ARTICLES

Relations of canonical and unitary transformations for a general time-dependent quadratic Hamiltonian system

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We consider general time-dependent quadratic Hamiltonian systems which are connected by canonical transformations and give the same classical equations of motion. In those systems, we demonstrate that canonical transformations in classical mechanics correspond to unitary transformations in quantum mechanics. The wave functions and the propagators are evaluated using the invariant operator method. However, the values of some functions of the canonical variables q and p are not equal to the values of the same functions of the other canonical variables Q and P, but the values of the functions of q and the kinetic momentum p_k are equal to those of the other Q and P_k in classical mechanics. We prove that these also hold in the quantum treatment. The uncertainty relations of momentum and position are evaluated for the two Hamiltonians. [S1050-2947(97)01206-7]

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I. INTRODUCTION

The nonconservative behavior of physical phenomena as contrasted with the conservative nature is an important science with a fascinating and apparently enduring problem [1]. In particular, the time evolution of dynamical systems with an explicitly time-dependent Hamiltonian has attracted considerable attention [2–6]. A general time-dependent quadratic Hamiltonian system has often been considered because of its various applications in quantum optics [7]. The most famous examples are the degenerate parametric amplifier [8] and Pauli trap [9]. The general form of this Hamiltonian, which describes a classical forced oscillator with a time-dependent frequency and velocity-dependent damping, has been shown by Havas [10], and this kind of system has also been considered by others [11,12].

A quantum-mechanical treatment of a time-dependent or nonconservative system must be treated carefully. First of all, classically there are numerous canonical momenta which correspond to one coordinate, but there is only one kinetic momentum as a simple product of mass and velocity, in which case the question arises as to which corresponds to the momentum operator in quantum mechanics and what is the interpretation of rest momenta. Second, there are numerous classical Hamiltonians which give one solution, and in this case one asks, which one is the quantum-mechanical Hamiltonian and how are the rest interpreted quantum mechanically?

What are the relations for the canonical transformation in classical mechanics and the unitary transformation in quantum mechanics? In this paper we choose a general time-dependent quadratic Hamiltonian which gives the same equations of motion as the linear nonhomogeneous time-dependent differential equation. The Hamiltonian systems which we shall develop are applicable to many physical systems such as a damped harmonic oscillator and a time-dependent harmonic oscillator [13–15]. The purpose of this paper is to address these issues by treating those Hamiltonians. In Sec. II, we consider time-dependent quadratic Hamiltonian systems which are related by canonical transformations and give the same coordinates but different canonical momenta. Though the canonically transformed Hamiltonian is represented by one Hamiltonian, it represents numerous Hamiltonians because it contains an arbitrary time-dependent function. Thus we treat numerous Hamiltonians for one equation of motion. We also find the classical quadratic invariant quantities for each Hamiltonian.

In Sec. III, we show that the canonical transformation of the two classical Hamiltonians corresponds to a unitary transformation in quantum mechanics. The quantum invariant operators which correspond to the classical invariant quantities are evaluated in this section for those Hamiltonian systems. From these, the Schrödinger solutions are found exactly with the auxiliary conditions as classical solutions. The propagator is also calculated from these wave functions. In Sec IV, we treat the quantum expectation values of function of the coordinate and the unitary transformed momentum operator, and we compare the expectation values of these functions and find their relations. We also evaluate the uncertainty relation of the coordinate and canonical momentum operators for two given Hamiltonians. In Sec. V, we give a summary and conclusion.

II. CLASSICAL TREATMENT OF THE SYSTEM

We consider the classical Hamiltonian of a general time-dependent quadratic system given as

$$H(p,q,t)=\tfrac{1}{2}\{A(t)p^2+2B(t)qp+C(t)q^2\}+D(t)q+E(t)p+F(t), \tag{2.1}$$

where q and p are the canonical coordinate and momentum, respectively, and $A(t)\neq 0$ and $B(t)$, $C(t)$, $D(t)$, $E(t)$, and $F(t)$ are real and piecewise continuously differentiable (with respect to t) functions. From Hamilton's equations of motion, the Hamiltonian Eq. (2.1) gives the nonhomogeneous time-dependent differential equation as

$$\ddot{q}+\zeta(t)\dot{q}+\xi(t)q=\chi(t), \tag{2.2}$$

where ζ, ξ, and χ are given by

$$\zeta(t)\equiv-\frac{\dot{A}(t)}{A(t)}, \tag{2.3}$$

$$\xi(t)\equiv\left\{A(t)C(t)+\frac{\dot{A}(t)B(t)}{A(t)}-B(t)^2-\dot{B}(t)\right\}, \tag{2.4}$$

$$\chi(t)\equiv-A(t)D(t)+\dot{E}(t)-\frac{\dot{A}(t)}{A(t)}E(t)+B(t)E(t). \tag{2.5}$$

There are bound and unbound systems in the solution of Eq. (2.2). However, we do not know whether or not our system is bound unless the time-dependent coefficients of the Hamiltonian are known. Here, we only choose the bound systems in the Hamiltonian Eq. (2.1). Without loss of generality, if the particular solution is q_{pa}, the general solution of Eq. (2.2) can be written as

$$q(t)=C_1\eta(t)\exp\{i\theta(t)\}+C_2\eta(t)\exp\{-i\theta(t)\}+q_{pa}. \tag{2.6}$$

Substituting Eq. (2.6) into Eq. (2.2), we obtain the two differential equations

$$\eta\ddot{\theta}+2\dot{\eta}\dot{\theta}+\xi\eta\dot{\theta}=0, \tag{2.7}$$

$$\ddot{\eta}-\eta\dot{\theta}^2+\zeta\dot{\eta}+\xi\eta=0. \tag{2.8}$$

From Eq. (2.7), we can find the first integral as

$$\Omega=\frac{\eta^2\dot{\theta}}{A}=\text{const.} \tag{2.9}$$

From Eq. (2.9) we know that the Wronskian determinant of two terms of the homogeneous solutions of Eq. (2.6) is not zero,

$$W=2i\eta^2\dot{\theta}=2iA\Omega\neq 0. \tag{2.10}$$

This provides proof that Eq. (2.6) is a general solution of Eq. (2.2).

We would like to find a classical invariant quantity I which depends on independent variables: canonical coordinate, momentum, and time. Since functions of I are also invariant, the invariant quantities are numerous. We are interested in the quadratic invariants form of canonical coordinate q and momentum p. From the Hamiltonian (2.1) we can readily find the invariant quantity as [4]

$$I(p,q,t)=\frac{1}{2}\left[\frac{\Omega^2}{\eta^2}(q-q_0)^2+\left\{\left(\frac{B}{A}\eta-\frac{\dot{\eta}}{A}\right)(q-q_0)+\eta(p-p_0)\right\}^2\right], \tag{2.11}$$

where q_0 and p_0 are the time-dependent form of the classical canonical coordinate and momentum. Since Eq. (2.11) is an ellipse in q and p space if the coefficients are fixed, we know that the system is bound.

We are also interested in the other Hamiltonian which gives the same classical solution Eq. (2.2). We try a canonical transformation from the variables (q,p) to new canonical variables (P,Q)

$$Q=q, \tag{2.12}$$

$$P=p-G(t)q, \tag{2.13}$$

where $G(t)$ is an arbitrary real differentiable function. Thus there are numerous canonical momenta of the form of Eq. (2.13). To find the Hamiltonian of the variables (Q,P), we introduce the time-dependent generating function

$$F(P,Q,t)=-\tfrac{1}{2}G(t)Q^2. \tag{2.14}$$

The new Hamiltonian, which gives the same equation of motion, becomes [16]

$$H'(P,Q,t)=H(p,q,t)-p\frac{\partial q}{\partial t}-\frac{\partial F(P,Q,t)}{\partial t}. \tag{2.15}$$

Substituting Eqs. (2.1) and (2.14) into Eq. (2.15), we find

$$H'(Q,P,t)=\tfrac{1}{2}A(t)P^2+\tfrac{1}{2}\{A(t)G(t)^2+C(t)+2B(t)G(t)+\dot{G}(t)\}Q^2+\{A(t)G(t)+B(t)\}QP+[D(t)+E(t)G(t)]Q+E(t)P+F(t). \tag{2.16}$$

Since $Q=q$, Eq. (2.16) gives the same form of the solution of the Hamiltonian Eq. (2.1). Thus we know that only one classical solution can be found from numerous different Hamiltonians.

With the same method, we can also easily find the quadratic invariant quantity of the new Hamiltonian Eq. (2.16) as

$$I'(P,Q,t)=\frac{1}{2}\left[\frac{\Omega^2}{\eta^2}(Q-Q_0)^2+\left\{\left(\frac{B}{A}\eta+G\eta-\frac{\dot{\eta}}{A}\right)(Q-Q_0)+\eta(P-P_0)\right\}^2\right]. \tag{2.17}$$

Since $Q=q$, the Hamiltonian of Eq. (2.16) represents a bound system, which can be deduced from Eq. (2.17).

III. SCHRÖDINGER SOLUTIONS AND PROPAGATORS

A. Schrödinger solutions

We define the quantum Hamiltonian of the system by replacing the classical canonical coordinates q and p by the operators $\hat{q}$ and $\hat{p}$ as

$$\hat{H}(\hat{p},\hat{q},t)=\tfrac{1}{2}[A(t)\hat{p}^2+B(t)(\hat{p}\hat{q}+\hat{q}\hat{p})+C(t)\hat{q}^2]+D(t)\hat{q}+E(t)\hat{p}+F(t). \tag{3.1}$$

The Schrödinger equation of the system can be written

$$i\hbar\frac{\partial\phi}{\partial t}=\hat{H}(p,q,t)\phi. \tag{3.2}$$

Since this equation has the arbitrary coefficients in the Hamiltonian Eq. (3.1), it cannot be solved directly. To solve the Schrödinger equation with auxiliary conditions, we can readily find the quantum invariant operator I which has a quadratic form with operators $\hat{p}$ and $\hat{q}$ as

$$I(\hat{p},\hat{q},t)=\frac{1}{2}\left\{\frac{\Omega^2}{\eta^2}(\hat{q}-q_0)^2+\left[\left(\frac{B}{A}\eta-\frac{\dot{\eta}}{A}\right)(\hat{q}-q_0)+\eta(\hat{p}-p_0)\right]^2\right\}. \tag{3.3}$$

Here we show that this result has the same form as the classical invariant quantity whose canonical variables are displaced by the quantum operators. The invariant quantity Eq. (3.3) can be replaced by creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{I}=\hbar\Omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2}), \tag{3.4}$$

where

$$\hat{a}=\left(\frac{A}{2\hbar\dot{\theta}}\right)^{1/2}\left[\frac{1}{A}\left\{\dot{\theta}+i\left(B-\frac{\dot{\eta}}{\eta}\right)\right\}(\hat{q}-q_0)+i(\hat{p}-p_0)\right], \tag{3.5}$$

and

$$\hat{a}^\dagger=\left(\frac{A}{2\hbar\dot{\theta}}\right)^{1/2}\left[\frac{1}{A}\left\{\dot{\theta}-i\left(B-\frac{\dot{\eta}}{\eta}\right)\right\}(\hat{q}-q_0)-i(\hat{p}-p_0)\right]. \tag{3.6}$$

If $[\hat{q},\hat{p}]=i\hbar$, then

$$[\hat{a},\hat{a}^\dagger]=1. \tag{3.7}$$

In Sec. II, we showed that classically there are numerous Hamiltonians which give only one classical equation but numerous canonical momenta. To find them, we introduce the unitary operator whose characters are equal to the canonical transformation in classical mechanics,

$$\hat{U}(\hat{p},\hat{q},t)=\exp\left\{-\frac{i}{2\hbar}G(t)\hat{q}^2\right\}, \tag{3.8}$$

$$\hat{U}^\dagger(\hat{p},\hat{q},t)=\exp\left\{\frac{i}{2\hbar}G(t)\hat{q}^2\right\}, \tag{3.9}$$

where $G(t)$ is the same as in the generating function Eq. (2.14). Using Eqs. (3.8) and (3.9), the new operators $\hat{Q}$ and $\hat{P}$ are defined from the operators $\hat{q}$ and $\hat{p}$ as

$$\hat{Q}=\hat{U}^\dagger\hat{q}\hat{U}=\hat{q}, \tag{3.10}$$

$$\hat{P}=\hat{U}\hat{p}\hat{U}^\dagger=\hat{p}+G(t)\hat{q}, \tag{3.11}$$

where Eqs. (3.10) and (3.11) correspond to the classical canonical transformation Eqs. (2.12) and (2.13). These kinds of unitary operators acting on the Hilbert space $\mathfrak{R}$ transform the Schrödinger operator

$$\hat{S}_0\equiv\hat{H}-i\frac{\partial}{\partial t} \tag{3.12}$$

of the first Hamiltonian system into the Schrödinger operator

$$S_1\equiv H'(Q,P,t)-i\frac{\partial}{\partial t}, \tag{3.13}$$

namely,

$$\hat{S}_1\equiv\hat{U}^\dagger\hat{S}_0\hat{U} \tag{3.14}$$

through Eqs. (3.11) and (3.12). We obtain the new quantum Hamiltonian as

$$H'(\hat{Q},\hat{P},t)=\tfrac{1}{2}A(t)\hat{P}^2+\tfrac{1}{2}\{A(t)G(t)^2+C(t)+2B(t)G(t)+\dot{G}(t)\}\hat{Q}^2+\tfrac{1}{2}\{A(t)G(t)+B(t)\}(\hat{P}\hat{Q}+\hat{Q}\hat{P})+[D(t)+E(t)G(t)]\hat{Q}+E(t)\hat{P}+F(t). \tag{3.15}$$

This result is the same form as the classical Hamiltonian, Eq. (2.16), whose canonical variables are displaced by quantum operators. Here we know that there are numerous quantum Hamiltonians and Schrödinger equations for one system.

We can obtain the new quantum invariant operator $\hat{I}'$, which is the quadratic form of $\hat{Q}$ and $\hat{P}$, as

$$\hat{I}'(\hat{P},\hat{Q},t)=\frac{1}{2}\left[\frac{\Omega^2}{\eta^2}(\hat{Q}-Q_0)^2+\left\{\left(\frac{B}{A}\eta+G(t)\eta-\frac{\dot{\eta}}{A}\right)\times(\hat{Q}-Q_0)+\eta(\hat{P}-P_0)\right\}^2\right]. \tag{3.16}$$

This Eq. (3.16) is also simplified by new creation and annihilation operator $(\hat{b},\hat{b}^\dagger)$ as

$$\hat{I}'=\hbar\Omega(\hat{b}+\hat{b}+\tfrac{1}{2}), \tag{3.17}$$

where

$$\hat{b}=\left(\frac{A}{2\hbar\dot{\theta}}\right)^{1/2}\left[\frac{1}{A}\left\{\dot{\theta}+i\left(AG+B-\frac{\dot{\eta}}{\eta}\right)\right\}(\hat{Q}-Q_0)+i(\hat{P}-P_0)\right], \tag{3.18}$$

$$\hat{b}^\dagger=\left(\frac{A}{2\hbar\dot{\theta}}\right)^{1/2}\left[\frac{1}{A}\left\{\dot{\theta}-i\left(AG+B-\frac{\dot{\eta}}{\eta}\right)\right\}(\hat{Q}-Q_0)-i(\hat{P}-P_0)\right]. \tag{3.19}$$

The spectra and eigenfunctions of invariant operators $\hat{I}'$ and $\hat{I}$ are found as

$$\hat{I}U_n(q,t)=\lambda U_n(q,t), \tag{3.20}$$

where

$$\lambda=\hbar\Omega(n+\tfrac{1}{2}),\quad n=0,1,2,\ldots, \tag{3.21}$$

$$U_n(q,t)=\left(\frac{\dot{\theta}}{\hbar A\pi}\right)^{1/4}\frac{(-1)^n}{\sqrt{n!}}\exp\left\{-\frac{\dot{\theta}}{2\hbar A}q_0^2-\delta\frac{q^2}{2}+\epsilon q\right\}\times H_n(\sqrt{\delta_r}(q-q_0)), \tag{3.22}$$

$$\delta=\frac{1}{\hbar A}\left\{\dot{\theta}+i\left(B-\frac{\dot{\eta}}{\eta}\right)\right\}=\delta_r+i\delta_i, \tag{3.23}$$

$$\epsilon=\delta q_0+i\frac{p_0}{\hbar}. \tag{3.24}$$

Using the eigenstates of the invariant operator Eq. (3.22), the solution of the Schrödinger equation is obtained as

$$\psi_n(q,t)=\frac{1}{\sqrt{n!2^n}}\left(\frac{\dot{\theta}}{\hbar A\pi}\right)^{1/4}\exp\left\{-\frac{\dot{\theta}}{2\hbar A}q_0^2-\frac{\epsilon^2}{2\delta}+i\Delta-i\left(n+\frac{1}{2}\right)\theta\right\}\exp\left\{-\frac{\delta}{2}\left(q-q_0-\frac{ip_0}{\delta\hbar}\right)^2\right\}\times H_n(\sqrt{\delta_r}(q-q_0)), \tag{3.25}$$

where

$$\frac{d\Delta}{dt}=-\frac{F}{\hbar}-\frac{1}{2\hbar A}\left\{\left(\dot{q}_0-\frac{\dot{\eta}}{\eta}q_0+\dot{\theta}q_0\right)\times\left(\dot{q}_0-\frac{\dot{\eta}}{\eta}q_0-\dot{\theta}q_0\right)-E^2\right\}. \tag{3.26}$$

Using the same method as above, we obtain the wave function of the Schrödinger equation, which comes from the new Hamiltonian Eq. (3.15), as

$$\psi_n^b(Q,t)=\frac{1}{\sqrt{n!2^n}}\left(\frac{\dot{\theta}}{\hbar A\pi}\right)^{1/4}\exp\left\{-\frac{\dot{\theta}}{2\hbar A}Q_0^2-\frac{\epsilon_b^2}{2\delta_b}+i\Delta_b-i\left(n+\frac{1}{2}\right)\theta\right\}\exp\left\{-\frac{\delta_b}{2}\left(Q-Q_0-\frac{iP_0}{\delta_b\hbar}\right)^2\right\}\times H_n(\sqrt{\delta_r}(Q-Q_0)), \tag{3.27}$$

where

$$\delta_b=\frac{1}{\hbar A}\left\{\dot{\theta}+i\left(GA+B-\frac{\dot{\eta}}{\eta}\right)\right\}=\delta_r+i\delta_i^b, \tag{3.28}$$

$$\epsilon_b=\delta_bQ_0+i\frac{P_0}{\hbar}, \tag{3.29}$$

$$\frac{d\Delta_b}{dt}=-\frac{F}{\hbar}-\frac{1}{2\hbar A}\left\{\left(\dot{Q}-\frac{\dot{\eta}}{\eta Q_0}+\dot{\theta}Q_0\right)\times\left(\dot{Q}-\frac{\dot{\eta}}{\eta}Q_0-\dot{\theta}Q_0\right)-E^2\right\}. \tag{3.30}$$

Having found the Schrödinger solutions of a different type for the Schrödinger equations for one system, we raise a question: What is the meaning of the numerous kinds of quantum states for one system? We will discuss this in Sec. IV.

B. Propagators

The propagator is defined for the bound system as

$$K(q,t;q',t')=\sum_{n=0}^{\infty}\phi_n(q,t)\phi_n^*(q',t'). \tag{3.31}$$

With the help of Mehler's formula,

$$\sum_{n=0}^{\infty}H_n(X)H_n(Y)\frac{Z^n}{n!2^n}=\sqrt{1-Z^2}\exp\left\{\frac{2XYZ-X^2-Y^2}{1-Z^2}+X^2+Y^2\right\}, \tag{3.32}$$

and Eq. (3.25), the propagator is given by

$$K(q,t;q',t')=\left(\frac{\dot\theta^{1/2}\dot\theta'^{1/2}}{2i\pi\hbar\ \sin(\theta-\theta')\,\eta^{1/2}\eta'^{1/2}}\right)^{1/2}\exp\{i(\Delta-\Delta')\}$$
$$\times\exp\left[i\left\{\left(B-\frac{\dot\eta}{\eta}\right)q_0^2+p_0q_0-\left(B'-\frac{\dot\eta'}{\dot\eta'}\right)q_0'^2-p_0'q_0'\right\}-p_0^2-p_0'^2\right]$$
$$\times\exp\left\{\frac{i}{2\hbar A}\,(q-q_0)^2\left[\dot\theta\ \cot(\theta-\theta')-B+\frac{\dot\eta}{\eta}\right]-\frac{i}{\hbar}\,p_0(q-q_0)-\frac{p_0^2}{2\hbar\,\delta}\right\}$$
$$\times\exp\left\{\frac{-i}{2\hbar A'}\,(q'-q_0)^2\left[\dot\theta'\ \cot(\theta-\theta')-B'+\frac{\dot\eta'}{\eta'}\right]+\frac{i}{\hbar}\,p_0'(q'-q_0')-\frac{p_0'^2}{2\hbar\,\delta^*}\right\}$$
$$\times\exp\left\{-\frac{i}{\hbar}\left(\frac{\dot\theta\dot\theta'}{AA'}\right)^{1/2}(q-q_0)(q'-q_0')/\sin(\theta-\theta')\right\},\tag{3.33}$$

where the prime means the quantities at time $t=t'$. With the same method and Eqs. (3.27) and (3.32), the propagator for the new Hamiltonian becomes

$$K(Q,t;Q',t')=\left(\frac{\dot\theta^{1/2}\dot\theta'^{1/2}}{2i\pi\hbar\ \sin(\theta-\theta')\,\eta^{1/2}\eta'^{1/2}}\right)^{1/2}\exp\left\{-P_0^2-P_0'^2-\frac{P_0^2}{2\hbar\,\delta_b}-\frac{P_0'^2}{2\hbar\,\delta_b^*}\right\}\exp\{\Delta_b-\Delta_b'\}$$
$$\times\exp\left[i\left\{\left(B+AG-\frac{\dot\eta}{\eta}\right)Q_0^2+P_0Q_0-\left(B'+A'G'-\frac{\dot\eta'}{\eta'}\right)Q_0'^2-P_0'Q_0'\right\}\right]$$
$$\times\exp\left[\frac{i}{\hbar}\left\{\frac{1}{2A}\,(Q-Q_0)^2\left(\dot\theta\ \cot(\theta-\theta')-B-AG+\frac{\dot\eta}{\eta}\right)-P_0(Q-Q_0)\right\}\right]$$
$$\times\exp\left\{-\frac{i}{\hbar}\left[\frac{1}{2A'}\,(Q'-Q_0)^2\left(\dot\theta\ \cot(\theta-\theta')-B'-A'G'+\frac{\dot\eta'}{\eta'}\right)\right]\right\}$$
$$\times\exp\left\{-\frac{i}{\hbar}\,P_0'(Q'-Q_0')-\frac{i}{\hbar}\left(\frac{\dot\theta\dot\theta'}{AA'}\right)^{1/2}(Q-Q_0)(Q'-Q_0')/\sin(\theta-\theta')\right\}.\tag{3.34}$$

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We know that propagators for different types of Hamiltonians for one system do not have the same form.

IV. QUANTUM AVERAGE AND UNCERTAINTY RELATIONS

A. Expectation of the momentum operators and kinetic momentum

Since the momentum operators of the quantum system correspond to the canonical momenta in the classical system, there are numerous momentum operators and Hamiltonians for a quantum treatment of the system. We would like to find their quantum averages, i.e., expectation values. To do this, we take two different Hamiltonians and their corresponding operators, $\hat q,\hat p$ and $\hat Q,\hat P$. We define the kinetic momentum operator, which corresponds to the classical kinetic momentum as

$$\hat p_k\equiv\dot{\hat q}=A(t)\hat p+B(t)\hat q+E(t).\tag{4.1}$$

From this definition, we can find the commutation relation between q and p_k as

$$[\hat q,\hat p_k]=i\hbar A(t),\tag{4.2}$$

which is quite different from the commutation relation of the operators corresponding to the canonical coordinate and momentum.

To calculate the expectation value and uncertainty, it is convenient to represent $\hat p_k,\hat q,\hat p$ in the form of lowering and raising operators for the first Hamiltonian system as

$$\hat p_k=\dot{\hat q}=\left(\frac{A\hbar}{2\dot\theta}\right)^{1/2}\left\{\left[\frac{\dot\eta}{\eta}-i\dot\theta\right]\hat a+\left[\frac{\dot\eta}{\eta}+i\dot\theta\right]\hat a^\dagger\right\}$$
$$+Ap_0+Bq_0+E,\tag{4.3}$$

$$\hat q=\left(\frac{\hbar\dot A}{2\dot\theta}\right)^{1/2}(\hat a+\hat a^\dagger)+q_0,\tag{4.4}$$

$$\hat p=-\frac{i}{A}\left(\frac{\hbar A}{2\dot\theta}\right)^{1/2}\left\{\dot\theta-i\left(B-\frac{\dot\eta}{\eta}\right)\right\}\hat a+\frac{i}{A}\left(\frac{\hbar\dot A}{2\dot\theta}\right)^{1/2}$$
$$\times\left\{\theta+i\left(B-\frac{\dot\eta}{\eta}\right)\right\}\hat a^\dagger+p_0,\tag{4.5}$$

and for the second Hamiltonian system as

$$\hat{P}_k=\hat{Q}=\left(\frac{A\hbar}{2\dot{\theta}}\right)^{1/2}\left\{\left[\frac{\dot{\eta}}{\eta}-i\dot{\theta}\right]\hat{b}+\left[\frac{\dot{\eta}}{\eta}+i\dot{\theta}\right]\hat{b}^{\dagger}\right\}+AP_0$$
$$+(B+AG)Q_0+E, \tag{4.6}$$

$$\hat{Q}=\left(\frac{\hbar\dot{A}}{2\dot{\theta}}\right)^{1/2}(\hat{b}+\hat{b}^{\dagger})+Q_0, \tag{4.7}$$

$$\hat{P}=-\frac{i}{A}\left(\frac{\hbar\dot{A}}{2\dot{\theta}}\right)^{1/2}\left\{\dot{\theta}+i\left(AG+B-\frac{\dot{\eta}}{\eta}\right)\right\}\hat{b}+\frac{i}{A}\left(\frac{\hbar\dot{A}}{2\dot{\theta}}\right)^{1/2}$$
$$\times\left\{\dot{\theta}-i\left(AG+B-\frac{\dot{\eta}}{\eta}\right)\right\}\hat{b}^{\dagger}+P_0. \tag{4.8}$$

Let us define the quantum eigenstates of the two Hamiltonians as $|n\rangle$ and $|n_b\rangle$. From the form of Eqs. (4.4), (4.5), (4.7), and (4.8), we can readily show that

$$\langle n|f(q,p,t)|n'\rangle\neq\langle n_b|f(Q,P,t)|n_b'\rangle, \tag{4.9}$$

and from Eqs. (4.3), (4.4), (4.6), and (4.7) we show that

$$\langle n|f(q,p_k,t)|n'\rangle=\langle n_b|f(Q,P_k,t)|n_b'\rangle. \tag{4.10}$$

Although there are numerous classical Hamiltonians and corresponding canonical momenta for the system, there is only one classical solution. Thus $f(q,p_k,t)$ is fixed regardless of the selection of the Hamiltonian, but $f(q,p,t)$ is different depending on the Hamiltonian. Like the classical results, although there are numerous Schrödinger equations and their solutions, the quantum average $\langle f(\hat{q},\hat{p}_k,t)\rangle$ is the same for all states of each different Hamiltonian, but $\langle f(\hat{q},\hat{p},t)\rangle$ depends on the states of the selected Hamiltonian.

B. Uncertainty relations

Generally, the uncertainty product of the two observables is determined by commutation relations. The commutation relations for the original $(\hat{q},\hat{p})$ and unitary transformed coordinate and momentum $(\hat{P},\hat{Q})$ are given as

$$[\hat{q},\hat{p}]=i\hbar, \tag{4.11}$$

$$[\hat{Q},\hat{P}]=i\hbar. \tag{4.12}$$

Thus the uncertainty product of $(\hat{q},\hat{p})$ and $(\hat{Q},\hat{P})$ is greater than $\hbar/2$. In this section, we evaluate the exact uncertainty of $(\hat{p},\hat{q})$ and $(\hat{P},\hat{Q})$ using Eqs (4.4), (4.5), (4.7), and (4.8) as

$$\langle n|\Delta q\Delta p|n\rangle=\left(n+\frac{1}{2}\right)\hbar\left[1+\frac{1}{\dot{\theta}^2}\left(B-\frac{\dot{\eta}}{\eta}\right)^2\right]^{1/2}, \tag{4.13}$$

$$\langle n+1|\Delta p\Delta q|n\rangle$$
$$=\frac{\hbar}{2}(n+1)\left\{1+\frac{1}{\dot{\theta}_2}\left(B-\frac{\dot{\eta}}{\eta}\right)^2\right\}^{1/4}$$
$$\times\left\{1+\left[\left(\frac{2A}{\hbar\dot{\theta}}\right)^{1/2}\frac{2p_0}{\sqrt{n+1}}+\frac{1}{\dot{\theta}}\left(B-\frac{\dot{\eta}}{\eta}\right)\right]^2\right\}^{1/4}$$
$$\times\left\{\left(\frac{2\dot{\theta}}{\hbar A}\right)^{1/2}\frac{2q_0}{\sqrt{n+1}}-1\right\}^{1/2}, \tag{4.14}$$

$$\langle n+2|\Delta p\Delta q|n\rangle=\frac{\hbar}{2}\sqrt{(n+2)(n+1)}$$
$$\times\left[1+\frac{1}{\dot{\theta}_2}\left(B-\frac{\dot{\eta}}{\eta}\right)^2\right]^{1/2}, \tag{4.15}$$

$$\langle n_b|\Delta Q\Delta P|n_b\rangle=\left(n_b+\frac{1}{2}\right)\hbar\left[1+\frac{1}{\dot{\theta}^2}\left(AG+B-\frac{\dot{\eta}}{\eta}\right)^2\right]^{1/2}, \tag{4.16}$$

$$\langle n_b+1|\Delta P\Delta Q|n_b\rangle$$
$$=\frac{\hbar}{2}(n_b+1)\left\{1+\frac{1}{\dot{\theta}^2}(EG+B-\dot{\eta}\eta)^2\right\}^{1/4}$$
$$\times\left\{1+\left[\left(\frac{2A}{\hbar\dot{\theta}}\right)^{1/2}\frac{2P_0}{\sqrt{n_b+1}}+\frac{1}{\dot{\theta}}\left(AG+B-\frac{\dot{\eta}}{\eta}\right)\right]^2\right\}^{1/4}$$
$$\times\left\{\left(\frac{2\dot{\theta}}{\hbar A}\right)^{1/2}\frac{2Q_0}{\sqrt{n_b+1}}-1\right\}^{1/2}, \tag{4.17}$$

$$\langle n_b+2|\Delta P\Delta Q|n_b\rangle$$
$$=\frac{\hbar}{2}\sqrt{(n_b+2)(n_b+1)}\left[1+\frac{1}{\dot{\theta}^2}\left(AG+B-\frac{\dot{\eta}}{\eta}\right)^2\right]^{1/2}, \tag{4.18}$$

where $|n\rangle$ are the number states for one Hamiltonian system, and $|n_b\rangle$ are those for the other Hamiltonian system. The uncertainty product of $\hat{q}$ and $\hat{p}$ is different from that of $\hat{Q}$ and $\hat{P}$ for any states. From Eq. (4.2), we know that the uncertainty of position and kinetic momentum does not satisfy Heisenberg's uncertainty principle for the case of $|A(t)|<1$. The uncertainty $\hat{q},\hat{p}_k,\hat{Q},\hat{P}_k$ can be calculated by Eqs. (4.3), (4.4), (4.6), and Eq. (4.7) as

$$\langle n|\Delta q\Delta p_k|n\rangle=\langle n_b|\Delta Q\Delta P_k|n_b\rangle$$
$$=\left(n+\frac{1}{2}\right)\hbar|A|\left[1+\left(\frac{\dot{\eta}}{\dot{\theta}\eta}\right)^2\right]^{1/2}. \tag{4.19}$$

This is the same for all Hamiltonian systems.

V. SUMMARY AND CONCLUSIONS

In this section, we summarize and discuss the results obtained in the previous sections. In Sec. II, we found that there are numerous Hamiltonians which give one classical equation of motion. They are related by a canonical transformation. Thus we have the same number of canonical momenta and invariant quantities. Although there are numerous kinds of canonical momenta for one equation of motion, there is only one coordinate $\hat{q}$ and kinetic momentum $\hat{p}_k$, and the functions of those variables are the same for all those Hamiltonian systems.

In Sec. III, we treated the system quantum mechanically, where we defined the quantum Hamiltonian derived from the classical Hamiltonian by substituting the quantum operator $(\hat{q},\hat{p})$. The other quantum Hamiltonian, corresponding to the canonical transformed classical Hamiltonian, was found by unitarily transforming one quantum Hamiltonian. In this case, the unitary operator has the classical generating function in its exponent. From both Hamiltonians, we obtained quantum quadratic invariant operators, the Schrödinger solution, and the propagator. The wave functions and propagators corresponding to the Hamiltonians have auxiliary conditions as classical equations of motion.

In Sec. IV, we treated the expectation values, which are different for the function of $\hat{q}$ and $\hat{p}$ and for that of $\hat{Q}$ and $\hat{P}$ for any system, but the expectation values of $\hat{q}$ and $\hat{p}_k$ are equal to those of $\hat{Q}$ and $\hat{P}$ for any system. This reflects the fact that the functions of coordinate and kinetic momentum are unique for every classical Hamiltonian which gives one equation of motion.

We conclude that displacing the classical canonical variable by a quantum-mechanical operator, we can uniquely obtain the quantum-mechanical Hamiltonian from any classical Hamiltonian system. That is, there are numerous kinds of classical Hamiltonians for one classical equation of motion, and any Hamiltonian for them can be selected as a quantum Hamiltonian substituting canonical variables by quantum operators.

Section V dealt with quantum uncertainty. The quantum uncertainty of the operators $\hat{q}$ and $\hat{p}$ corresponding to classical canonical variables are not equal to the operators $\hat{Q}$ and $\hat{P}$. However, these satisfy Heisenberg's uncertainty principle. The uncertainty of $\hat{q}$ and kinetic momentum $\hat{p}_k$ is the same for all quantum Hamiltonians which correspond to classical Hamiltonians giving one classical equation of motion. However, these can not satisfy Heisenberg's uncertainty principle.

In this paper, we treated only quadratic Hamiltonian systems. While it is very difficult to deal with more general Hamiltonian systems, we expect in the future to make progress along these lines, which will be reported in a later paper.

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