Finite-dimensional coherent-state generation and quantum-optical nonlinear oscillator models

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(Received 27 June 1996)

We discuss a system comprising a cavity with a nonlinear medium of k-order and an external coherent field excitation. We assume that the cavity field was initially in vacuum state. We show that for the case of weak external field our system behaves similarly to one described in finite-dimensional Hilbert space. Moreover, we perform numerical calculations simulating the dynamics of our system and compare the results with those of analytical attempts. [S1050-2947(97)01905-7]

PACS number(s): 42.50.Dv, 42.50.Ar, 42.65.−k

I. INTRODUCTION

The most commonly used states of quantum optics are coherent states defined by Glauber [1] by application of a displacement operator $\hat{D}$ acting on the vacuum state $|0\rangle$. This definition concerns infinite-dimensional Hilbert space. Recently, we observe great interest in the problem of coherent states defined in finite-dimensional Hilbert space (FDHS). For instance, Bužek *et al.* [2] discussed a coherent state analogous to that defined by Glauber; however, the state proposed in [2] was defined in $(s+1)$-dimensional space. The same problem was a subject of Ref. [3], where the analytical solutions for the coherent state were found. Moreover, a Wigner representation of the coherent states in FDHS was discussed in [4], whereas some aspects of the problem of harmonic-oscillator states in FDHS were dealt with in [5].

In this paper we propose a group of general models that, we believe, can lead to the generation of quantum states very close to the coherent states in FDHS. As we will show, our states correspond to those discussed in [3]. The models combine the evolution of a nonlinear medium in a cavity and a weak external coherent excitation. We show that for a sufficiently weak external field, resonance effects start to play a significant role, whereas nonresonant couplings become negligible. The nonlinear quantum evolution of the cavity field in the nonlinear medium is crucial for obtaining a state corresponding to the finite-dimensional coherent state in such a system. The effectiveness of the process is, however, considerably diminished by the cavity losses. Nevertheless, it seems important to us that a cavity with a nonlinear medium (with a field initially in vacuum state) and a sufficiently weak external excitation can lead with high accuracy to finite-dimensional coherent-state generation. For this situation we will derive analytical formulas for the probabilities corresponding to the Fock states we are interested in. Moreover, we will perform numerical calculations in which we simulate the dynamics of our system and compare the results with those of an analytical attempt.

II. MODEL AND AN ANALYTICAL SOLUTION

Glauber [1] defined a coherent state in infinite-dimensional Hilbert space (IDHS) by means of a displacement operator $\hat{D}(\alpha,\alpha^*)$ defined as

$$\hat{D}(\alpha,\alpha^*)=e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}. \tag{1}$$

Thus the coherent state $|\alpha\rangle_\infty$ in IDHS is defined as

$$|\alpha\rangle_\infty=\hat{D}(\alpha,\alpha^*)|0\rangle. \tag{2}$$

A problem arises when we try to define the coherent state for FDHS. There are two ways to determine such a state. One definition concerns *truncated coherent states* in FDHS. It was introduced by Kuang, Wang, and Zhou [6,7] and is based on the normalized truncation of the Fock expansion of the Glauber infinite-dimensional coherent state $|\alpha\rangle_\infty$. This procedure is equivalent to the action of the operator $\exp(\alpha\hat{a}^\dagger)$ (with proper normalization) on the vacuum state $|0\rangle$ [5]. Obviously, the operator $\exp(\alpha\hat{a}^\dagger)$ is truncated. The alternative attempt discussed by Bužek *et al.* [2] and Miranowicz *et al.* [3] concerns $(s+1)$-dimensional Hilbert space and is based on the action of the operator $\hat{D}^{(s)}$ (analogous to the Glauber displacement operator) on the vacuum state. This method differs from the commonly used definition of the coherent state for IDHS in that all states and operators are defined for the FDHS first and then applied to the Glauber-like definition. The properties of these states were discussed in Ref. [2] where a numerical analysis was proposed. Moreover, Miranowicz *et al.* [3] found and discussed analytical results for the coherent states defined in $(s+1)$-dimensional Hilbert space. Thus, in this paper, we shall deal with the kind of definition for the coherent state defined in FDHS. Since we are interested in finding physical models leading to the generation of the coherent states defined in FDHS rather than in the investigation of the properties of those states, we shall proceed to discuss a particular group of Hamiltonians. Therefore, we propose a group of quantum-optical models based on the following Hamiltonian (in the interaction picture):

$$\hat{H}=\frac{\lambda_k}{k}(\hat{a}^\dagger)^k\hat{a}^k+\epsilon(\hat{a}^\dagger+\hat{a}), \tag{3}$$

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where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, and λ_k denotes the constant of kth-order nonlinearity. The second term in the Hamiltonian (3) (proportional to ϵ) corresponds to the coherent excitation of the cavity by the external field. We use units of $\hbar=1$. Obviously, one should keep in mind that our models concern real physical situations (although they naturally involve certain limitations) and are defined in IDHS.

Let us express the wave function for our system in a Fock basis:

$$|\Psi(t)\rangle=\sum_{j=0}^{\infty} a_j(t)|j\rangle. \tag{4}$$

This wave function obeys a Schrödinger equation with the Hamiltonian expressed by Eq. (3):

$$i\frac{d}{dt}|\Psi(t)\rangle=\left(\frac{\lambda_k}{k}(\hat{a}^\dagger)^k\hat{a}^k+\epsilon(\hat{a}^\dagger+\hat{a})\right)|\Psi(t)\rangle. \tag{5}$$

Applying the standard procedure to our wave function (4) and the Hamiltonian (3) we obtain a set of equations for the probability amplitudes $a_j(t)$. They are of the form

$$\begin{aligned} i\frac{d}{dt}a_j(t)=&\frac{\lambda_k}{k}[j(j-1)\cdots(j-k+1)]a_j(t)+\epsilon[\sqrt{j}a_{j-1}(t)\\ &+\sqrt{j+1}a_{j+1}(t)], \end{aligned} \tag{6}$$

where k denotes the order of the nonlinear process and j corresponds to the j-photon state. Obviously, one should keep in mind that for $j<0$ we have $a_j=0$. We see from Eq. (6) that the set of equations for a_j is infinite. This is obvious, since we deal with the IDHS. Nevertheless, our aim here is to show that under special conditions our system behaves as one defined in FDHS. The first step is to assume that the coupling is weak, i.e., $\epsilon\ll\lambda_k$. As a consequence, we can treat our problem perturbatively. However, the main point of our considerations is the fact that the part of the Hamiltonian (3) corresponding to the evolution of the nonlinear medium,

$$\hat{H}_{\rm NL}=\frac{\lambda_k}{k}(\hat{a}^\dagger)^k\hat{a}^k, \tag{7}$$

produces degenerate states (corresponding to $j=0,1,\ldots,k-1$). As we take into account not only the first part of the Hamiltonian (3) but the second part too, we see that resonance arises between the interaction described by the latter and the degenerate states generated by $H_{\rm NL}$. This resonance effect and the assumption concerning weak coupling between the external and cavity fields ($\epsilon\ll\lambda_k$) leads to the closed-form dynamics and cuts some subspace of states out of all of the Fock states. As a consequence, assuming that the dynamics of the physical process starts from the vacuum $|0\rangle$, the evolution of the system is restricted to the states $|m\rangle$, where ($m=0,1,2,\ldots,k-1$). This situation resembles that for k-degenerate atomic levels coupled by a zero-frequency field, where this resonant interaction selects, from the whole set of levels, only those that lead to a closed system dynamics. Interaction with the remaining atomic levels can be treated as a negligible perturbation [8]. Obviously, one should note that the resonance discussed in this paper differs in character from the resonances commonly discussed in various papers where the cavity field and the difference between the energies of the atomic levels (or cavity frequencies) have identical values.

Thus, we write the following equations of motion for the probability amplitudes:

$$\begin{aligned}
i\frac{d}{dt}a_0(t)&=\epsilon a_1(t),\\
i\frac{d}{dt}a_1(t)&=\epsilon[a_0(t)+\sqrt{2}a_2(t)],\\
&\vdots\\
i\frac{d}{dt}a_{k-2}(t)&=\epsilon[\sqrt{k-2}a_{k-3}(t)+\sqrt{k-1}a_{k-1}(t)],\\
i\frac{d}{dt}a_{k-1}(t)&=\epsilon[\sqrt{k-1}a_{k-2}(t)+\sqrt{k}a_k(t)],\\
i\frac{d}{dt}a_k(t)&=\lambda_k(k-1)!a_k(t)+\epsilon[\sqrt{k}a_{k-1}(t)\\
&\quad+\sqrt{k+1}a_{k+1}(t)],\\
&\vdots
\end{aligned} \tag{8}$$

Since we have assumed $\lambda_k\gg\epsilon$, the last of the above equations indicates that the amplitude $a_k(t)$ is rapidly oscillating in comparison with the amplitudes $a_j(t)$, where $j<k$. Therefore, similar to the rotating-wave approximation applied to the atomic systems [8], we can neglect the influence of the amplitudes $a_m(t)$ ($m\geqslant k$) on the dynamics of our system described by the Eq. (8). Moreover, these amplitudes for the time $t=0$ are assumed to be equal to zero. Therefore, we neglect them and our equations of motion become

$$\begin{aligned}
i\frac{d}{dt}a_0(t)&=\epsilon a_1(t),\\
i\frac{d}{dt}a_1(t)&=\epsilon[a_0(t)+\sqrt{2}a_2(t)],\\
&\vdots\\
i\frac{d}{dt}a_{k-2}(t)&=\epsilon[\sqrt{k-2}a_{k-3}(t)+\sqrt{k-1}a_{k-1}(t)],\\
i\frac{d}{dt}a_{k-1}(t)&=\epsilon\sqrt{k-1}a_{k-2}(t).
\end{aligned} \tag{9}$$

We see that the dynamics of the system is closed within a finite subspace of the Fock states. As a consequence, we deal here with a finite-dimensional space. Of course, one should keep in mind that the set of equations (9) gives zero-order solutions for our perturbative treatment. It is obvious that we are in a position to find the solution for arbitrary values of the parameter k. For instance, for $k=2$ we get the following equations of motion:

$$i\frac{d}{dt}a_0(t)=\epsilon a_1(t),$$
$$i\frac{d}{dt}a_1(t)=\epsilon a_0(t), \tag{10}$$

and their solution:

$$a_0(t)=i\cos(\epsilon t),$$
$$a_1(t)=\sin(\epsilon t). \tag{11}$$

Clearly, this result resembles that for a two-level atom in an external field [8] and the dynamics of our system exhibits well-known oscillatory behavior. Moreover, this solution [Eq. (10)] is identical to that derived for the Fock-state expansion of the coherent state defined in FDHS for two-dimensional Hilbert space [2,3].

Analogously we can write the appropriate formulas for the case $k=3$. Here, the equations of motion become

$$i\frac{d}{dt}a_0(t)=\epsilon a_1(t),$$
$$i\frac{d}{dt}a_1(t)=\epsilon[a_0(t)+\sqrt{2}a_2(t)], \tag{12}$$
$$i\frac{d}{dt}a_2(t)=\epsilon\sqrt{2}a_1(t),$$

and the solutions corresponding to these equations are

$$a_0(t)=\tfrac{1}{3}[2+\cos(\sqrt{3}\epsilon t)],$$
$$a_1(t)=\frac{-i}{\sqrt{3}}\sin(\sqrt{3}\epsilon t), \tag{13}$$
$$a_2(t)=\frac{\sqrt{2}}{3}[\cos(\sqrt{3}\epsilon t)-1].$$

The solutions [Eqs. (12)] are identical to those derived in [3] defined for the coherent state in three-dimensional FDHS. Of course, we can write the appropriate formulas for arbitrary values of the parameter k (corresponding to arbitrary dimension of FDHS). The only limitation on our considerations resides in the handling of large sets of equations. As mentioned earlier, ours are zero-order perturbation solutions. Obviously, it is possible to find higher-order formulas that correspond to the Fock states for numbers of photons higher than the dimension of the FDHS discussed. To obtain the probability amplitudes corresponding to the higher states we should apply the standard perturbation procedure again. Thus we apply the zero-order solutions to the extended equations of motion including terms corresponding to interaction with the higher Fock states. For instance, for the case $k=3$ we get the following formulas for the probability amplitudes corresponding to the three-photon and four-photon states:

$$a_3(t)=\frac{\epsilon}{\lambda_3\sqrt{6}}[\cos(\sqrt{3}\epsilon t)-1], \tag{14}$$
$$a_4(t)=\frac{\epsilon^2}{4\sqrt{6}\lambda_3^2}[\cos(\sqrt{6}\epsilon t)-1].$$

From Eqs. (13) we see that the probability for the state $|3\rangle$ is proportional to ϵ^2, whereas that corresponding to the four-photon state is proportional to ϵ^4. As a consequence, for the case discussed in this paper where the external field–cavity field coupling is weak ($\epsilon\ll\lambda_k$), the influence of the higher Fock state is negligible.

It is visible, as we compare our results [Eq. (13)] with those of Miranowicz *et al.* [3], that the photon distribution of the states generated by our system is identical with the distribution corresponding to the three-dimensional FDHS. We see that physical systems described by the Hamiltonians defined in Eq. (3) can exhibit a dynamics that leads to FDHS states' generation. This is a main result of our considerations — FDHS states seem to be not only a mathematical concept. Obviously, one should always keep in mind that the accuracy of the expansion derived in this paper is limited by the accuracy of the perturbation procedure.

Moreover, at this point we should mention the losses in our system that may destroy the effects discussed here. This problem has already been discussed in the previous paper [9] where a similar model (with kicked nonlinear oscillator) has been studied. It was shown that the damping constant γ should be much smaller than the nonlinearity λ_k. We realize that this is a very strong requirement for experiment. Nevertheless, various experiments, for instance, those involving the very tiny effect of ''vacuum Rabi splitting'' [10,11] give us some hope for the practical realization of our models.

III. NUMERICAL APPROACH

The second part of this paper is devoted to numerical calculations and a comparison of their results with those based on our formulas. Thus, analogously to [9], we define the following unitary evolution operator:

$$\hat{U}=e^{-i[(\lambda_k/k)(\hat{a}^\dagger)^k\hat{a}^k+\epsilon(\hat{a}^\dagger+\hat{a})]t}. \tag{15}$$

This operator acts on the initial vacuum state $|0\rangle$ giving the wave function $|\Psi(t)\rangle$ for arbitrary time t:

$$|\Psi(t)\rangle=\hat{U}(t)|0\rangle. \tag{16}$$

As we compare Eq. (15) with those for the Glauber definition of the coherent state [Eq. (2)], we see that the operator $\hat{U}$ plays the same role as the displacement operator $\hat{D}$, albeit for finite-dimensional space. Of course we should keep in mind that the condition $\epsilon\ll\lambda_k$ should be satisfied and we deal here with the following correspondence:

$$\hat{U}|_{\epsilon\ll\lambda}\leftrightarrow\hat{D}. \tag{17}$$

Similar to the previous section we start here from the case of $k=2$. For this situation our evolution operator $\hat{U}$ has the following form:

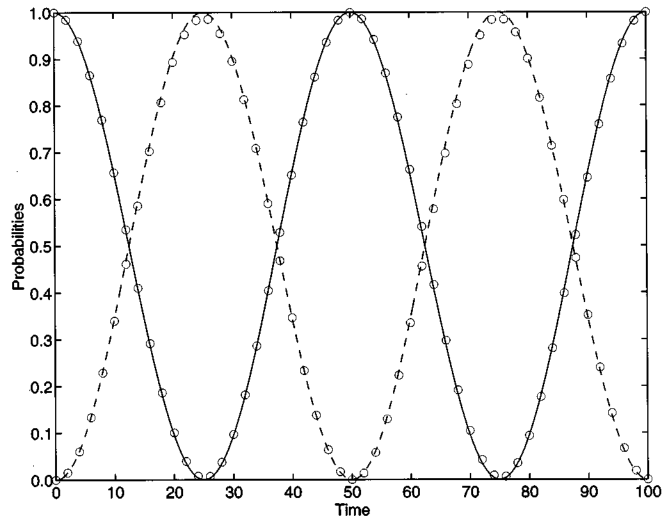

FIG. 1. Analytical solutions for the probabilities for the vacuum (solid line) and two-photon (dashed line) states in the case of $k=2$. The parameter $\epsilon=\pi/50$ (all parameters are measured in units of $\lambda=1$). Circle marks correspond to the probabilities found in our numerical calculations.

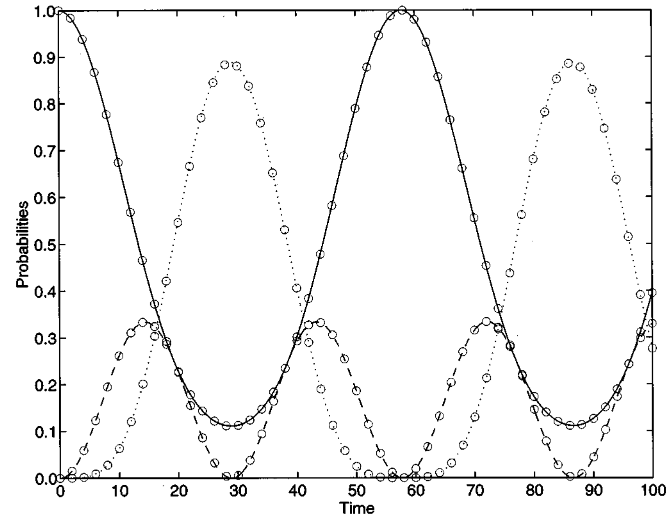

FIG. 2. The same as in Fig. 1, but for $k=3$. Analytical solutions for the probabilities corresponding to the successive one-photon state (dashed line) and two-photon state (dotted line) are compared to the results of our numerical attempt (circles).

$$\hat{U}=e^{-i[(\lambda_2/2)(\hat{a}^\dagger)^2\hat{a}^2+\epsilon(\hat{a}^\dagger+\hat{a})]t}. \tag{18}$$

Moreover, we assume that for the time $t=0$ the field was in the vacuum state $|0\rangle$. Figure 1 shows the probabilities for the Fock states versus the time t (we use units of $\lambda_2=1$) obtained from the analytical formulas (10) and from our numerical approach. These two results exhibit very good agreement. Moreover, one notes that the probabilities for the vacuum and one-photon states oscillate regularly and the population flows only between these two states. As a consequence, the influence of the two-photon and higher states on the dynamics of our system is negligible. In addition, for appropriately chosen time t we get the one-photon state ($t=25n$, $n=1,2,\ldots$ for $\epsilon=\pi/50$), similarly to [9]. Obviously, for the same time the probability for the vacuum state is equal to zero. Hence, we can treat the unitary evolution operator $\hat{U}$ as a switching operator applicable in quantum computation theory. Of course, one should keep in mind the condition $\epsilon\ll\lambda_2$.

Figure 2 corresponds to the case of $k=3$ and shows the probabilities obtained from the numerical calculations and from the equations (12). Obviously, for this situation the unitary evolution operator $\hat{U}$ has the following form:

$$\hat{U}=e^{-i[(\lambda_3/3)(\hat{a}^\dagger)^3\hat{a}^3+\epsilon(\hat{a}^\dagger+\hat{a})]t}. \tag{19}$$

Again, we see very good agreement between the analytical and numerical attempts. Moreover, the dynamics of our system is restricted to the three states: the vacuum and the one- and two-photon states. In addition, contrary to the case of $k=2$, we are not able to generate a pure n-photon state. However, for some values of the time t, we get a mixture of $\sim$90% of the vacuum state $|0\rangle$ and $\sim$10% of $|2\rangle$.

The time evolution of the probability for the three-photon state $|3\rangle$ is shown in Fig. 3, where it is seen to take significantly smaller values than its counterparts corresponding to the lower Fock states. Its maximum value is $\sim 1.2\times 10^{-3}$. Moreover, similarly to Fig. 2, we observe good agreement between the numerical calculations based on the unitary operator $\hat{U}$ (19) and the analytical formula (14).

As the value of the external-cavity fields interaction increases our analytical model based on the perturbation procedure can no longer be correct and differs significantly from that based on the numerical calculations. Figure 4 shows the numerical results for $k=3$, but for higher values of the coupling constant $\epsilon=\pi/5$. We see that the influence of higher Fock states becomes significant (for this case the influence of $|4\rangle$). Obviously, as ϵ increases the higher and higher states start to play a significant role. For this case our model cannot be treated as one corresponding to finite-dimensional space.

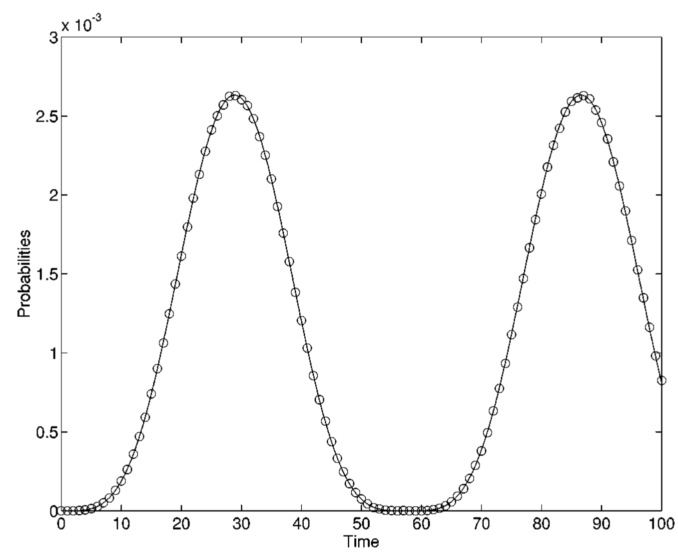

FIG. 3. Analytical solution (solid line) and numerical results (circles) for the evolution of the probability corresponding to the three-photon state. All parameters are the same as in Fig. 2.

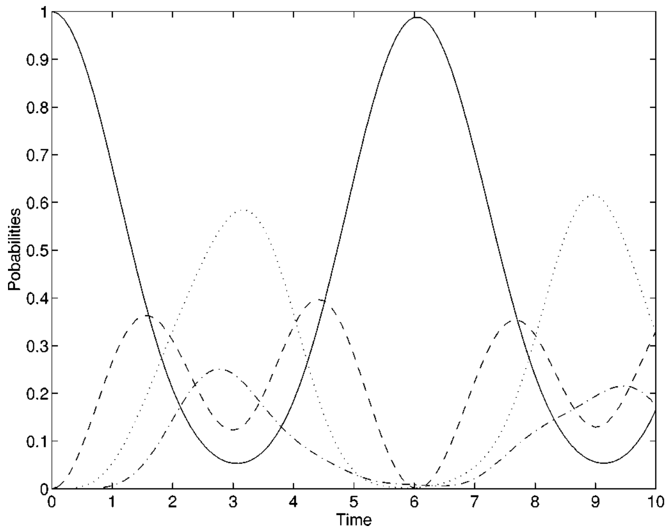

FIG. 4. The numerical results for $k=3$ and $\epsilon=\pi/5$. The probabilities for the states are given by $|0\rangle$ (solid line), $|1\rangle$ (dashed line), $|2\rangle$ (dotted line), and $|3\rangle$ (dashed-dotted line).

IV. CONCLUSIONS

In this paper we dealt with a group of models of a general nature combining the evolution of a nonlinear medium in a cavity and a weak external coherent excitation. We have shown that these models can lead to the generation of quantum states very close to the coherent states in FDHS corresponding to those discussed in [3]. We have shown that for a sufficiently weak external–cavity fields interaction, the resonance effects become significant, and nonresonant couplings become negligible. The nonlinear quantum evolution of the cavity field in the nonlinear medium is crucial for the preparation of a state corresponding to the finite-dimensional coherent state in such a system. For this situation we derived analytical formulas for the probabilities corresponding to the Fock states we are interested in. Moreover, we proposed and performed numerical calculations in which we simulate the dynamics of our system. A comparison of our numerical attempt with our analytical results shows very good agreement. Of course, this agreement is visible only for the case of weak field coupling ($\epsilon \ll \lambda_k$), when the perturbation method can be satisfactorily applied. Moreover, we have shown that for the case $k=2$ we get the one-photon Fock state, whereas for $k=3$ we are able to obtain an almost pure (90%) two-photon state $|2\rangle$. In addition, as the dynamics of the system is closed within two-dimensional space the unitary evolution operator $\hat{U}$ acts as a switching operator that can be applied in the quantum computer systems theory.

Obviously, we should mention the influence of the losses in the system on the dynamics of our models. This problem has already been discussed in Ref. [9] for the case of $k=2$. Although the model discussed there concerned pulsed excitations, the considerations discussed in [9] can be applied to the model discussed in this paper. Of course, to investigate the effects of the losses we could perform appropriate numerical calculations; however, the problem appears to be of sufficient interest for a separate paper. For instance, such investigations could be based on the quantum trajectories method [12]. Alternatively, one might apply the density matrix ρ method as done in Ref. [13].

ACKNOWLEDGMENT

The author wishes to thank Professor Tanaś for carefully reading this text and for his valuable discussions and suggestions.

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