lambda/2)}{\hbar}+\gamma_j+\gamma_{j'}\right]\\ &\sim 2\langle\Delta E\rangle B_{j,j'}(E_0)\cos\left[\frac{S_j(E_0+\hbar\lambda/2)+S_{j'}(E_0-\hbar\lambda/2)}{\hbar}+\gamma_j+\gamma_{j'}\right].\end{aligned} \tag{B10}$$

Now consider the terms in Eq. (B4) involving a difference of actions.

Case 2: Of the terms with $j\neq j'$, some will have stationary phase points in the interval $E_0-\epsilon/2\leqslant E\leqslant E_0+\epsilon/2$, and others will not. For those terms that do not have stationary phase points,

$$\begin{aligned}\int_{E_0-\epsilon/2}^{E_0+\epsilon/2}&dE\ B_{j,j'}(E)\cos\left[\frac{S_j(E+\lambda\hbar/2)-S_{j'}(E-\hbar\lambda/2)}{\hbar}+\gamma_j-\gamma_{j'}\right]\\ &\sim\frac{2\hbar B_{j,j'}(E_0)}{T_j(E_0)-T_{j'}(E_0)}\cos\left[\frac{S_j(E_0+\hbar\lambda/2)-S_{j'}(E_0-\hbar\lambda/2)}{\hbar}+\gamma_j-\gamma_{j'}\right]\sin[(T_j(E_0)-T_{j'}(E_0))\epsilon/\hbar]\sim 2\epsilon B_{j,j'}(E_0)\\ &\quad\times\cos\left[\frac{S_j(E_0+\hbar\lambda/2)-S_{j'}(E_0-\hbar\lambda/2)}{\hbar}+\gamma_j-\gamma_{j'}\right]\mathrm{sinc}\{[T_j(E_0)-T_{j'}(E_0)]\epsilon/\hbar\},\end{aligned} \tag{B11}$$

as one can easily show by applying Eq. (B6). In the limit as $\epsilon\rightarrow\langle\Delta E\rangle$ we find that

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ B_{j,j'}(E)\cos\left[\frac{S_j(E+\hbar\lambda/2)-S_{j'}(E-\hbar\lambda/2)}{\hbar}+\gamma_j-\gamma_{j'}\right]$$
$$\sim 2\langle\Delta E\rangle B_{j,j'}(E_0)\sin\left[\frac{S_j(E_0+\hbar\lambda/2)-S_{j'}(E_0-\hbar\lambda/2)}{\hbar}+\gamma_j-\gamma_{j'}\right]. \tag{B12}$$

Consider those terms that do have stationary phase points $E'\in[E_0-\epsilon/2,E_0+\epsilon/2]$. We define $\phi(E)=S_j(E+\hbar\lambda/2)-S_{j'}(E-\hbar\lambda/2)$. Expanding $\phi(E)$ about the stationary phase point E' gives $\phi(E)\sim\phi(E')+\frac{1}{2}\phi''(E')(E-E')^2$. Now consider that

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ B_{j,j'}(E)e^{i\phi(E)/\hbar}\sim B_{j,j'}(E')e^{i\phi(E')/\hbar}\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ e^{i\phi''(E')(E-E')^2/2\hbar}. \tag{B13}$$

This approximation resembles the stationary phase approximation [46] except that we have retained the original limits on the remaining integral instead of replacing them by $\pm\infty$. Evaluation of the remaining integral in Eq. (145) then yields [47]

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ B_{j,j'}(E)e^{i\phi(E)/\hbar}\sim\frac{1}{2}\sqrt{\frac{ih}{\phi''(E')}}B_{j,j'}(E')e^{i\phi(E')/\hbar}$$
$$\times\left\{\mathrm{erf}\left[\sqrt{\frac{-i\phi''(E')}{2\hbar}}(E_0+\epsilon/2-E')\right]-\mathrm{erf}\left[\sqrt{\frac{-i\phi''(E')}{2\hbar}}(E_0-\epsilon/2-E')\right]\right\}. \tag{B14}$$

In the limit as $\epsilon\to\langle\Delta E\rangle$ we obtain

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ B_{j,j'}(E)e^{i\phi(E)/\hbar}\sim\langle\Delta E\rangle B_{j,j'}(E')e^{i\phi(E')/\hbar}e^{i\phi''(E')(E_0-E')^2/2\hbar} \tag{B15}$$

or

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ B_{j,j'}(E)\cos\left[\frac{S_j(E+\hbar\lambda/2)-S_{j'}(E-\hbar\lambda/2)}{\hbar}+\gamma_j-\gamma_{j'}\right]\sim\langle\Delta E\rangle B_{j,j'}(E')\cos(\Phi), \tag{B16}$$

where $\Phi=\phi(E')/\hbar+\phi''(E')(E_0-E')^2/2\hbar+\gamma_j-\gamma_{j'}$.

Case 3: Finally, we consider the terms in Eq. (B4) where $j=j'$. First note that there are no true stationary phase points, i.e., $T_j(E+\hbar\lambda/2)-T_j(E-\hbar\lambda/2)\neq 0$ for $\lambda\neq 0$. Second, note that the action $S_j(E)$ changes by $S_j(E_0+\epsilon/2)-S_j(E_0-\epsilon/2)\equiv\Delta S_j(E_0)\sim T_j(E_0)\epsilon$ over the interval $E_0-\epsilon/2\leqslant E\leqslant E_0+\epsilon/2$. Since $\epsilon\sim h/T_{\min}$ it follows that $\Delta S_j(E_0)/\hbar\sim 2\pi T_j(E_0)/T_{\min}$. In general $T_j(E_0)\gg T_{\min}$ and so $S/\hbar$ changes dramatically over the interval $E_0-\epsilon/2\leqslant E\leqslant E_0+\epsilon/2$. Similarly, $\Delta[S_j(E_0+\hbar\lambda/2)-S_j(E_0-\hbar\lambda/2)]/\hbar\sim 2\pi[T_j(E_0+\hbar\lambda/2)-T_j(E_0-\hbar\lambda/2)]/T_{\min}$, and since even $|T_j(E_0+\hbar\lambda/2)-T_j(E_0-\hbar\lambda/2)|\gg T_{\min}$ in general, it follows that the phase of the cosine factor will oscillate many times over the interval $E_0-\epsilon/2\leqslant E\leqslant E_0+\epsilon/2$. By contrast, over the interval $E_0-hk_j/2T_j(E_0)\leqslant E\leqslant E_0+hk_j/2T_j(E_0)$, $\Delta[S_j(E_0+\hbar\lambda/2)-S_j(E_0-\hbar\lambda/2)]/\hbar\sim 2\pi k_j[T_j(E_0+\hbar\lambda/2)-T_j(E_0-\hbar\lambda/2)]/T_j(E_0)$. In general $|T_j(E_0+\hbar\lambda/2)-T_j(E_0-\hbar\lambda/2)|/T_j(E_0)\ll 1$ for h small and so the phase is effectively stationary over the small interval $E_0-hk_j/2T_j(E_0)\leqslant E\leqslant E_0+hk_j/2T_j(E_0)$ about E_0, and we will therefore break the integral into three parts:

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ B_{j,j}(E)\cos\left(\frac{S_j(E+\lambda\hbar/2)-S_j(E-\hbar\lambda/2)}{\hbar}\right)=\left[\int_{E_0-hk_j/2T_j(E_0)}^{E_0+hk_j/2T_j(E_0)} dE+\int_{E_0-\epsilon/2}^{E_0-hk_j/2T_j(E_0)} dE\right.$$
$$\left.+\int_{E_0+hk_j/2T_j(E_0)}^{E_0+\epsilon/2} dE\right]B_{j,j}(E)\cos\left(\frac{S_j(E+\hbar\lambda/2)-S_j(E-\hbar\lambda/2)}{\hbar}\right). \tag{B17}$$

The first term can be approximated via

$$\int_{E_0-hk_j/2T_j(E_0)}^{E_0+hk_j/2T_j(E_0)} dE\ B_{j,j}(E)\cos\left(\frac{S_j(E+\hbar\lambda/2)-S_j(E-\hbar\lambda/2)}{\hbar}\right)\sim\frac{k_jhB_{j,j}(E_0)}{T_j(E_0)}\cos\left(\frac{S_j(E_0+\hbar\lambda/2)-S_j(E_0-\hbar\lambda/2)}{\hbar}\right), \tag{B18}$$

while the other two terms can be evaluated [46] via Eq. (B6) to give

$$\int_{E_0-\epsilon/2}^{E_0-hk_j/2T_j(E_0)} dE\ B_{j,j}(E)\cos\left(\frac{S_j(E+\hbar\lambda/2)-S_j(E-\hbar\lambda/2)}{\hbar}\right)$$
$$\sim \frac{2\hbar B_{j,j}(E_0)}{T_j(E_0+\hbar\lambda/2)-T_j(E_0-\hbar\lambda/2)}\cos\left(\frac{S_j(E_0+\hbar\lambda/2)-S_j(E_0-\hbar\lambda/2)}{\hbar}\right)$$
$$\times\sin\{[\epsilon/2+hk_j/2T_j(E_0)][T_j(E_0+\hbar\lambda/2)-T_j(E_0-\hbar\lambda/2)]/\hbar\}, \quad \text{(B19)}$$

and

$$\int_{E_0+hk_j/2T_j(E_0)}^{E_0+\epsilon/2} dE\ B_{j,j}(E)\cos\left(\frac{S_j(E+\hbar\lambda/2)-S_j(E-\hbar\lambda/2)}{\hbar}\right)$$
$$\sim \frac{2\hbar B_{j,j}(E_0)}{T_j(E_0+\hbar\lambda/2)-T_j(E_0-\hbar\lambda/2)}\cos\left(\frac{S_j(E_0+\hbar\lambda/2)-S_j(E_0-\hbar\lambda/2)}{\hbar}\right)$$
$$\times\sin\{[\epsilon/2-hk_j/2T_j(E_0)][T_j(E_0+\hbar\lambda/2)-T_j(E_0-\hbar\lambda/2)]/\hbar\}. \quad \text{(B20)}$$

Combined, these last two terms contribute

$$4\epsilon B_{j,j}(E_0)\cos\left(\frac{S_j(E_0+\hbar\lambda/2)-S_j(E_0-\hbar\lambda/2)}{\hbar}\right)\mathrm{sinc}\{\epsilon[T_j(E_0+\hbar\lambda/2)-T_j(E_0-\hbar\lambda/2)]/\hbar\}$$
$$\times\cos\{2\pi k_j[T_j(E_0+\lambda\hbar/2)-T_j(E_0-\lambda\hbar/2)]/T_j(E_0)\}. \quad \text{(B21)}$$

Therefore, in the limit as $\epsilon\to\langle\Delta E\rangle$ the total contribution for case 3 is

$$\int_{E_0-\epsilon/2}^{E_0+\epsilon/2} dE\ B_{j,j}(E)\cos\left(\frac{S_j(E+\hbar\lambda/2)-S_j(E-\hbar\lambda/2)}{\hbar}\right)$$
$$\sim \frac{hk_jB_{j,j}(E_0)}{T_j(E_0)}\cos\left(\frac{S_j(E_0+\hbar\lambda/2)-S_j(E_0-\hbar\lambda/2)}{\hbar}\right)$$
$$+4\langle\Delta E\rangle B_{j,j}(E_0)\cos\left(\frac{S_j(E_0+\hbar\lambda/2)-S_j(E_0-\hbar\lambda/2)}{\hbar}\right)\cos\{2\pi k_j[T_j(E_0+\lambda\hbar/2)-T_j(E_0-\lambda\hbar/2)]/T_j(E_0)\}. \quad \text{(B22)}$$

In summary, evaluation of the three types of integrals in Eq. (95) in the correspondence limit, yields the contributions of Eqs. (B10), (B12), (B16) and (B22), which combined make a total contribution of the form

$$\frac{hk_jB_{j,j}(E_0)}{T_j(E_0)}\cos\left(\frac{S_j(E_0+\lambda\hbar/2)-S_j(E_0-\hbar\lambda/2)}{\hbar}\right)+O(h^{s-1}e^{iz/\hbar}). \quad \text{(B23)}$$

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$$(\mathbf{q},\mathbf{q}'|n,m)(n,m|\mathbf{q}_0,\mathbf{q}_0')=\langle\mathbf{q}'|m\rangle\langle n|\mathbf{q}\rangle\langle\mathbf{q}_0|n\rangle\langle m|\mathbf{q}_0'\rangle$$
$$=(\mathbf{q}_0,\mathbf{q}|n,n)(m,m|\mathbf{q}',\mathbf{q}_0').$$

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[40] Technically, the uniform energy average employed here will not guarantee convergence since the resultant damping is proportional to $1/T_j$ while the periodic orbits proliferate [34,39] as the exponential of T_j. However, the arguments of Sec. III B are independent of the method of averaging, and can be restated in terms of a Gaussian exponential average which guarantees convergence [9]. We adopt the uniform energy average for simplicity.

[41] Here $\epsilon\to\langle\Delta E\rangle$ whereas in the eigenfunction section the limit was $\pi\epsilon\to\langle\Delta E\rangle$. The difference of π can be traced to the different choice of energy smoothing in two sections.

[42] To obtain Eq. (93) note that $d_\mu(E)=\Sigma_n\Omega_\mu(E-E_n) = \int dE' = \Omega_\mu(E-E')d(E')$. By the convolution theorem this equals $h^{-1}\int dt\ e^{iEt/\hbar}\tilde{\Omega}_\mu(t)\ \tilde{d}(t)$, where $\tilde{\Omega}_\mu(t)$ and $\tilde{d}(t)$ are the Fourier transforms of $\Omega_\mu(E)$ and $d(E)$, respectively. This can be written as $d_\mu(E)=h^{-1}\int dt\ \tilde{\Omega}_\mu(t)\ \mathrm{Tr}[e^{i(E-\hat{H})t/\hbar}]$, where $\tilde{\Omega}_\mu(t)=\mathrm{sinc}(\mu t/2\hbar)$. Adopting the semiclassical approximation of Berry [30] for the trace in this expression, and performing the time integral by stationary phase, then gives Eq. (93).

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