Theory of a semiconductor laser with phase-conjugate optical feedback

Erik Bochove
Phillips Laboratory, LIDA, KAFB, Albuquerque, New Mexico 87117
and Center for High Technology Materials, UNM, Albuquerque, New Mexico 87131
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The effects of phase-conjugate feedback on semiconductor laser mode structure and dynamics are studied, beginning with a derivation of the system's coherent optical rate equation. For fast-responding phase-conjugating mirror (PCM) a multimode external cavity longitudinal mode spectrum arises that is in many ways similar to that of conventional external feedback lasers, except that the spacing of the external resonator modes equals half that of a conventional external cavity of equal length. A linearized stability analysis based on rate equations for longitudinal modes shows that the external cavity modes alternate in stability as function of increasing frequency shift from the pump. These modes also appear as spikes in the random intensity noise and phase noise spectra. It is shown that the locked phase of the field must be controlled to achieve stable operation, and may be used to maximize the fraction of energy in the central ''spike'' of the laser spectrum. Finally, the effects of pump noise on laser linewidth, and the finite PCM response time, are described. [S1050-2947(97)03905-X]

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I. INTRODUCTION

Conventional optical feedback in semiconductor lasers [1] has been studied extensively over the past 15 years, partly because feedback may catastrophically affect the stability and noise properties of the field, and also because under controlled conditions it may be used to enhance performance, for example, to achieve significant narrowing of the line shape [2]. More recently, phase conjugate feedback (PCF) has received experimental and theoretical attention [3–6]. In this, the radiation reflected by the external device, a phase-conjugating mirror (PCM), is wave-front inverted [7]. Many of the issues that are important in conventional external feedback are also of interest here.

The physics of PCF is for various reasons inherently more complex than that of conventional feedback employing a passive mirror, since in the former, the wave-front inversion must be carried out in an active medium. The phase-reversal property of the reflected radiation causes a frequency shift when the frequency of the incident radiation differs slightly from that of the pump laser (nondegenerate four-wave mixing [7]), reverting back to the original frequency after each double pass around the cavity. The PCM possesses also a finite response time or bandwidth, and potential nonlinearity, which both tend to limit its performance. For example, a four-wave mixing PCM is subject to trade-off between reflectivity and bandwidth, and the response time of commonly used conjugating media is much larger than those describing semiconductor laser dynamics. Still, Kerr-type and photorefractive PCM's with faster than nanosecond response times have been used in the past, and those should not be incompatible with bandwidths in the tens's of gigahertz range.

Any mathematical model of the phase conjugator must entail the assumption that it is either externally pumped or self-pumped. In this paper, the former is assumed. With this restriction, a large class of phase-conjugating devices would fit into the rate equation model assumed here, although, as in previous theoretical treatments [4–6], an idealized PCM that responds infinitely fast and runs absolutely ''quiet'' will be assumed for most of the paper. The effects of pump noise and response lag are discussed later.

We begin with a derivation of an optical rate equation that retains the full modal structure of the compound etalon. This method applies to multiple external resonator models and to strongly coupled, high-reflectivity PCM's. For a single round trip, the resulting rate equation reduces to the phase-conjugate equivalent of the single-mode rate equation of conventional feedback [8]. However, the generality of the approach rests on the property that this method applied to *any* kind of laser cavity yields solutions that are rigorously equivalent to those of the complete set of Maxwell equations. In past rate equation treatments, cavity loss effects on the modal structure were approximated, resulting in wrong predictions of the linewidth. An interesting consequence of the present approach is that empirical Petermann-K corrections to the theoretical laser linewidth should arise in a logical manner from the theory. The larger issue of the appropriateness of a local rate equation is not believed to be different than in semiconductor laser modeling, generally. It is certainly required that the adiabatic assumption of the medium is applicable, and that the transverse structure of the field is inherently stable, at least throughout the regions of stability of the longitudinal modes. These conditions are most easily satisfied in smaller devices [9].

The remainder of the paper treats the problem of noise and fluctuations. The treatment of fluctuations is along the lines of the Langevin noise analysis of Agrawal and Gray [6], but differs significantly in three aspects. First, we expand to include the derivation and analysis of higher-order phase-conjugate external cavity modal rate equations. Second, their assumption of phase locking to an apparently imposed value of the phase [i.e., Eq. (15) in [6]] is here shown to be in conflict with other steady-state requirements of the rate equation, and therefore it is not adopted in this paper. And finally, their approximation to expand time-delayed quantities to first order in the external cavity round-trip time is not made, as it has important consequences for the predicted stability of the system. For example, the stated approximation leads to the prediction that the system is *always* dynamically unstable when the product of the injection frequency of feedback and the external cavity round-trip time (denoted by η) equals or exceeds unity. Instead, we find that stability may persist for $\eta > 1$, under certain conditions, provided that the response time of the PCM is small compared to other time constants.

In addition, the results not covered in previous work can be summarized as follows. We find that at moderately high feedback levels (for which $\eta \approx 1$) almost all (>99%) of the laser spectrum is included in the sharp central ''spike'' also discussed in Ref. [6]. An expression is derived for the width of this spike when the spectral linewidth of the laser pumping the PCM is taken into account. Assuming a *thin* four-wave mixing medium for the PCM [7], it is found that the minimum linewidth of the spike is identical with the pump linewidth. Another refinement of the model, the inclusion of finite PCM response time, shows it to diminish the stability at higher feedback levels. Lastly, rate equations and steady-state conditions are derived, and a stability analysis performed, of modes whose frequencies differ by discrete values from the PCM pump frequency. These modes are not described by the rate equations of Ref. [6], but are consistent with previous (Fox-Li-type) treatments, and our predictions in agreement with already well-known properties [10,11].

II. COHERENT OPTICAL RATE EQUATION

Although derivations of rate equations exist in the literature [8,12–14], the generality of the present derivation implies that a universal connection can be established between the rate equation for *any* laser cavity and the logarithm of a basic operator, the Fox-Li round-trip propagator [15]. Thus the rate equation is determined once the round-trip propagator is calculated, and its solutions expressible in terms of its eigenfunctions and eigenvalues. The described method may be used to derive rate equations for other types of laser cavities (e.g., external feedback and injection locking geometries, unstable laser resonators, and broad area semiconductor lasers) [16]. The ''spirit'' of the approach we used is similar to that of Lang and Yariv [13], who obtained an approximate rate equation using not unsimilar arguments from ours.

The laser cavity is taken to be uniformly filled with a gain medium, and it is assumed that other conditions are likewise appropriate (e.g., plane mirrors or end facets) for all fields to be approximated by to the right- and left-propagating plane waves. Using ω_0 to denote the pump frequency of the PCM pump laser, the slowly varying factor of the complex field inside the laser cavity (defined by $0 < z < L_c$), at frequency components $\omega_0 \pm \nu$, is taken as

$$E(z,t) = \tilde{\mathcal{E}}(\nu)[\sqrt{R_L}e^{ik(\nu)z} + e^{-ik(\nu)z}]e^{-i\nu t} + \tilde{\mathcal{E}}(-\nu)[\sqrt{R_L}e^{ik(-\nu)z} + e^{-ik(-\nu)z}]e^{i\nu t}, \quad (1)$$

where the notation $\mathcal{E}(\pm\nu), k(\pm\nu)$ refers to the field components at $\omega_0 \pm \nu$, and R_L is the reflectivity of the left mirror or facet. The α or line-broadening parameter [17], $\alpha = -\Delta n_r / \Delta n_i$, describes the connection between the real and imaginary parts of the active medium's nonlinear index. Finally, the wave vector is given by $k(\nu) = n(\nu)[(\omega_0 + \nu)/c + (-i+\alpha)G(\nu)/(2c)]$, in terms of $n(\nu)$ and $G(\nu)$, the refractive index and gain, respectively.

Application of the boundary conditions at the right end of the laser cavity yields

$$\tilde{\mathcal{E}}(\nu)e^{-ik(\nu)L_c} = r(\nu)\sqrt{R_L}e^{ik(\nu)L_c}\tilde{\mathcal{E}}(\nu) + p(-\nu)\sqrt{R_L}e^{-ik^*(-\nu)L_c}\tilde{\mathcal{E}}^*(-\nu), \quad (2)$$

where $r(\nu)$ and $p(\nu)$ denote conventional and phase-conjugate effective reflectivity functions, respectively. Equation (2) can be written in matrix form

$$\tilde{x}(\nu)\exp(-i\nu\tau_c) = \tilde{L}(\nu)\tilde{x}(\nu), \quad (3)$$

where $\tilde{x}(\nu)$ is the column vector of components $\{\mathcal{E}(\nu), \tilde{\mathcal{E}}(-\nu)^*\}$ and the components of the round-trip transfer matrix $\tilde{L}(\nu)$ are given by

$$\tilde{L}_{11}(\nu) = \tilde{L}_{22}(-\nu)^* = \sqrt{R_L}e^{i[\delta - (1/2)(\alpha+i)G(\nu)]\tau_c}r(\nu),$$

$$\tilde{L}_{12}(\nu) = \tilde{L}_{21}(-\nu)^* = \sqrt{R_L}e^{[-(1/4)i\alpha\Delta G(\nu) + (1/2)\overline{G(\nu)}]\tau_c}p(-\nu), \quad (4)$$

in which $\omega_c = 2\pi q/\tau_c$ (q = integer) is the cold-cavity mode frequency, $\delta = \omega_0 - \omega_c$ is the detuning of the pump from the cavity frequency, τ_c is the cold-cavity round-trip delay, and the remaining functions of the gain are defined as $\overline{G(\nu)} = \frac{1}{2}[G(\nu) + G(-\nu)]$, $\Delta G(\nu) = G(\nu) - G(-\nu)$. The wave vector $k(\nu)$ was expanded in ν, in which terms of order ν^2 and in the difference between group and phase delays were omitted.

Equation (3) shows that the matrix $\tilde{L}(\nu)$ determines the evolution of the field over discrete intervals of time τ_c. Our objective is to turn this into an equivalent first-order differential equation in time. There are other ways of approaching this; here we follow a kind of analytical continuation argument.

During a time Δt, containing $N = \Delta t/\tau_c$ intervals $-\tau_c$, Eq. (3) predicts that the field evolves to

$$\left.\begin{matrix}\mathcal{E}(t+\Delta t)\\ \mathcal{E}^*(t+\Delta t)\end{matrix}\right\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{L}(\nu)^{\Delta t/\tau_g} \times \left\{\begin{matrix}\tilde{\mathcal{E}}(\nu)\\ \tilde{\mathcal{E}}^*(-\nu)\end{matrix}\right\} \times e^{-i\nu t}d\nu. \quad (5)$$

It is now assumed that this expression is valid also for fractional multiples of the cavity round-trip time. The justification of this procedure is given below. Expanding both sides to first order in Δt, yields then the coherent rate equation in general form:

$$\frac{dx(t)}{dt} = \frac{1}{2\pi\tau_c}\int_{-\infty}^{+\infty}\ln L(\nu)\tilde{x}(\nu)e^{-i\nu t}d\nu. \quad (6)$$

This expression is the exact rate equation satisfied by the field, and it is fully equivalent to the integral equation form, Eq. (3). Note that it represents the rate equation for *any* laser cavity satisfying an integral equation in which $\tilde{L}(\nu)$ plays the role of round-trip propagator, because the explicit form of $\tilde{L}(\nu)$ need not be specified for Eq. (6) to be valid. The extension to inclusion of transverse spatial dimensions is straightforward, since considering $\tilde{L}(\nu)$ an operator in the transverse plane does not affect the formal form of Eq. (6).

Equation (6) applies also to the case of strong phase-conjugate interaction, say, for which $p(\nu)\geqslant 1$, although then a more practical rate equation would be based on double round trips. For experimental reasons the motivation for studying that case appears to be minimal, however, since highly nonlinear media with response times faster than that of the laser are currently not available.

Fourier transforming Eqs. (3) and (6), we note that both equations yield the following characteristic equation for the modal frequencies of the resonator in the presence of the phase-conjugating mirror:

$$\det[e^{-i\nu\tau_c}\hat{1}-\tilde{L}(\nu)]=0. \tag{7}$$

Since Eqs. (3) and (6) have identical solutions, they must be regarded as equivalent descriptions of the cavity field, and the solutions of Eq. (6) are rigorous solutions of Maxwell's equations in the same sense as the solutions of Eq. (3). This result is the justification for the above ''analytical continuation'' procedure.

Using Eq. (4), Eq. (7) yields the characteristic equation of the cavity under study:

$$\Delta(\nu)=1-\sqrt{R_L}r^*(-\nu)e^{-2iL_ck^*(-\nu)}-\sqrt{R_L}r(\nu)e^{2iL_ck(\nu)} \\ -e^{2iL_c[k(\nu)-k^*(-\nu)]}\sqrt{R_L}[p(-\nu)p^*(\nu) \\ -r(\nu)r^*(-\nu)]=0. \tag{8}$$

By taking the complex conjugate of this equation an equivalent expression is obtained, and since either transforms into the other by the substitution $\nu\rightarrow-\nu^*$, it follows that the cavity modes come in pairs arranged symmetrically about the pump frequency, having identical loss constant.

Limiting now the number of external resonator round trips to one, then $r(\nu)\approx\sqrt{R_R}$, and $p(-\nu)\approx(1-R_R)\rho_{\rm PC}(-\nu)e^{2i\nu L/c}$, where R_R is the right facet reflectivity, L is the external cavity length, and $\rho_{\rm PC}(\nu)$ the complex PCM reflectivity. The logarithm of the matrix in Eq. (6) may be calculated using Cauchy's theorem. Expanding to first order in $\rho_{\rm PC}$, and suppressing the dispersion of G and $\rho_{\rm PC}$, yields the following simplification of Eq. (6):

$$\dot{\mathcal{E}}(t)=i\Delta\omega\mathcal{E}(t)+\tfrac{1}{2}(1-i\alpha)[G(N)-1/\tau_p]\mathcal{E}(t) \\ +\kappa e^{i\phi_{\rm PC}}\mathcal{E}^*(t-\tau), \tag{9}$$

where κ is the PCF coupling parameter, $\phi_{\rm PC}=\arg(\rho_{\rm PC})$, τ_p is the cavity lifetime of the solitary laser, $\tau=2L/c$ is the external cavity round-trip time, $\Delta\omega$ is the detuning of the pump from the solitary laser frequency, and their defining relations are

$$\Delta\omega=\delta-\tfrac{1}{2}\alpha\tau_p^{-1}, \quad \tau_p=-\tau_c/\ln(R_LR_R),$$

$$\kappa=(1-R_R)\xi|\rho_{\rm PC}|/(\sqrt{R_R}\tau_c\sin\xi), \quad \xi=(2\delta-\alpha G)\tau_c. \tag{10}$$

In steady-state operation, ξ is of the order of magnitude of $\kappa\tau_c\ll 1$, so it is justified to let $h\equiv\xi/\sin\xi\rightarrow 1$ in the definition of κ. The fluctuations of h are likewise negligible, since $h\approx 1+\xi^2/6\approx 1+G_n^2\tau_c^2\Delta N^2/6$, where G_n is the derivative of the gain with respect to carrier number N and ΔN the fluctuation in carriers. Using $\Delta N\approx N$, for strong oscillations, yields $\Delta h\equiv\overline{h}-1\approx\tau_c^2/6\tau_p^2\approx 0.08$, for typical values, τ_p being the cavity lifetime, while for fluctuations in normal operation it would be orders of magnitude smaller still.

Equation (9) has the form of the rate equation used in the previous work on PCF [4–6,13], and differs from that of conventional feedback [14] by the presence of the complex conjugate of the field on the right-hand side of the equation, and the conjugator phase, $\phi_{\rm PC}$. In the limit $\xi\rightarrow 0$ this equation is identical to van Tartwijk's [8], whose derivation is based on the standard approximation [14] $E(t+\tau_c)\approx E(t)+\tau_c\dot{E}(t)$. In the present approach, a weak PCM reflectivity approximation was applied to the exact expressions, Eq. (6). That these results still differ by the factor h, even in the limit $\tau_c\rightarrow 0$, is due to the fact that κ approaches infinity in the same limit, so that the assumption of slowly varying field required for van Tartwijk's expansion is not guaranteed.

We would like to express the range of validity of Eq. (8) in terms of an inequality, and it would seem that $\kappa\tau_c\ll 1$ represents a natural condition. This condition may not be sufficiently stringent, since, by analogy with conventional feedback, the injection rate of PCF may be expected to be necessarily smaller than the cavity loss rate of the solitary laser. This leads to the stronger requirement

$$2\kappa\tau_p<1, \tag{11}$$

as will be justified below [viz., Eq. (16)].

It will be shown that the above approximate rate equation yields external cavity sidebands to the solitary laser-mode frequencies, as in the conventional external cavity laser [14], but with the expected difference that the spacing of these additional modes is approximately half that in the conventional resonator having equal external cavity length. However, the well-known transverse modal degeneracy and aberration-correction properties [10] of phase-conjugate cavities having near-unity reflectivity of the PCM, do not follow from Eq. (8). To this end, it is possible to derive from Eq. (6) a simpler rate equation covering this case.

III. DEGENERATE MODE PROPERTIES

A. Rate equations and steady-state conditions

The state of operation under which the laser frequency is locked to the PCM pump frequency (ω_0) will be referred to as the fundamental, or degenerate mode. The rate equations for the frequency-shifted modes will be derived and discussed in Sec. IV of this paper.

Writing the normalized field in the form that is appropriate for description of the fundamental mode, i.e., $\mathcal{E}(t)=\sqrt{P(t)}e^{i\phi(t)}$, in which P is the number of photons in the proper laser cavity and ϕ the phase, then substitution into Eq. (8) yields rate equations for $P(t)$ and $\phi(t)$ [6]:

$$\dot{P}(t)=[G(N)-1/\tau_p]P(t)+2\kappa\sqrt{P(t)P(t-\tau)}\times\cos[\phi_{PC}-\phi(t)-\phi(t-\tau)]+R_{sp}+F_P(t), \quad (12a)$$

$$\dot{\phi}(t)=\Delta\omega-\tfrac{1}{2}\alpha[G(N)-1/\tau_p]+\kappa\sqrt{P(t-\tau)/P(t)}\times\sin[\phi_{PC}-\phi(t)-\phi(t-\tau)]+F_\phi(t). \quad (12b)$$

A spontaneous emission rate, $R_{sp}=fG(N)$, where f is the product of the internal quantum efficiency and the ''excess spontaneous emission'' (''Petermann K'') factor, was added to the right-hand side of the power equation, and the stochastic Langevin forces $F_{p,\phi}(t)$ to describe noise processes. These equations are basically those of Ref. [6], allowing for the omission of the nonlinear gain dependence here [18], which has generally little effect at typical power levels.

These rate equations are similar to those of conventional feedback, except that there the phase difference, $\phi(t)-\phi(t-\tau)$, appears in both expressions, while here it is their *sum*. In steady-state operation the phases cancel in the former case (provided $\Delta\omega$ is then defined as the laser frequency shift measured from the solitary laser frequency), so that the phase is free to assume any value, while an even number of external cavity mode frequencies, clustered about the central mode's frequency, are obtained as solutions. In the PCF case the laser frequency is fixed (at the pump frequency of the PCM for the case of the degenerate mode), and the phase equation fixes the phase of the field at steady state as opposed to the frequency. Since Eq. (11) depends on ϕ only through the difference $2\phi-\phi_{PC}$, it follows that ϕ must lock in relation to the PCM phase whenever steady-state operation is achieved.

The medium dynamics are described by the fluctuations in the electron carrier number $N(t)$, as modeled by

$$\dot{N}(t)=\frac{I}{q}-\frac{N(t)}{\tau_c}-G(N)P(t)+F_N(t), \quad (13)$$

where I is the injection current, q the electron charge. The second term describes carrier relaxation, of rate τ_e^{-1}, the third term describes stimulated emission, and the last term is, again, a Langevin force. The dependence of the gain on N is assumed linear in fluctuations about steady state.

Neglecting the time averages of all higher powers of the small fluctuating quantities, Eqs. (12) yield steady-state equations for the gain and phase:

$$G=-R_{sp}/P+1/\tau_p-2\kappa\cos\psi,$$

$$\Delta\omega=\tfrac{1}{2}\alpha(G-1/\tau_p)-\kappa\sin\psi, \quad (14)$$

where $\psi=\phi_{PC}-2\phi$. Combining the two results in

$$\Delta\omega=-\kappa\sqrt{1+\alpha^2}\sin(\psi+\beta)-\tfrac{1}{2}\alpha R_{sp}/P, \quad (15)$$

where $\beta=\arctan\alpha$. Equation (15) defines a locking range for $\Delta\omega$ [5], analogous to injection locking [19]. Since $\Delta\omega$ is an experimentally controlled parameter, for given $\Delta\omega$ within the locking bandwidth Eq. (15) narrows the available range of ϕ from $(-\pi,\pi)$ for the solitary laser down to just two discrete values, which may be controlled by fixing the values of $\Delta\omega$ and κ. Thus when ψ is treated as a free parameter in this paper, it is implied that the detuning $\Delta\omega$ is allowed to range freely.

A relation giving the steady-state value of P in terms of the solitary laser parameters is readily obtained from Eqs. (12), which is relatively simple if the spontaneous emission term is neglected:

$$\frac{P}{P_0}=\frac{\omega_{r0}^2\tau_e+2\kappa\cos\psi}{\omega_{r0}^2\tau_e(1-2\kappa\tau_p\cos\psi)}\approx 1+\frac{2\kappa\cos\psi}{\omega_{r0}^2\tau_e}, \quad (16)$$

where P_0 is the solitary laser value of P, and $\omega_{r0}=\sqrt{G_nP_0/\tau_p}$ is the solitary laser relaxation frequency, in which $G_n\equiv\partial G/\partial N$. Note that modulation of ϕ_{PC} does not result in modulation of P, since, according to Eq. (15), ψ stays constant, so that the laser phase ϕ is modulated instead. Also, the power does not depend on the length of the external resonator when the laser operates in the degenerate mode, as there is no round-trip phase accrual in the external cavity. The term in parentheses in the denominator of the first equation in Eq. (16) was approximated by unity, in view of the inequality (11).

B. Fluctuation and dynamical stability analysis of the degenerate mode

The fluctuations about steady-state values of P, ϕ, and N can be studied using linearized field and carrier rate equations with added Langevin forces. According to Henry [20], the effect of spontaneous emission on the laser linewidth attributed to the Langevin force in the equation for N is about 0.1% that of the Langevin forces in the other equations, so that neglecting it is justified. The remaining Langevin forces are taken to satisfy [21] $\langle|\tilde{f}_P(\omega)|^2\rangle=4P^2\langle|\tilde{f}_\phi(\omega)|^2\rangle=2R_{sp}P$, $\langle|\tilde{f}_P(\omega)\tilde{f}_\phi^*(\omega)\rangle=0$, where $\tilde{f}_i(\omega)=\mathcal{F}[F_i(t)]$, (where $\mathcal{F}$ denotes the Fourier transform), and the angular brackets denote ensemble averages. The laser's three dynamical variables are written as the sum of their steady-state values and a small fluctuating part, i.e., $P(t)=P+p(t)$, $\phi(t)=\phi+\varphi(t)$, and $N(t)=N+n(t)$, and Eqs. (12) and (13) linearized in the fluctuations. Writing their Fourier transforms $\tilde{p}(\omega)$, etc., and after elimination of $\tilde{n}(\omega)$, the equations for the fluctuating power and phase are given by

$$i\omega\tilde{p}(\omega)=-\left[\Gamma_P+\frac{\omega_r^2}{\gamma_e-i\omega}+(1-e^{i\omega\tau})\kappa\cos\psi\right]\tilde{p}(\omega)-2P(1+e^{i\omega\tau})\kappa\sin\psi\tilde{\varphi}(\omega)+\tilde{f}_P(\omega)-4P\kappa\sin\psi\tilde{\varphi}_p(\omega), \quad (17a)$$

$$-i\omega\tilde{\varphi}(\omega)=\left[\frac{-\alpha\omega_r^2}{\gamma_e-i\omega}+(1-e^{i\omega\tau})\kappa\sin\psi\right]\frac{\tilde{p}(\omega)}{2P}-(1+e^{i\omega\tau})\kappa\cos\psi\tilde{\varphi}(\omega)+\tilde{f}_\phi(\omega)-2\kappa\cos\psi\tilde{\varphi}_p(\omega), \quad (17b)$$

where $\omega_r=\sqrt{GG_nP}$, $\lambda_e=\tau_e^{-1}+G_nP$ is the damping constant, and $\Gamma_P=R_{sp}/P$, all defined at their steady-state values in the presence of feedback and $\varphi_P(\omega)$ represents the fluc-

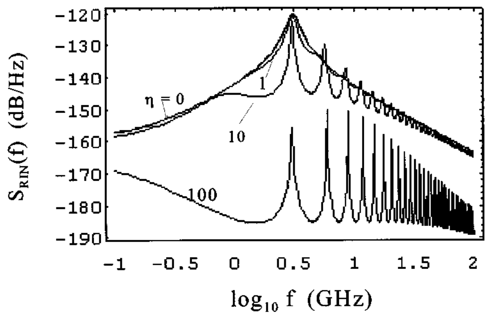

FIG. 1. RIN spectra for various values of $\eta=\kappa\tau$. The phase value $\psi=0$ was assumed.

tuation in the PCM pump phase. The form of the terms containing $\varphi_P(\omega)$ is based on the property that the PCM reflection constant is proportional to the product of the two complex pump-field amplitudes. Equations (17) reduce to those in Ref. [6] under the substitution $\exp(i\omega\tau)\rightarrow 1+i\omega\tau$, and omitting the pump-noise terms. The pump fluctuations will be neglected until Sec. III D, where their effects on phase and amplitude fluctuations are calculated.

The definitions of the random intensity noise (RIN) and frequency noise spectra (FNS), $S_{\rm RIN}(\omega)=\langle|\tilde{p}(\omega)|^2\rangle/P^2$ and $S_{\rm FNS}(\omega)=\langle|\omega\tilde{\varphi}(\omega)|^2\rangle$, respectively, are used to characterize intensity and frequency noise properties. A typical plot of the former is shown in Fig. 1 for the arbitrarily chosen phase $\psi=0$. Note that PCF reduces the level of both types of noise over most of the frequency range. Another feature is the presence of prominent spikes, associated with the excitation of phase conjugate external resonator modes, which were not predicted earlier. For this plot, and others in this paper, we used parameter values from [6] appropriate for a GaAs semiconductor laser operating at 5 mW: $\tau_e=2$ ns, $\tau_p=1.5$ ps, $\tau=9$ ps, $G_n=4500\ \mathrm{s}^{-1}$, $R_{\rm sp}=1.7$ G, and $P=125\,000$. The fixed value $L=5$ cm for the external cavity length was adopted, unless otherwise indicated.

From the system determinant of Eqs. (17) (where $s=-i\omega$ and $\eta=\kappa\tau$), we have

$$[(\Gamma_P+s)(\gamma_e+s)+\omega_r^2]s\tau+\eta[\{\Gamma_P(\gamma_e+s)+\omega_r^2\}(1+e^{-s\tau}) \\ +2s(\gamma_e+s)]\cos\psi-\alpha\omega_r^2\sin\psi(1+e^{-s\tau})\eta \\ +(\gamma_e+s)\tau^{-1}(1-e^{-2s\tau})\eta^2=0. \tag{18}$$

Some limiting cases are readily obtained from Eq. (18), beginning with the solution for the solitary laser [22], $s_0=-\frac{1}{2}(\gamma_{e0}+\Gamma_{p0})+i\sqrt{\omega_{r0}^2+\frac{1}{4}(\gamma_{e0}-\Gamma_{p0})^2}$. In the presence of PCF, unstable solutions are possible. At low feedback levels one of the roots is real, and approaches zero as $\kappa\rightarrow 0$. Using the notation $\Gamma=-s$ for the relaxation rate, for $\eta\ll 1$ this root is given by

$$\Gamma=\frac{2\kappa[\gamma_e\Gamma_P\cos\psi+\omega_r^2\sqrt{1+\alpha^2}\cos(\psi+\beta)]}{\gamma_e\Gamma_P+\omega_r^2} \\ \approx 2\sqrt{1+\alpha^2}\kappa\,\cos(\psi+\beta). \tag{19}$$

The numerical solution of Eq. (18) shows that for $\cos(\psi+\beta)<0$, the root is unstable at *all* levels of feedback, where the point defined by the condition $\Gamma=0$ in (19), or $\psi+\beta\approx 0$, is a ''limit point'' of the system. This instability in the real root has been referred to as a ''fold instability,'' while that associated with positive real values of s and nonvanishing imaginary part is called a ''Hopf instability [4].''

The effect of weak PCF on the damping rate and oscillation frequency of the relaxation oscillation is found by solving Eq. (18) to first order in $\eta=\kappa\tau$ about the unperturbed complex solution s_0, given above. The result is

$$s_0'=s_0-\frac{(1-e^{-s_0\tau})s_0(\gamma_{e0}+s_0)\cos\psi-\alpha\omega_{r0}^2(1+e^{-s_0\tau})\sin\psi}{\tau s_0(\gamma_{e0}+\Gamma_{p0}+2s_0)}\,\eta. \tag{20}$$

Equation (20) shows that the magnitudes of both quadratures of the constant s_0' may be increased or decreased by adjusting the phase, making their control with PCF appear feasible.

For a short external cavity ($\tau\rightarrow 0$), Eq. (18) reduces to a cubic polynomial in s:

$$s^3+(\gamma_e+\Gamma_p+2\kappa\,\cos\psi)s^2 \\ +[\omega_r^2+\gamma_e\Gamma_p+2(\gamma_e+\Gamma_p)\kappa\,\cos\psi]s+2\kappa[\omega_r^2\sqrt{1+\alpha^2} \\ \times\cos(\psi+\beta)+\Gamma_p\gamma_e\cos\psi]=0. \tag{21}$$

Three independent stability conditions are obtained by means of the Hurwitz criterion as necessary and sufficient. Denoting c_n as the coefficient of s^n in Eq. (21), these are $c_n/c_3>0$, and $c_1c_2-c_0>0$, or, for the latter condition,

$$\kappa[\Gamma_p\gamma_e\cos\psi+\omega_r^2\sqrt{1+\alpha^2}\cos(\psi+\beta)] \\ <(\gamma_e+\Gamma_p+2\kappa\,\cos\psi)[\omega_r^2+\gamma_e\Gamma_p \\ +2(\gamma_e+\Gamma_p)\kappa\,\cos\psi]. \tag{22}$$

The stability condition is not trivially satisfied in the limit of vanishing external resonator length, as it is for conventional feedback. For example, if $2\kappa>\gamma_e+\Gamma_p$, then the laser is unstable for $\pi/2<\psi<3\pi/2$.

The short external cavity laser cavity becomes unstable when one of the above conditions is violated, while the others are satisfied. This allows for the possibility of either a limit point, defined by $c_0=0$, at which the fold instability sets in, or at the Hopf bifurcation point, satisfying $c_1c_2-c_0=0$. The former condition is consistent with Eq. (19).

In the long external resonator limit, Eq. (18) reduces to the solution of the uncoupled laser equation for rapidly vary-

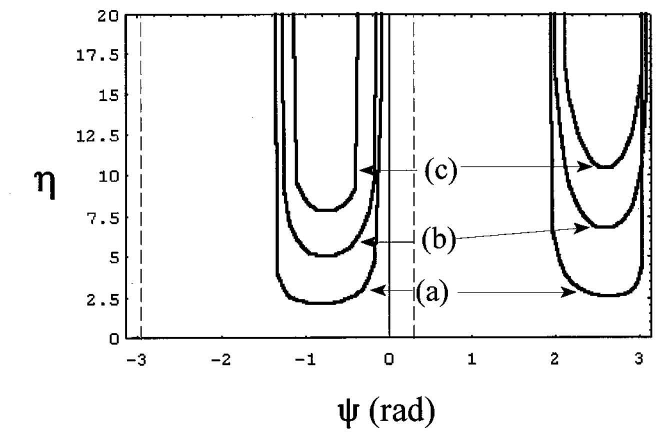

FIG. 2. Regions of stable operation of the mode locked to the PCM pump frequency. Plots are for three different values of external cavity length L, (a) 5 cm, (b) 10 cm, and (c) 15 cm. The region below each curve is stable against Hopf instability; that between the two dashed lines is stable against the fold instability. $\alpha = 3$ is assumed.

ing oscillations, and, in addition, has slowly oscillating solutions related to external cavity resonances, given by

$$s\tau = \pm iq\pi + \ln\left|\frac{\gamma_e\kappa}{\gamma_e(\kappa+\Gamma_p\cos\psi)+\sqrt{1+\alpha^2}\,\omega_r^2\cos(\psi+\beta)}\right|, \tag{23}$$

where q runs over even integers if the function whose absolute value is taken is positive, and it runs over odd integers otherwise. We note that the fold instability is the only instability in this limit.

The results of numerical solution of Eq. (18) are plotted in Fig. 2, which shows the primary stable regions in the κ-ψ plane with respect to the Hopf instability. Only the boundaries of the stable regions of weakest feedback level are indicated, since as κ increases, alternating stable and unstable regions are encountered. The value $\alpha = 3$ was assumed, and the values of the other constants are as before. Curves for three values of the external cavity length are shown. The boundaries of the fold instability are indicated by dashed lines. Calculations carried out with L as variable supported Eq. (23), in that at large L the mode was found to be stable in every instance.

Another feature is that ''unlimited'' feedback strength [subject to Eq. (11)] appears possible for large ranges of phase values. This is not predicted by the linearized model of Ref. [6] from which an absolute upper limit of $\eta = 1$ is obtained for stability if the phase is ranged freely. The larger magnitudes for the coupling constant assumed in Fig. 2 may be experimentally achieved by antireflection coating the laser facet, or by using a PCM with gain.

Such apparent ''unlimited'' dynamical stability could possibly be interpreted as contradicting Refs. [4], [5], where self-exciting oscillations and chaos at higher feedback levels are predicted by direct numerical solution of field and carrier rate equations. In Sec. IV of this paper, however, it is shown that half of the nondegenerate higher-order modes are un-

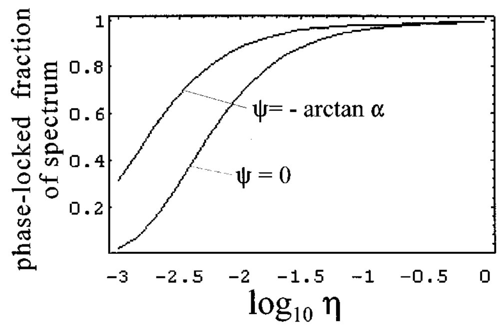

FIG. 3. Relative power in the central spike of the spectrum as function of feedback strength for two values of the phase (all other parameters have the same values as in previous figures).

stable. The relative stability at high PCF level contrasts with the sensitivity of conventional external feedback cavities, which become unstable at the critical value of the coupling, given by $\kappa_c \approx \omega_r^2/2\gamma_e\sqrt{1+\alpha^2}$ [23].

C. Spectral line shape

In the presence of PCF, a portion of the emitted radiation is locked in phase with the pump, and thus does not fluctuate (assuming a perfectly quiet pump). This results in a sharp centrally located spike on the laser spectrum [5,6]. Using the general expression for the spectral line-shape function [6], it may be shown that the fraction f_L of the spike in the spectrum relative to the total integrated intensity is given by the long-time limit of the variance:

$$f_L = \lim_{t\to\infty}\langle|\varphi(t)|^2\rangle = \frac{1}{\pi}\int_{-\infty}^{+\infty}\langle|\varphi(\omega)|^2\rangle\, d\omega. \tag{24}$$

This quantity rapidly approaches unity for $\eta \geqslant 1$, as illustrated in Fig. 3 for two values of the phase. In weak PCF, for which the phase variance relaxes exponentially as described by a single relaxation rate Γ [Eq. (19)], the following simple expression is found:

$$f_L = e^{-\Gamma_0/\Gamma}, \tag{25}$$

where Γ_0 is the relaxation rate of the solitary laser, corresponding to a Lorentzian line-shape function of width $\Delta\nu_0 = \Gamma_0/\pi$. However, its applicability is restricted to small f_L. Figure 3 shows that most of the laser radiation becomes phase locked at moderate feedback level.

D. Effects of pump fluctuations

The finite linewidth of the pump laser adds to the RIN and FNS noise levels of the laser radiation. An example of the results of a small signal analysis based on Eqs. (17) is illustrated in Fig. 4. The pump linewidth was taken equal to that of the free running slave laser, the phase $\psi = -\beta$, and all the other parameters are those of Fig. 2, *et seq*. The laser line shape is affected by pump phase fluctuations primarily through broadening of the central ''spike.'' The width of the latter is determined using the expression $\Delta\nu = (2\pi)^{-1}\lim_{\omega\to 0}\langle|\omega\tilde{\varphi}(\omega)|^2\rangle$, where the spectrum of phase fluctuations is determined using Eqs. (17). This leads then to the relation, $\langle|\tilde{\varphi}(0)|^2\rangle = \langle|\tilde{\varphi}_p(0)|^2\rangle$, between the phase fluc-

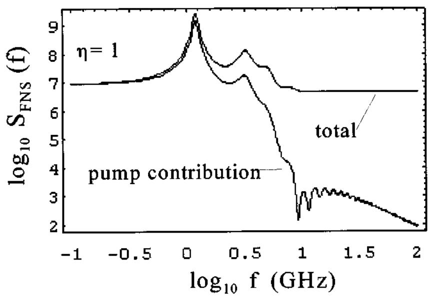

FIG. 4. Contribution to RIN due to finite linewidth of the pump laser for phase $\psi = -\arctan\alpha = -1.25$.

tuations of the laser field and those of the pump laser, so that the linewidth of the spiked part of the laser spectrum just equals the linewidth of the pump laser [5,6].

E. Effects of finite PCM response time

The steady state excepting, the above conclusions are valid only for an ideal conjugating medium having zero response time. Fluctuating fields are conjugated by the mirror only if the latter responds quickly enough to form a grating during a typical fluctuation period. The effect of a finite response time may be modeled by assuming that the response of the PCM is exponential in time, which is taken into account, e.g., by substituting $\kappa = \kappa_0/(1+s\tau_{PC})$ for κ in Eqs. (18). Using Eq. (19), the relaxation rate Γ for weak feedback is thus obtained from the solution of

$$\Gamma \approx \frac{2\sqrt{1+\alpha^2}\kappa_0 \cos(\psi+\beta)}{1-\Gamma\tau_{PC}}. \tag{26}$$

This yields two values for Γ for small κ or short τ_{PC}, but for $8\kappa\tau_{PC}\sqrt{1+\alpha^2}\cos(\psi+\beta) > 1$, a single relaxation rate $\Gamma = 1/(2\tau_{PC})$ accompanied by relaxation oscillations is predicted.

The PCM delay can evidently cause instability in the presence of strong feedback. In Fig. 5 are plotted the trajectories traced by a few of the poles of the system response function [solutions of Eq. (18)] as the PCM response time ranges from 0 to $10^5\tau$. Initially all poles are located in the left half of the complex plane, but some cross the imaginary axis as the response delay increases, showing that the system becomes unstable.

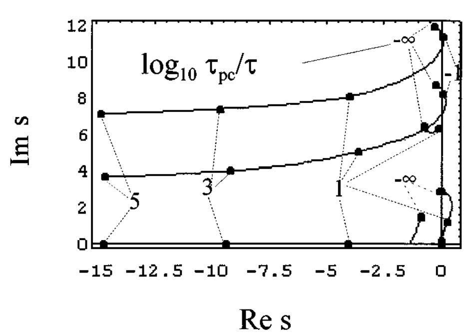

FIG. 5. Solutions of characteristic equation, Eq. (18), as function of PCM response time modeled according to $\kappa \to \kappa(s) = \kappa_0/(1+s\tau_{PC})$. At one extremity of each curve, $\tau_{PC}=0$, at the other $\tau_{PC}=10^5\tau$. The feedback strength is $\kappa_0\tau=10$, phase $\psi=0$.

IV. RATE EQUATION ANALYSIS OF HIGHER-ORDER MODES

Higher-order phase conjugate external resonator modes, consisting of mutually phase- and amplitude-locked (by the PCM medium) pairs of fields at frequencies $\omega=\omega_0 \pm \nu$, are observable provided that $\nu\tau_{PC} \ll 1$. For the purposes of this analysis it will be assumed that only one pair of such coupled modes is present, so that we substitute the following steady-state expression for the field into Eq. (9):

$$\mathcal{E}(t) = \sqrt{P_1(t)}\exp[i(\omega_0+\nu)t + i\phi_1(t)] + \sqrt{P_2(t)}\exp[i(\omega_0-\nu)t + i\phi_2(t)]. \tag{27}$$

Neglecting the small difference in gain between the two frequencies $\pm\nu$, this yields

$$\frac{\dot{P}_1(t)}{P_1(t)} = G - \tau_p^{-1} + 2\kappa\sqrt{P_2(t-\tau)/P_1(t)} \times \cos[\phi_{PC} - \phi_1(t) - \phi_2(t-\tau) + \nu\tau] + R_{sp}/P_1(t) + F_{P1}(t)/P_1(t), \tag{28a}$$

$$\frac{\dot{P}_2(t)}{P_2(t)} = G - \tau_p^{-1} + 2\kappa\sqrt{P_1(t-\tau)/P_2(t)}\cos[\phi_{PC} - \phi_2(t) - \phi_1(t-\tau) - \nu\tau] + R_{sp}/P_2(t) + F_{P2}(t)/P_2(t), \tag{28b}$$

$$\dot{\phi}_1(t) = \Delta\omega + \nu - \tfrac{1}{2}\alpha\left(G - \frac{1}{\tau_p}\right) + \kappa\sqrt{P_2(t-\tau)/P_1(t)} \times \sin[\phi_{PC} - \phi_1(t) - \phi_2(t-\tau) + \nu\tau] + F_{\phi 1}(t), \tag{28c}$$

$$\dot{\phi}_2(t) = \Delta\omega - \nu - \tfrac{1}{2}\alpha\left(G - \frac{1}{\tau_p}\right) + \kappa\sqrt{P_1(t-\tau)/P_2(t)} \times \sin[\phi_{PC} - \phi_2(t) - \phi_1(t-\tau) - \nu\tau] + F_{\phi 2}(t), \tag{28d}$$

where $P(t) = P_1(t) + P_2(t)$ is the normalized total power.

Solution of the steady-state equations yields the following equations for ν, ψ, laser threshold, and power:

$$\nu = \tfrac{1}{2}\kappa[\sqrt{P_1/P_2}\sin(\psi-\nu\tau) - \sqrt{P_2/P_1}\sin(\psi+\nu\tau)], \tag{29}$$

$$\Delta\omega = -\tfrac{1}{2}\alpha\left(G - \frac{1}{\tau_p}\right) - \tfrac{1}{2}\kappa[\sqrt{P_1/P_2}\sin(\psi-\nu\tau) + \sqrt{P_2/P_1}\sin(\psi+\nu\tau)]. \tag{30}$$

$$G-\frac{1}{\tau_p}=-R_{\rm sp}-2\kappa\sqrt{P_1/P_2}\cos(\psi-\nu\tau)$$
$$=-R_{\rm sp}-2\kappa\sqrt{P_2/P_1}\cos(\psi+\nu\tau), \tag{31}$$

$$\frac{P_1}{P_2}=\frac{\cos(\psi+\nu\tau)}{\cos(\psi-\nu\tau)}, \tag{32}$$

where $\psi=\phi_{\rm PC}-2\phi$, $\phi\equiv\frac{1}{2}(\phi_1+\phi_2)$. The phase difference $\phi_1-\phi_2$ remains undetermined. These equations satisfy $P_1 = P_2$ when $\nu=0$ or $\psi=0, \pm\pi$. Note that Eq. (32) predicts the unphysical $P_1P_2<0$ when $\psi=\pm\frac{1}{2}\pi$ and $\nu\neq 0$, and indeed, no solution consistent with the other equations, other than $\nu=0$, is found at these phase angles.

Equation (29) confirms the known result, that the mode spacing of the external resonator modes is approximately

$$\Delta\nu=\nu_{n+1}-\nu_n\approx\frac{\pi}{\tau}, \tag{33}$$

i.e., half that of the conventional external resonator mode spectrum.

As in conventional optical feedback, a minimum feedback strength must be present for the first higher-order PC mode to appear. This property is illustrated in Fig. 6. It is notable that for some values of the phase (i.e., $\psi\approx\pm\pi/2$) the only solution is $\nu=0$, and at other values (i.e., $\psi\approx\pm\pi/4$) higher-order mode solutions are found for any nonzero η value. For the negative value of the former case it is recalled that the fundamental mode is stable for all values of the feedback level (viz, Fig. 2), and hence it appears that the laser, when operated within this narrow phase ranges would be globally stable. Using Eqs. (33) and (34), this requires a detuning $\delta\omega$ equal to $\delta\omega=\frac{1}{2}[\kappa+\alpha(R_{\rm sp}+\tau_p^{-1})]$.

Dynamical stability is determined by linearizing Eqs. (28) in the usual way, leading to the system determinant:

$$\det\begin{bmatrix} -s\tau-\mu_1-C_p & \mu_2+e^{-s\tau}C_p & 2S_p & 2e^{-s\tau}S_p \\ \mu_2+e^{-s\tau}C_m & -s\tau-\mu_1-C_m & 2e^{-s\tau}S_m & 2S_m \\ \frac{1}{2}(\alpha\mu_1-S_p) & \frac{1}{2}(\alpha\mu_2+e^{-s\tau}S_p) & -s\tau-C_p & -e^{-s\tau}C_p \\ \frac{1}{2}(\alpha\mu_2+e^{-s\tau}S_m) & \frac{1}{2}(\alpha\mu_1-S_m) & -e^{-s\tau}C_m & -s\tau-C_m \end{bmatrix}=0, \tag{34}$$

where $\psi_{p/m}=\psi+/-\nu\tau$, $\omega_{r1,2}^2=GG_nP_{1,2}$, $\mu_{1,2}=\omega_{r1,2}^2\tau/(\gamma_e+s)$, $C_p=\eta(P_2/P_1)^{1/2}\cos\psi_P$, $S_p=\eta(P_2/P_1)^{1/2}\sin\psi_p$, $C_m=\eta(P_1/P_2)^{1/2}\cos\psi_m$, and $S_m=\eta(P_1/P_2)^{1/2}\sin\psi_p$.

Numerical evaluation of the above determinant showed that, when considered as a function of increasing ν, the modes are alternatively stable and unstable. Labeling them $n=0,1,\ldots,$ where $n=0$ represents the $\nu=0$ solution, then, for example, for $\eta=5$ and $\psi=-1$ rad, the $n=0,2,4,\ldots,$ modes are stable and the odd n modes are unstable. For $\psi=-2$ the odd-n modes are stable and the $n=0$ and even modes are unstable. However, in experiments it is $\Delta\omega$ that is controlled, and the phase will have a different value for each external cavity mode. This situation is illustrated in Table I, from which it is seen that in this case also the modes are alternatingly stable and unstable, with the higher gain mode in each pair being stable.

TABLE I. Frequencies and other properties of higher-order external cavity modes. Fixed parameter values are $\eta=10$, $\Delta\omega=0$, $\alpha=3$, and the values of the other parameters the same as elsewhere in this paper.

$\nu\tau/\pi$	ϕ (rad)	P_1/P_2	$(G-1/\tau_p)\tau$	$\Gamma\tau$
0	-4.39	1	6.32	6.07
0.968	-4.36	0.567	-6.55	unstable
2.078	-4.26	3.151	7.23	6.07
2.875	-4.13	0.224	-7.81	unstable

V. CONCLUSION

The main conclusions of this paper are summarized as follows. A method [16] that is a broad generalization of Lang and Yariv's approach [13] was used to derive the rate equation for a laser subject to PCF. The weak feedback version of this equation agreed with an earlier derivation [8] in the appropriate limits, and was used to study the dynamics of a semiconductor laser. The rate equation for the strong interaction limit has been given elsewhere [16], which predicts the well-known degeneracy and aberration-correction properties of phase-conjugate cavities (Au Yeung *et al.* [10]).

The steady-state mode distribution is described as a fundamental mode that is locked to the PCM pump in frequency

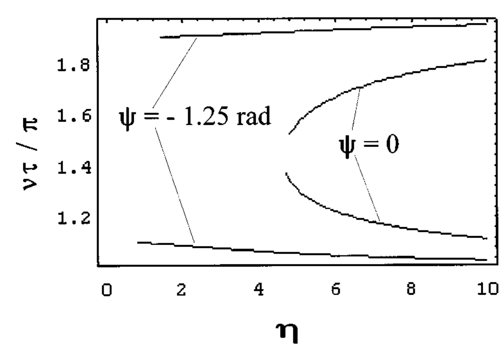

FIG. 6. Frequency offset from pump frequency vs feedback parameter η of first two higher-order cavity modes plotted for two phase values.

and in phase, and higher-order laser cavity and closely spaced external-resonator modes that consist of frequency and phase-locked pairs of fields having their respective frequencies symmetrically placed on each side of the pump frequency. The actual presence of each mode, including the fundamental, is subject to conditions that are controllable by tuning the pump frequency relative to the solitary laser frequency.

The stability of PCF external resonator modes is affected by the response time of the PCM under higher feedback levels. If the PCM response time is short compared to all time constants describing the laser dynamics, by tuning the phase inside an appropriate range, the system is stable even for high feedback levels, but for longer response times and sufficient feedback strength instability sets in.

The line shape of the laser with PCF consists of a sharp spike located on top of a broad background, as demonstrated previously [6]. For an external cavity a few centimeters long the area under the spike approaches 100% of the total area as $\eta \rightarrow 1$, while for a given η value the phase yielding the maximum spike is just that assumed by Agrawal and Gray, i.e., $\phi = \frac{1}{2}(\phi_{PC} + \arctan\alpha)$. It was also shown that the width of this spike is just equal to the linewidth of the PCM's pump laser.

The higher-order phase-conjugate longitudinal external cavity modes increase the available locking frequency range, but also reduce the range of feedback strength available for stable operation. An exception is a narrow phase band within which the basic mode is stable for all levels of feedback (for infinitely fast responding PCM), and where no higher-order external cavity modes appear. In addition, these modes contribute to the RIN and FNS noise. Also importantly, the presence of such modes, as in the conventional external cavity laser, introduces ''ringing'' effects in the modulation transfer function, so that in practice the modulation bandwidth of the laser places an upper limit on the length of the external cavity, which would be half as great as that of the conventional external cavity.

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