Harmonic generation by scattering circularly polarized light of arbitrary intensity from free electrons of arbitrary initial velocity

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We derive a general analytic expression for the harmonic power generated per unit solid angle as a result of scattering plane-wave, circularly polarized light of arbitrary intensity from free electrons moving initially with arbitrary velocity. The relativistic derivation is carried out fully in the laboratory frame. [S1050-2947(97)04304-7]

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I. INTRODUCTION

The problem of classical harmonic generation as a result of the interaction of a plane-wave laser field with a free electron that is initially at rest at the origin lends itself to a simple solution when tackled in a frame of reference in which the electron is *on average at rest*. A complete discussion of this restricted problem was given by Sarachik and Schappert [1] in an important 1970 paper. Physical quantities of interest to a laboratory observer, such as the frequency of the scattered radiation and the scattered power cross section, are then obtained using the appropriate Lorentz transformation. The parallel geometry, one in which the electron initially moves parallel to the (laser) radiation field direction of propagation, is only a simple generalization of the situation just described.

However, in most experiments, a beam of relativistic electrons is made to cross a beam of super intense light at some angle θ_0 relative to its direction of propagation. A Lorentz transformation between the frame in which the electron stays on average at rest, the R frame, and the laboratory, the L frame, is in general very cumbersome in analytic form.

In a recent publication [2], we derived analytic classical expressions for the power cross section of radiation scattered from relativistic electrons. The parallel geometry and the geometry corresponding to an electron initially at rest were studied directly in the L frame, while the perpendicular case had to be done in the R frame first.

In this paper, we present a general derivation for the nth harmonic power cross section, generated by scattering plane-wave, circularly polarized, superintense, laser light from a relativistic electron moving initially at an *arbitrary* velocity, directly in the L frame and without using the notion of an R frame first. We show that our general result reproduces the older ones in the appropriate limits [1,2].

II. PRELIMINARIES

The plane-wave, circularly polarized radiation field, frequency ω_0, and propagation vector $\mathbf{k}=(\omega_0/c)\hat{\mathbf{k}}$, will be modeled by the vector potential

$$\mathbf{A}(\eta)=\frac{a}{\sqrt{2}}(\hat{\mathbf{i}}\cos\eta+\hat{\mathbf{j}}\sin\eta), \tag{1}$$

where a is a constant amplitude, $\eta=\omega_0 t-\mathbf{k}\cdot\mathbf{r}$ is the phase, t is the time, $\mathbf{r}$ is the position vector of the electron, and c is the speed of light. The initial velocity vector, scaled by the speed of light, will be given by

$$\boldsymbol{\beta}_0=\beta_0(\hat{\mathbf{i}}\sin\theta_0+\hat{\mathbf{k}}\cos\theta_0), \tag{2}$$

where θ_0 is the angle $\boldsymbol{\beta}_0$ makes with $\mathbf{k}$. We further let the unit vector $\hat{\mathbf{n}}=(n_1,n_2,n_3)=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ point in the direction of observation of the scattered radiation, in a spherical polar coordinate system with origin at the point of intersection of the laser and electron beams. In the far field approximation, the energy scattered per unit solid angle $d\Omega$ and per unit frequency $d\omega$ is given by [3]

$$\begin{aligned}\frac{d^2E}{d\Omega\, d\omega}&=\frac{(e\omega)^2}{4\pi^2c^3}\left|\int_{-\infty}^{\infty}\hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\frac{d\mathbf{r}}{dt}\right)\exp\left\{i\omega\left[t-\frac{\hat{\mathbf{n}}\cdot\mathbf{r}(t)}{c}\right]\right\}dt\right|^2\\&=\frac{(e\omega)^2}{4\pi^2c^3}\{(1-n_1^2)|K_x|^2+(1-n_2^2)|K_y|^2\\&\quad+(1-n_3^2)|K_z|^2-2[n_1n_2\mathrm{Re}(K_xK_y^*)\\&\quad+n_1n_3\mathrm{Re}(K_xK_z^*)+n_2n_3\mathrm{Re}(K_yK_z^*)]\},\end{aligned} \tag{3}$$

where Re stands for the real part of its argument, and

$$\mathbf{K}=\int_{-\infty}^{\infty}\frac{d\mathbf{r}}{d\eta}\exp\left\{i\frac{\omega}{\omega_0}\left[\eta+\frac{\omega_0}{c}[z-\hat{\mathbf{n}}\cdot\mathbf{r}(\eta)]\right]\right\}d\eta. \tag{4}$$

For the electron trajectory, we use the expression we have recently derived [2],

$$\mathbf{r}(\eta)=\mathbf{r}_0+\frac{c}{\omega_0}\int_{\eta_0}^{\eta}\left[\frac{\gamma_0 mc\boldsymbol{\beta}_0+(e/c)\mathbf{A}(\eta')}{\gamma_0 mc(1-\hat{\mathbf{k}}\cdot\boldsymbol{\beta}_0)}\right]d\eta'+\hat{\mathbf{k}}\left(\frac{c}{\omega_0}\right)\int_{\eta_0}^{\eta}\left[\frac{\frac{1}{2}(e\mathbf{A}(\eta')/\gamma_0 mc^2)^2+(e\mathbf{A}(\eta')/\gamma_0 mc^2)\cdot\boldsymbol{\beta}_0}{(1-\hat{\mathbf{k}}\cdot\boldsymbol{\beta}_0)^2}\right]d\eta'. \quad (5)$$

Employing η as a parameter, we get from Eq. (5) the following parametric equations for the trajectory:

$$x(\eta)=\frac{c}{\omega_0}(a_1\eta+b_1\sin\eta), \quad (6)$$

$$y(\eta)=-\frac{c}{\omega_0}b_2\cos\eta, \quad (7)$$

$$z(\eta)=\frac{c}{\omega_0}(a_3\eta+b_3\sin\eta), \quad (8)$$

where

$$a_1=\frac{\beta_0\sin\theta_0}{1-\beta_0\cos\theta_0}, \quad (9)$$

$$b_1=b_2=\frac{(q/\gamma_0\sqrt{2})}{1-\beta_0\cos\theta_0}, \quad (10)$$

$$a_3=\frac{\beta_0\cos\theta_0}{1-\beta_0\cos\theta_0}+\frac{(q/2\gamma_0)^2}{(1-\beta_0\cos\theta_0)^2}, \quad (11)$$

$$b_3=\frac{(q/\gamma_0\sqrt{2})\beta_0\sin\theta_0}{(1-\beta_0\cos\theta_0)^2}, \quad (12)$$

and where $q=ea/mc^2$, m is the electron mass, and $\gamma_0=(1-\beta_0^2)^{-1/2}$. Note that, in writing down Eqs. (6)–(8) from Eq. (5), η_0 and $\mathbf{r}_0$ have been dropped, the reason being that they enter into Eq. (3) for the scattered power only through an unimportant phase factor.

III. HARMONIC GENERATION

We now present a systematic derivation of a general expression for the scattered power per unit laboratory solid angle. Using Eqs. (6)–(8) in Eq. (4), we get

$$K_x=\frac{2\pi c}{V}\sum_{n=-\infty}^{\infty}i^n\sum_{\ell=-\infty}^{\infty}(-i)^{\ell}J_{\ell+n}(X)\left[a_1J_{\ell}(Y)+\frac{b_1}{2}[J_{\ell+1}(Y)+J_{\ell-1}(Y)]\right]\delta\left(\omega-n\frac{\omega_0}{V}\right), \quad (13)$$

$$K_y=\frac{2\pi c}{V}\sum_{n=-\infty}^{\infty}i^n\sum_{\ell=-\infty}^{\infty}(-i)^{\ell+1}J_{\ell+n}(X)\left[\frac{b_2}{2}\right]\times[J_{\ell+1}(Y)-J_{\ell-1}(Y)]\delta\left(\omega-n\frac{\omega_0}{V}\right), \quad (14)$$

$$K_z=\frac{2\pi c}{V}\sum_{n=-\infty}^{\infty}i^n\sum_{\ell=-\infty}^{\infty}(-i)^{\ell}J_{\ell+n}(X)\left[a_3J_{\ell}(Y)+\frac{b_3}{2}[J_{\ell+1}(Y)+J_{\ell-1}(Y)]\right]\delta\left(\omega-n\frac{\omega_0}{V}\right), \quad (15)$$

where

$$V=1-n_1a_1-(n_3-1)a_3, \quad (16)$$

and

$$X=n_2b_2\frac{\omega}{\omega_0}, \quad (17)$$

$$Y=[n_1b_1+(n_3-1)b_3]\frac{\omega}{\omega_0}. \quad (18)$$

Thus, in view of the presence of the δ function above, it follows that the radiation is emitted only at the *nth harmonic* frequency:

$$\omega=\omega^{(n)}=\frac{n\omega_0}{1-n_1a_1-(n_3-1)a_3}. \quad (19)$$

The algebra leading to Eqs. (13)–(15) involves the following. First, the trigonometric functions in $d\mathbf{r}/dt$ are expressed in exponential form. Second, the generating function of the Bessel functions,

$$e^{is\sin\xi}=\sum_{n=-\infty}^{\infty}J_n(s)e^{in\xi}, \quad (20)$$

is then used in part of the integrand. Third, the integrations over η are carried out giving δ functions. Fourth, and finally, the dummy summation indices are changed in such a way as to allow for extraction of a common δ function.

Equations (13)–(15) can be simplified further. To accomplish this, we let

$$u=\sqrt{\frac{X+iY}{X-iY}}=e^{i\zeta}, \quad \zeta=\tan^{-1}\frac{Y}{X}, \quad (21)$$

and employ the Graf addition theorem [4]

$$\sum_{\ell=-\infty}^{\infty}(-i)^{\ell}J_{\ell+n}(X)J_{\ell}(Y)=u^nJ_n(\sqrt{X^2+Y^2}). \quad (22)$$

Equations (13)–(15) now take on the following simplified form, involving only a single infinite sum over *the harmonics*:

$$K_x=\frac{2\pi c}{V}\sum_{n=-\infty}^{\infty}i^n\left\{a_1u^nJ_n(n\Theta)+i\frac{b_1}{2}[u^{n-1}J_{n-1}(n\Theta)-u^{n+1}J_{n+1}(n\Theta)]\right\}\delta\left(\omega-n\frac{\omega_0}{V}\right), \quad (23)$$

$$K_y=\frac{2\pi c}{V}\sum_{n=-\infty}^{\infty}i^n\left[\frac{b_2}{2}\right]\{u^{n-1}J_{n-1}(n\Theta)+u^{n+1}J_{n+1}(n\Theta)\}\delta\left(\omega-n\frac{\omega_0}{V}\right), \quad (24)$$

$$K_z=\frac{2\pi c}{V}\sum_{n=-\infty}^{\infty} i^n\left\{ a_3u^nJ_n(n\Theta)+i\frac{b_3}{2}[u^{n-1}J_{n-1}(n\Theta) - u^{n+1}J_{n+1}(n\Theta)]\right\}\delta\left(\omega-n\frac{\omega_0}{V}\right). \tag{25}$$

In the above equations, we have used the relation

$$\Theta=\frac{\sqrt{X^2+Y^2}}{n}=\frac{\sqrt{(n_2b_2)^2+[n_1b_1+(n_3-1)b_3]^2}}{1-n_1a_1-(n_3-1)a_3}, \tag{26}$$

which holds under the restriction of the δ functions, with ω given by Eq. (19). Next, we transform the energy expression, Eq. (3), into one involving the scattered power, which is in turn defined by

$$P=\lim_{T\to\infty}\frac{E}{T}, \tag{27}$$

where T is a measure of time. To accomplish this, an integral representation for one of the δ functions resulting from substitution of Eqs. (23)-(25) in Eq. (3) is used, whereby

$$\delta(\omega-\omega')=\lim_{T\to\infty}\int_{-T/2}^{T/2} e^{i(\omega-\omega')t}\frac{dt}{2\pi} = \frac{T}{2\pi}\ \text{ only for }\ \omega=\omega'. \tag{28}$$

Making use of the remaining δ function, we then integrate the expression obtained from Eq. (3), after the operations implied by Eqs. (27) and (28) have been carried out, over all frequencies in order to get the power scattered per unit laboratory solid angle. The result of doing so, and after some simplification and hindsight, is

$$\frac{dP}{d\Omega}=\frac{(e\omega_0)^2}{2\pi c}\sum_{n=1}^{\infty}\frac{n^2}{V^4}\{(1-n_1^2)|F_1^n|^2+(1-n_2^2)|F_2^n|^2 +(1-n_3^2)|F_3^n|^2-2[n_1n_2\mathrm{Re}(F_1^nF_2^{n*}) +n_1n_3\mathrm{Re}(F_1^nF_3^{n*})+n_2n_3\mathrm{Re}(F_2^nF_3^{n*})]\}, \tag{29}$$

where

$$F_1^n=\left(a_1+\frac{b_1}{\Theta}\sin\zeta\right)J_n(n\Theta)+i(b_1\cos\zeta)J'_n(n\Theta), \tag{30}$$

$$F_2^n=\left(\frac{b_2}{\Theta}\cos\zeta\right)J_n(n\Theta)-i(b_2\sin\zeta)J'_n(n\Theta), \tag{31}$$

$$F_3^n=\left(a_3+\frac{b_3}{\Theta}\sin\zeta\right)J_n(n\Theta)+i(b_3\cos\zeta)J'_n(n\Theta), \tag{32}$$

where J' is the derivative of the Bessel function J with respect to its argument. In arriving at Eqs. (30)–(32), the well-known recurrence relations of the Bessel functions [5] have been used. Equation (29), together with Eqs. (30)–(32), gives the total average power scattered per unit solid angle. The term *total* here means *summed over all the harmonics from* $n=1$ *to* $n=\infty$. The terms in the sum corresponding to negative (and zero) values of the index n have been dropped, since frequencies can only be positive. Contribution to the total power of the harmonic of order n may be read off of Eq. (29) simply by dropping the summation sign. Generally, this contribution may now be cast in the following form:

$$\frac{dP^{(n)}}{d\Omega}=\frac{(e\omega_0)^2}{2\pi c}\frac{n^2}{[1-a_1\sin\theta\cos\phi+2a_3\sin^2(\theta/2)]^4}\left\{J_n^2(n\Theta)\left[(1-\sin^2\theta\cos^2\phi)\left(a_1+\frac{b_1}{\Theta}\sin\zeta\right)^2+(1-\sin^2\theta\sin^2\phi)\right.\right.$$
$$\times\left(\frac{b_2}{\Theta}\cos\zeta\right)^2+\sin^2\theta\left(a_3+\frac{b_3}{\Theta}\sin\zeta\right)^2-2\ \sin^2\theta\sin\phi\cos\phi\left(a_1+\frac{b_1}{\Theta}\sin\zeta\right)\left(\frac{b_2}{\Theta}\cos\zeta\right)-2\sin\theta\cos\theta\cos\phi\left(a_1+\frac{b_1}{\Theta}\sin\zeta\right)$$
$$\left.\times\left(a_3+\frac{b_3}{\Theta}\sin\zeta\right)-2\sin\theta\cos\theta\sin\phi\left(\frac{b_2}{\Theta}\cos\zeta\right)\left(a_3+\frac{b_3}{\Theta}\sin\zeta\right)\right]+J'^2_n(n\Theta)[(1-\sin^2\theta\cos^2\phi)(b_1\cos\zeta)^2+(1$$
$$-\sin^2\theta\sin^2\phi)(b_2\sin\zeta)^2+\sin^2\theta(b_3\cos\zeta)^2+2b_1b_2\sin^2\theta\sin\phi\cos\phi\sin\zeta\cos\zeta-2b_1b_3\sin\theta\cos\theta\cos\phi\cos^2\zeta$$
$$\left.+2b_2b_3\sin\theta\cos\theta\sin\phi\ \cos\zeta\sin\zeta]\right\}. \tag{33}$$

Equation (33) is the centerpiece of the present paper. It applies regardless of what initial conditions are imposed on the magnitude and direction of the electron velocity. Two special cases will be taken up shortly, and Eq. (33) will be used to obtain simplified expressions for the average power scattered into the nth harmonic per unit solid angle in each case.

Note at this point that an expression for the total scattered power per unit solid angle may be obtained from Eq. (33) with the help of [1]

$$\sum_{n=1}^{\infty} n^2J_n^2(n\Theta)=\frac{\Theta^2(4+\Theta^2)}{16(1-\Theta^2)^{7/2}}, \tag{34}$$

$$\sum_{n=1}^{\infty} n^2J'^2_n(n\Theta)=\frac{(4+3\Theta^2)}{16(1-\Theta^2)^{5/2}}. \tag{35}$$

Moreover, dividing the scattered power by the incident intensity $I_0=(e\omega_0q)^2/8\pi c r_0^2$, where r_0 is the classical electron radius, gives an expression for the differential scattering cross section,

$$\frac{1}{r_0^2}\frac{d\sigma}{d\Omega}=\frac{8\pi c}{(e\omega_0 q)^2}\frac{dP}{d\Omega}. \tag{36}$$

IV. SPECIAL CASES

The special cases we consider now, namely, one in which the electron is initially at rest and the parallel geometry, i.e., one in which the initial velocity vector is along the laser field direction of propagation, are related via a simple Lorentz boost along the z axis, direction of the laser propagation vector. For these situations, Eqs. (9)–(12) yield

$$a_1=b_3=0,\quad b_1=b_2\equiv b,\quad a_3\equiv a,\zeta=\frac{\pi}{2}-\phi,$$

$$V=1+2a\sin^2(\theta/2),\quad \Theta=\frac{b\sin\theta}{1+2a\sin^2(\theta/2)}. \tag{37}$$

In other words, we distinguish one case from the other by the values taken by a and b. Substituting the parameter values given by Eq. (37) in Eq. (33) gives

$$\frac{dP^{(n)}}{d\Omega}=\frac{(e\omega_0 b)^2}{4\pi c}\frac{2n^2}{[1+2a\sin^2(\theta/2)]^4}\times\left\{\frac{[\cos\theta-2a\sin^2(\theta/2)]^2}{b^2\sin^2\theta}J_n^2(n\Theta)+J'^2_n(n\Theta)\right\}. \tag{38}$$

The case of an electron initially at rest follows from Eq. (38) by setting $a=q^2/4$ and $b=q/\sqrt{2}$ and simplifying. The result is

$$\frac{dP^{(n)}}{d\Omega}=\frac{(e\omega_0 q)^2}{8\pi c}\frac{2n^2}{[1+\frac{1}{2}q^2\sin^2(\theta/2)]^4}\times\left\{\frac{2[\cos\theta-\frac{1}{2}q^2\sin^2(\theta/2)]^2}{q^2\sin^2\theta}J_n^2(n\Theta)+J'^2_n(n\Theta)\right\}. \tag{39}$$

This case has been considered by Sarachik and Schappert [1] in the moving frame first and the result was then Lorentz transformed to the laboratory frame. It has also been worked out by us [2] directly in the laboratory frame with identical results.

For the parallel geometry, we have

$$a=\frac{\beta_0}{1-\beta_0}+\frac{q^2}{4}\left[\frac{1+\beta_0}{1-\beta_0}\right],\quad b=\frac{q}{\sqrt{2}}\sqrt{\frac{1+\beta_0}{1-\beta_0}}. \tag{40}$$

In this case, inserting the parameter values given by Eq. (40) in Eq. (33) leads to

$$\frac{dP^{(n)}}{d\Omega}=\frac{(e\omega_0 q)^2}{8\pi c}\left[\frac{1+\beta_0}{1-\beta_0}\right]\frac{2n^2}{[1+2a\sin^2(\theta/2)]^4}\times\left\{\frac{2[\cos\theta-2a\sin^2(\theta/2)]^2}{q^2\sin^2\theta}\left[\frac{1-\beta_0}{1+\beta_0}\right]\times J_n^2(n\Theta)+J'^2_n(n\Theta)\right\}, \tag{41}$$

This case has also been considered before [2] and Eq. (41) agrees exactly with the earlier result. Note here as well that Eq. (41) reduces to Eq. (39) in the limit of $\beta_0\rightarrow 0$, as expected.

V. CONCLUSION

We have obtained, classically and relativistically, an expression for the power generated into the harmonic of order n due to the scattering of circularly polarized light of arbitrary intensity from an electron initially moving with an arbitrary velocity. The derivation has been carried out in the laboratory reference frame and the main result has been shown to reproduce the earlier results corresponding to an electron initally at rest at the origin and the one in which the electron initally moves parallel to the radiation field direction of propagation. In both cases a simplified expression is obtained from Eq. (33). Doing the same thing for *the perpendicular geometry*, the case in which the electron is initially moving perpendicular to the laser direction of propagation, does not result in a great simplification. For this and other desired initial geometries, the general analytic expression, Eq. (33), ought to be used.

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