Inverting a time-dependent harmonic oscillator potential by a unitary transformation and a class of exactly solvable oscillators

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A time-dependent unitary (canonical) transformation is found that maps the Hamiltonian for a harmonic oscillator with time-dependent real mass and real frequency to that of a generalized harmonic oscillator with time-dependent real mass and imaginary frequency. The latter may be reduced to an ordinary harmonic oscillator by means of another unitary (canonical) transformation. A simple analysis of the resulting system leads to the identification of a previously unknown class of exactly solvable time-dependent oscillators. Furthermore, it is shown how one can apply these results to establish a canonical equivalence between some real and imaginary frequency oscillators. In particular it is shown that a harmonic oscillator whose frequency is constant and whose mass grows linearly in time is canonically equivalent with an oscillator whose frequency changes from being real to imaginary and vice versa repeatedly. [S1050-2947(97)07205-3]

The solution of the Schrödinger equation, $H\psi = i\dot{\psi}$, for a harmonic oscillator with time-dependent mass m and frequency ω, i.e.,

$$H(t) = \frac{1}{2m(t)} p^2 + \frac{m(t)\omega^2(t)}{2} x^2, \qquad (1)$$

has been the subject of continuous investigation since the late 1940's, [1–4]. The main reason for the interest in this problem is its wide range of application in the description of physical systems. Although by now there exist dozens of articles on the subject, a closed analytic expression for the time-evolution operator is still missing. Recently, Ji and Kim [4] showed that using the Lewis-Riesenfeld method [2] one can construct an invariant operator in terms of the (two independent) solutions of the classical dynamical equations:

$$\frac{d}{dt}\left[m(t) \frac{d}{dt} x_c(t) \right] + m(t)\omega^2(t)x_c(t) = 0, \qquad (2)$$

and therefore reduce the solution of the Schrödinger equation to that of Eq. (2), for $\omega(t), m(t) \in \mathbb{R}$.[1] The case where the frequency ω is imaginary has been considered only in the time-independent case [5].

The purpose of this paper is to study the implications of the recently developed method of adiabatic unitary transformation of the Hilbert space [6] for this problem. The basic idea of this method is to use the inverse of the adiabatically approximate time-evolution operator to transform to a moving frame. This transformation has proven to lead to some interesting results for the system consisting of a magnetic dipole in a changing magnetic field. The analogy between the dipole system and the time-dependent harmonic oscillator is best described in terms of the relation between their dynamical groups, namely, SU(2) and SU(1,1) [7].

Let us first concentrate on the real frequency case, $\omega(t) \in \mathbb{R}$. Then the Hamiltonian (1) and its eigenvectors $|n;t\rangle$ can be expressed in terms of the creation ($a^\dagger$) and annihilation (a) operators, namely,

$$H(t) = \omega(t)[a^\dagger(t)a(t) + 1/2], \qquad (3)$$

$$|n;t\rangle = \frac{1}{\sqrt{n!}} a^{\dagger n}(t)|0;t\rangle, \qquad (4)$$

where

$$a(t) := \frac{1}{\sqrt{2}} (e^{\kappa(t)}x + ie^{-\kappa(t)}p),$$

$$\kappa(t) := \tfrac{1}{2} \ln[m(t)\omega(t)], \qquad (5)$$

and $|0;t\rangle$ corresponds to the ground state at time t, i.e., $a(t)|0;t\rangle = 0$.[2] Similarly to the time-independent case the eigenvalue equation

$$H(t)|n;t\rangle = E_n(t)|n;t\rangle \qquad (6)$$

is satisfied for $E_n(t) = E_n(\omega(t)) = \omega(t)(n+1/2)$. Therefore the eigenvalues are nondegenerate and the adiabatically approximate time-evolution operator [6] may be expressed as

$$U_0(t) := \sum_n e^{i\alpha_n(t)}|n;t\rangle\langle n;0|, \qquad (7)$$

where

$$\alpha_n(t) := \delta_n(t) + \gamma_n(t), \quad \delta_n(t) := -\int_0^t E_n(t')dt',$$

$$\gamma_n(t) := i\int_0^t A_{nn}(t')dt', \qquad (8)$$

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[1]Note that the case $m=0$ is a singularity of the problem. The results of this article are only valid for the cases where the mass does not vanish. These can, however, be treated in the limiting sense $m\to 0$. In other words, I shall only consider time periods during which the mass is nonzero but can be arbitrarily small.

$$A_{mn}(t) := \left\langle m;t \left| \frac{d}{dt} \right| n;t \right\rangle. \qquad (9)$$

Following the ideas developed in Ref. [6], let us next use $U_0^{-1}(t) = U_0^\dagger(t)$ to perform a unitary transformation of the Hilbert space. Recall that under a general unitary transformation $|\psi(t)\rangle \to |\psi'(t)\rangle = \mathcal{U}(t)|\psi(t)\rangle$, the Hamiltonian H and the corresponding time-evolution operator transform according to

$$H(t) \to H'(t) = \mathcal{U}(t)H(t)\mathcal{U}^\dagger(t) - i\mathcal{U}(t)\dot{\mathcal{U}}^\dagger(t), \qquad (10)$$

$$U(t) \to U'(t) = \mathcal{U}(t)U(t)\mathcal{U}^\dagger(0). \qquad (11)$$

These equations are obtained by demanding that the time evolution in the transformed frame is governed by the Schrödinger equation corresponding to the transformed Hamiltonian H'. Note that unlike other observables that transform linearly, i.e., $\mathcal{O} \to \mathcal{U}(t)\mathcal{O}\mathcal{U}^\dagger(t)$, the Hamiltonian undergoes an affine transformation. In particular, its spectrum generally changes if the transformation is time dependent.

Substituting $U_0^\dagger$ for $\mathcal{U}$ in (10), one can show [6] that

$$H'(t) = -i \sum_{n \neq m} e^{-i[\alpha_m(t) - \alpha_n(t)]} A_{mn}(t)|m;0\rangle\langle n;0|. \qquad (12)$$

In order to express the transformed Hamiltonian H' in a closed form, one must first compute A_{mn}. This can be done directly by substituting Eq. (4) in (9) or indirectly using the identity

$$A_{mn}(t) = \frac{\langle m;t| [dH(t)/dt] |n;t\rangle}{E_n(t) - E_m(t)} \quad \text{for } m \neq n,$$

which is obtained by differentiating both sides of Eq. (6) and making use of the orthonormality of $|n;t\rangle$. The latter method turns out to be much simpler. It yields

$$A_{mn}(t) = \frac{\dot{\kappa}(t)}{2} [\sqrt{n(n-1)}\delta_{m,n-2} - \sqrt{m(m-1)}\delta_{m-2,n}]. \qquad (13)$$

Here use is made of the well-known relations

$$a(t)|n;t\rangle = \sqrt{n}|n-1;t\rangle, \quad a^\dagger(t)|n;t\rangle = \sqrt{n+1}|n+1;t\rangle,$$

$$\dot{a}(t) = \dot{\kappa}(t)a^\dagger(t). \qquad (14)$$

Note that Eq. (13) is also valid for $m = n$ since due to the fact that $|n;t\rangle$ can be chosen to be real, $A_{nn}(t) = 0$. In particular $\gamma_n(t) = 0$. Hence, $\alpha_n(t) = \delta_n(t) = (n + 1/2)\delta(t)$, where

$$\delta(t) := -\int_0^t \omega(\tau)d\tau. \qquad (15)$$

Substituting the expressions for α_n and A_{mn} in Eq. (12) and performing the summation over m, one finds

$$H'(t) = \frac{-i\dot{\kappa}(t)}{2} \sum_n [\sqrt{n(n-1)}e^{2i\delta(t)}|n-2;0\rangle\langle n;0|$$

$$- \sqrt{(n+1)(n+2)}e^{-2i\delta(t)}|n+2;0\rangle\langle n;0|]$$

$$= \frac{-i\dot{\kappa}(t)}{2} [e^{2i\delta(t)}a^2(0) - e^{-2i\delta(t)}a^{\dagger 2}(0)]$$

$$= \frac{\dot{\kappa}(t)}{2} \{\sin[2\delta(t)](e^{-2\kappa(0)}p^2 - e^{2\kappa(0)}x^2)$$

$$+ \cos[2\delta(t)](xp + px)\}. \qquad (16)$$

Next consider the Schrödinger equation in the transformed frame: $H'(t)|\psi'(t)\rangle = i|\dot{\psi}'(t)\rangle$. The presence of a total derivative $\dot{\kappa}$ on the right-hand side of Eq. (16), suggests a redefinition of the time $t \to t' := \kappa(t)$. Note that for $\dot{\kappa}(t) = 0$, i.e., $\omega(t) = c/m(t)$ for some constant $c \in \mathbb{R}$, $H'(t)$ vanishes identically and the adiabatic approximation is exact. Furthermore, for $\dot{\kappa} < 0$, one can consider the time-reversed system for which $\dot{\kappa} > 0$. The time-evolution operator for the original system is obtained from that of the time-reversed system by inversion. Therefore, without loss of generality, one can assume $\dot{\kappa}(t) > 0$.[3] The latter allows for the above-mentioned redefinition of time $t \to t'$. This leads to the Schrödinger equation $h(t')|\psi'(t')\rangle = i|\dot{\psi}'(t')\rangle$ for the Hamiltonian

$$h(t') := \tfrac{1}{2} \{\sin[2\widetilde{\delta}(t')](e^{-2\kappa_0}p^2 - e^{2\kappa_0}x^2)$$

$$+ \cos[2\widetilde{\delta}(t')](xp + px)\},$$

$$= \frac{1}{2m'(t')} p^2 + \frac{m'(t')\omega'^2(t')}{2} x^2$$

$$+ \tfrac{1}{2} \beta(t')(xp + px), \qquad (17)$$

where

$$\widetilde{\delta}(t') := \delta(t(t')), \quad \kappa_0 := \kappa(0),$$

$$m'(t') := \frac{e^{2\kappa_0}}{\sin[2\widetilde{\delta}(t')]}, \quad \omega'(t') := i\sin[2\widetilde{\delta}(t')], \qquad (18)$$

$$\beta(t') := \cos[2\widetilde{\delta}(t')] = \sqrt{1 + \omega'^2(t')}. \qquad (19)$$

Hence, the transformed Hamiltonian is a generalized harmonic oscillator [7]:

$$h(t') = \tfrac{1}{2}[\alpha(t')p^2 + \beta(t')(xp + px) + \gamma(t')x^2],$$

$$\alpha(t') := 1/m'(t'), \quad \gamma(t') := m'(t')\omega'^2(t') \qquad (20)$$

with a real mass $m'(t')$ and an imaginary frequency $\omega'(t') = ie^{2\kappa_0}/m'(t')$. Note that as a result of the adiabatic

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[2]Note that in the expression for a and in what follows only $\exp(\kappa)$ and the time derivative of κ appear. These are well-defined functions even if κ itself happens to be ill defined.

[3]In general, $\dot{\kappa}$ can change sign. In this case, one must divide the time axis into the intervals during which $\dot{\kappa}$ has definite sign or is zero. During the intervals where $\dot{\kappa} = 0$, the adiabatic approximation yields the exact time-evolution operator. For the intervals during which $\dot{\kappa} > 0$ the above results apply directly. For the intervals where $\dot{\kappa} < 0$, one can use the results obtained for the time-reversed system.

unitary transformation and redefinition of time, the two arbitrary functions $m(t)$ and $\omega(t)$ have been reduced to a single function, namely, $\widetilde{\delta}(t')$.

It is well known that one can transform the generalized harmonic oscillator to an ordinary harmonic oscillator by the time-dependent canonical transformation [7]

$$x \to x, \quad p \to p + \left[\frac{\beta(t')}{\alpha(t')} \right] x. \tag{21}$$

This leads to the Hamiltonian

$$h'(t') = \frac{1}{2} \left\{ \alpha(t')p^2 + \left[\gamma(t') - \frac{\beta^2(t')}{\alpha(t')} - \frac{d}{dt'} \left(\frac{\beta(t')}{\alpha(t')} \right) \right] x^2 \right\}$$

$$= \frac{1}{2m'(t')} p^2 + \frac{m'(t')\Omega'^2(t')}{2} x^2, \tag{22}$$

where

$$\Omega'(t') := \sqrt{-1 + \frac{2}{\sin[2\widetilde{\delta}(t')]} \frac{d\widetilde{\delta}(t')}{dt'}}. \tag{23}$$

Note that Ω' can be real or imaginary depending on the form of $\widetilde{\delta}(t')$. Clearly for $\Omega' = 0$, the problem reduces to a free particle with a variable mass whose solution can be given exactly. In terms of the original functions, the condition $\Omega' = 0$ is expressed as

$$m(t) = \left[\frac{m_0}{\omega(t)} \right] \tan^2 \int_0^t \omega(\tau)d\tau, \tag{24}$$

where m_0 is a real constant.

Equation (24) determines a new class of exactly solvable cases that is the analog of the exactly solvable magnetic dipole Hamiltonians obtained in [6]. The only difference is that here an additional canonical transformation (21) is also performed. It is not difficult to see that this transformation corresponds to the action of the unitary operator $\mathcal{U}'(t') := \exp[i\beta(t')x^2/2\alpha(t')]$ on the Hilbert space. In fact, the Hamiltonian h' can be obtained from h by substituting h for H and $\mathcal{U}'$ for $\mathcal{U}$ in Eq. (10). Similarly in view of Eq. (11), one has the following relation between the evolution operators:

$$u(t') = e^{-i[\beta(t')/2\alpha(t')]x^2} u'(t') e^{i[\beta(\kappa_0)/2\alpha(\kappa_0)]x^2}, \tag{25}$$

where $u(t')$ and $u'(t')$ are the evolution operators for the Hamiltonians $h(t')$ and $h'(t')$, respectively. A further application of Eq. (11) leads to the expression

$$U(t) = U_0(t)u(t'(t)) \tag{26}$$

for the time-evolution operator associated with the original Hamiltonian (1). Here $U_0(t)$ is the adiabatically approximate expression for the time-evolution operator given by Eq. (7).

For the cases where the condition (24) holds, $u'(t')$ is the evolution operator for a free particle with a variable mass $m'(t')$, i.e.,

$$u'(t') = \exp\left[\frac{-i}{2} \int_{\kappa_0}^{t'} \frac{d\tau}{m'(\tau)} p^2 \right]. \tag{27}$$

If Ω' does not vanish but satisfies $\Omega'(t') = \Omega_0'/m'(t')$ for some constant Ω_0', then the Schrödinger equation for h' is still exactly solvable. In fact in this case

$$u'(t') = \exp\left[\frac{-i}{2} \int_{\kappa_0}^{t'} \frac{d\tau}{m'(\tau)} (p^2 + \Omega_0'^2 x^2) \right].$$

The condition $\Omega'(t') = \Omega_0'/m'(t')$ is a generalization of $\Omega'(t') = 0$. It corresponds to a larger class of exactly solvable time-dependent harmonic oscillators. In terms of the original parameters m and ω this condition is expressed as

$$m(t) = \frac{m_0 f^{1/\sqrt{\zeta-1}}(t)}{\omega(t)g^{1/\sqrt{\zeta}}(t)}, \tag{28}$$

where m_0 and $\zeta := 1 + e^{4\kappa_0}/\Omega_0^2$ are constants and

$$f(t) := \frac{\sqrt{\zeta - z(t)} - \sqrt{\zeta - 1}}{\sqrt{\zeta - z(t)} + \sqrt{\zeta - 1}}, \quad g(t) := \frac{\sqrt{\zeta - z(t)} - \sqrt{\zeta}}{\sqrt{\zeta - z(t)} + \sqrt{\zeta}},$$

$$z(t) := 1 - \frac{1}{\zeta - 1} \sin^2\left(\int_0^t \omega(\tau)d\tau \right).$$

Equation (24) is a special case of Eq. (28) where $\Omega_0' \to 0$.

Finally I would like to emphasize the following points:

(1) As one can see from the above analysis some of the quantities introduced in this paper may diverge at certain values of time t. This will in general happen for a discrete set of values of t. The appearance of such discrete singularities is an indication of the fact that one cannot proceed using the same time-dependent unitary transformation and the redefinition of time at the singular points. This does not, however, reduce the effectiveness of the results of this paper. In the presence of discrete singularities, one needs to confine the analysis of the problem to the intervals of time separated by the singular points. The above analysis can be pursued within each such interval. The results are, however, valid for the whole range of time. This is because, being an initial value first-order differential equation, the Schrödinger equation has no discontinuous solutions. The method of solution may fail at a certain discrete (measure zero) set of points, but due to the continuity of the solution of the Schrödinger equation this does not affect the form of the solution, i.e., the value of the time-evolution operator at a singular point is the same as its right or left limit. Hence, in practice one starts at $t = 0$ and integrates the Schrödinger equation with $U(0) = 1$ until one hits a singular point, say, t_1. Then, one sets $U(t_1) = \lim_{t \to t_1^-} U(t)$ and continues integrating the Schrödinger equation for $t > t_1$. Clearly, the same can be done for the next singular point and so on. This argument will only fail if the set of singular points become dense in an open time interval. This is a highly special case to which the analysis of this paper does not apply.

(2) For the case that Ω' is real, one can repeat the above analysis by replacing H of Eq. (1) by h' of Eq. (22). In principle this may lead to yet other exactly solvable cases. If the iteration of this procedure yields oscillators with real frequency at each step, then it can be repeated indefinitely. This
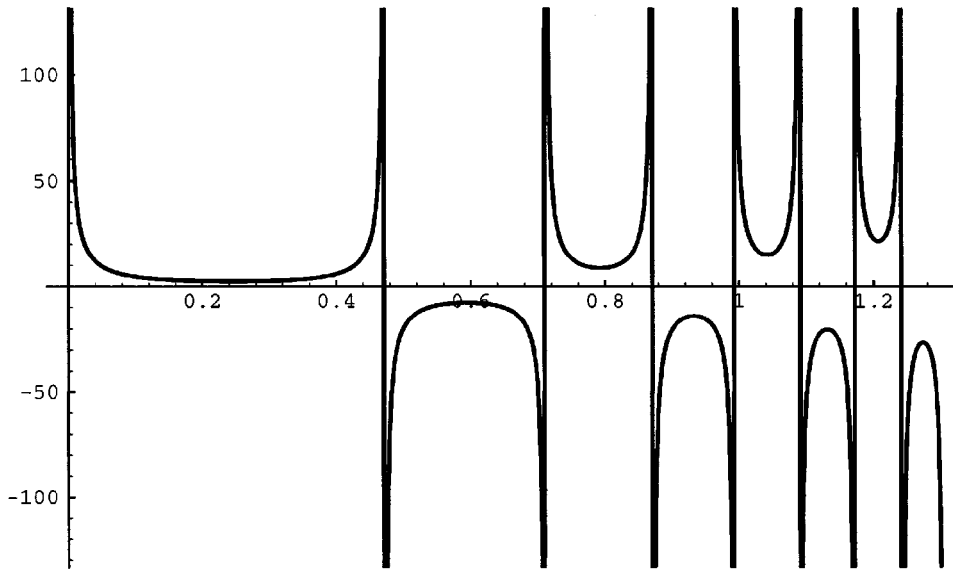
leads to a product expansion for the time-evolution operator, which is analogous to what is called an adiabatic product expansion in Ref. [6].

(3) The above analysis also indicates that the time-dependent harmonic oscillators whose mass M and frequency Ω are related according to

$$\Omega = \sqrt{-1 - \frac{\dot{M}}{\sqrt{M^2 - M_0^2}}} \qquad (29)$$

play a universal role. This is because as shown above the problem for the most general real frequency harmonic oscillator (1) can be reduced to this case by means of a series of unitary (canonical) transformations. Equation (29) is obtained from Eq. (23) by expressing the right-hand side of Eq. (23) in terms of the mass. The parameter M_0 is an arbitrary positive constant corresponding to $e^{2\kappa_0}$.

(4) It is tempting to seek applications of the known results for the real frequency oscillators to the time-dependent imaginary frequency oscillators satisfying Eq. (29). In view of the above analysis, there is a class of imaginary frequency oscillators of this form that is canonically equivalent to some real frequency oscillators. One must, however, be aware that by performing the canonical transformation described in this paper in the reverse order, one might not be able to transform an arbitrary imaginary frequency oscillator to a real frequency one, even if it satisfies Eq. (29). In this case the transformation associated with the adiabatic approximation would not be the same as the one obtained above. Nevertheless, the above scheme is consistent in the sense that if the frequency of the oscillator obtained by transforming back an imaginary frequency oscillator satisfying Eq. (29) turns out to be real then one has the desired result. Otherwise, the method fails to transform the imaginary frequency oscillator to a real frequency one. A simple example of a case where Eq. (29) is satisfied but the transformation to a real frequency oscillator is not possible is the time-independent oscillator with $M = \text{const}$ and $\Omega = i$. Another example is the case where the mass is decaying in time t' according to $M = m'(t') = M_0(1 + e^{-\mu t'})$ with $\mu > 1$ and the frequency

$\Omega = \Omega'(t')$ is given by Eq. (29), i.e., $\Omega^2 = -1 + \mu(1 + 2e^{\mu t'})^{-1/2}$. This system is particularly interesting since the potential $V := M\Omega^2 x^2/2$ changes sign at $t' = \kappa_* := (1/\mu) \ln[(\gamma^2 - 1)/2]$. It is positive for $t' < \kappa_*$ and negative for $t' > \kappa_*$. Hence, at $t' = \kappa_*$ the energy spectrum undergoes a ''phase transition.'' However, it is not difficult to see that transforming this system to a real frequency oscillator is not possible. This is because making the necessary canonical transformations, one obtains an oscillator with the mass $m(t)$ and frequency $\omega(t)$ satisfying

$$[m(t)\omega(t)]^{-2/\mu} = -1 - \frac{1}{\sin\left[2\int_0^t \omega(\tau)d\tau\right]}. \qquad (30)$$

Thus one should seek functions $m:[0,\infty) \to \mathbb{R}^+$ and $\omega:[0,\infty) \to \mathbb{R}^+$, which satisfy Eq. (30) and are positive for $t \in [0,T]$ for some $T > t_*$, where t_* is defined according to $\kappa(t_*) = \kappa_*$. That such functions do not exist can be directly inferred from the fact that for $t \to 0$ the right hand side of Eq. (30) tends to $-\infty$, whereas the left-hand side remains positive. A simple example of a real frequency oscillator, which is canonically equivalent to an oscillator whose frequency fluctuates between real and imaginary values, is

$$m(t) = m_0 + \mu t, \quad \omega(t) = \omega_0, \qquad (31)$$

where m_0, μ, and ω_0 are positive constants. Applying the canonical transformations introduced in this paper to this oscillator, one arrives at another canonically equivalent oscillator with mass $m'(t')$ and frequency $\Omega'(t')$ given by

$$m'(t') = \frac{m_0\omega_0}{\sin[(2/\mu)(m_0\omega_0 - e^{2t'})]},$$

$$\Omega'^2(t') = -1 - \frac{2e^{2t'}}{\mu \sin[(2/\mu)(m_0\omega_0 - e^{2t'})]}, \qquad (32)$$

where $t' = \kappa(t) = \ln[\omega_0(m_0 + \mu t)]/2$, and use is made of Eqs. (18) and (23). Figure 1 shows a plot of Ω'^2 as a function of

t', for $m_0 = \mu = \omega_0 = 1$. As seen from this plot, Ω'^2 changes sign repeatedly. However, the original oscillator (31) is a very simple time-dependent oscillator with positive real mass and frequency. In fact, one might try to apply the results of Refs. [3, 4] to obtain exact solution of the Schrödinger equation for this oscillator. This would immediately lead to the exact solution of the Schrödinger equation for the oscillator (32), at least for the periods of time during which m' and Ω' are continuous functions of t'. This is particularly interesting since as Ω'^2 changes sign from positive to negative, the spectrum of the corresponding oscillator (32) changes from being discrete to continuous and vice versa, while the spectrum of the canonically equivalent oscillator (31) always remains discrete.

In conclusion, the time-dependent unitary (canonical) transformations not only give rise to the identification of a large class of exactly solvable real frequency time-dependent oscillators, but also they allow for the study of some imaginary frequency oscillators by mapping them to the canonically equivalent real frequency oscillators. Although the energy spectra of these real and imaginary frequency oscillators are respectively continuous and discrete, the corresponding dynamical problems are equivalent. Note that a general time-dependent canonical transformation does not preserve the structure of the energy spectrum of the quantum system. This is mainly because of the fact that under such a transformation the Hamiltonian undergoes an affine transformation, not a linear one.

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