Eikonal formula for tensor forces

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The eikonal formula for the scattering amplitude is investigated for the most general local potential between two spin one-half particles, i.e., that which includes a tensor interaction. The analysis isolates the main contribution to high-energy scattering near the forward direction, constructs from it eikonal solution approximations, and indicates the way to obtain successive corrections. The method is also applied to include a spin-orbit potential contribution. Closed-form expressions that depend on the potential components are given for the Green's function and the scattering amplitude for these approximations. [S1050-2947(97)07904-3]

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I. INTRODUCTION

The eikonal formula is a useful approximation for the t matrix or the Green's function of a system involved in scattering characterized by a large energy as compared to typical parameters of the system. This approximation has been widely used in its scalar version but its extension to account for tensor forces has remained mostly unexplored. In fact, the presence of spin among particles requires that tensor forces be included in the description of their interaction, and these play an important role for various physical systems.

In atomic physics, a magnetic dipole force component between electrons is a relevant relativistic correction which has a tensor form [1]. In nuclear physics the tensor interaction is a necessary component in the description of the nucleon-nucleon force and even for quarks the relevance of this effective component has been pointed out in relation to the problem of the spin carried by them [2]. The eikonal formula has been investigated in the context of momentum expansions of the scattering amplitude [3] and it has been successfully applied in the study of both high energy nucleon-nucleon (NN) collision and that of hadron-nucleus collisions where multiparticle collision events are accounted for by the Glauber theory, which is based on summing two-body contributions into a linearized formula [4]. Moreover, as the Glauber approximation is approached in the high-energy and small angle limit, useful information is obtained about general multiple scattering (whose single- and two-scattering contributions evaluated on-shell are equivalent to the eikonal approximation for a composite system) [5]. An eikonal approach is also useful in various other areas such as quantum field theory [6] and the response function at large momentum transfers [7,8].

In recent years an increasingly comprehensive experimental study of spin observables in electron scattering of nuclei and in NN collisions has produced data on the cross section at various energy ranges with more detailed information on spin correlations [9]. In addition, calculations of the response function of nuclear matter, whose values can be extrapolated from data on electron scattering on nuclei, point to the importance of accounting for spin degrees of freedom and tensor forces [10]. It is hence desirable to search for a more formal but practical way of describing these degrees of freedom, and to have a simple way to relate the potential input to these observables. In the present study we derive an eikonal formula providing for such a description, namely, one which accounts for spin degrees of freedom.

As a way of introducing the subject, we now turn to the derivation of the scalar eikonal formula by considering a system of two particles interacting through a local potential, $\underline{V}(r)$. The nonrelativistic expression for the Green's function G with energy argument ω and relative coordinates in the center of mass system and reduced mass m (we use $\hbar=1$) is given by

$$\overline{G}=\frac{1}{\omega-\dfrac{p^2}{2m}-V+i\epsilon}. \tag{1}$$

The t matrix can be expressed in terms of the Green's function using

$$t=V+V\overline{G}V, \tag{2}$$

and it describes the scattering amplitude when evaluated in momentum space at on-shell momenta. Writing the initial momentum $\mathbf{k}_i$ and the final momentum $\mathbf{k}_f$ as

$$\mathbf{k}_i=\mathbf{q}-\frac{1}{2}\mathbf{\Delta},$$

$$\mathbf{k}_f=\mathbf{q}+\frac{1}{2}\mathbf{\Delta}, \tag{3}$$

we assume q is large enough so that $\Delta \ll q$. For on-shell scattering $2m\omega=|\mathbf{k}_i|^2=|\mathbf{k}_f|^2$ so $\mathbf{\Delta}\cdot\mathbf{q}=0$, implying $\omega\approx q^2/(2m)$. The eikonal approximation is obtained when essentially the potential in Eq. (1) is smooth enough on a scale of $1/q$ so that relevant contributions to the Green's function come only from values of the momentum close to $\mathbf{q}$ [4]. Consequently, the resulting denominator in Eq. (1) $q^2-p^2=2\mathbf{q}\cdot(\mathbf{q}-\mathbf{p})-(\mathbf{q}-\mathbf{p})^2$ is well approximated by its linear contribution, which leads to the eikonal propagator

$$\overline{G}_{\rm eik}=\frac{1}{v}\,\frac{1}{q-\hat{\mathbf{q}}\cdot\mathbf{p}-V/v+i\epsilon}, \tag{4}$$

with $v=q/m$. The latter equation allows for a closed formula. Explicitly, the eikonal formula for the Green's function in this case, expressed in coordinate space, and choosing the $\hat{\mathbf{z}}$ axis along the $\hat{\mathbf{q}}$ direction, with $\mathbf{r}=(\mathbf{b},z)$, is given by

$$\langle \mathbf{r}'|\overline{G_{\text{eik}}}|\mathbf{r}\rangle = \frac{-i}{v}\theta(z'-z)\,\delta^2(\mathbf{b}'-\mathbf{b})\exp\left[iq(z'-z) - \frac{i}{v}\int_z^{z'} d\zeta V(\mathbf{b},\zeta\hat{\mathbf{z}})\right]. \tag{5}$$

For the case of spin one-half particles, the most general rotationally invariant, spin-dependent local interaction $\mathbf{V}_{NN}$ (which involves matrix components) of two particles labeled 1 and 2, can be shown to consist of

$$\begin{aligned}\mathbf{V}_{NN}(r) &= V_c(r) + V_{\sigma'}(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + V_{S'}(r)\mathbf{S}_{12}(\hat{\mathbf{r}}) \\ &= V_c(r) + V_\sigma(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + V_S(r)\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2,\end{aligned} \tag{6}$$

corresponding, respectively, to the scalar, spin, and tensor components, where the tensor part $\mathbf{S}_{12}(\hat{\mathbf{r}}) = 3\hat{\mathbf{r}}\cdot\sigma_1\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, and the second expression uses

$$V_\sigma(r) = V_{\sigma'}(r) - V_{S'}(r), \quad V_S(r) = 3V_{S'}(r). \tag{7}$$

While a simple extension of the eikonal formula has been considered for a potential with a scalar spin component $V_\sigma(r)$ and a spin-orbit term [11] (by approximating a momentum operator with a constant term) and a spin-orbit extension has been treated more formally [12], to our knowledge this has not been the case for the tensor force. In this paper we shall analyze G_{eik} in Eq. (4), generalizing it to the case in which it includes the spin dependent interaction $\mathbf{V_{NN}}$ in Eq. (6). In Sec. II we examine an expansion leading to the usual eikonal formula and we analyze its extension to the tensor case, from which we obtain a first approximation valid near the forward direction. In Sec. III we find a rotation of coordinates which makes the formula useful for computations. In Sec. IV we consider further corrections, which brings us to an improved formula applicable in a larger range of small angles around the forward direction. We also consider a spin-orbit additional component in the potential. In Sec. V we derive closed-form expressions for the scattering amplitude and in Sec. VI we summarize this work.

II. TENSOR CONTRIBUTION

The possibility of giving the closed-form expression in Eq. (5) to the nontensor propagator G_{eik} in Eq. (4) is both a consequence of the fact that it satisfies an inhomogeneous first order differential equation and the fact that the potential commutes with itself at different points. In the tensor case, we are interested in calculating

$$G_{\text{eik}} = \frac{1}{v}\,\frac{1}{q - \hat{\mathbf{q}}\cdot\mathbf{p} - \mathbf{V}_{NN}/v + i\epsilon}, \tag{8}$$

and there is no simple explicit expression accounting for all terms as in Eq. (5). To see this, we expand G_{eik} in a Born-like series, from which Eq. (5) can be also derived,

$$\overline{G_{\text{eik}}} = G_0 + G_0VG_0 + G_0VG_0VG_0 + \cdots. \tag{9}$$

In this equation G_0 is the eikonal purely kinetic term

$$\langle \mathbf{r}'|G_0|\mathbf{r}\rangle = \frac{-i}{v}\theta(z'-z)\,\delta^2(\mathbf{b}'-\mathbf{b})\exp[iq(z'-z)]. \tag{10}$$

If we pick up the third term on the right-hand side (rhs) of Eq. (9) we find that the equation

$$\begin{aligned}\langle \mathbf{r}'|G_0VG_0VG_0|\mathbf{r}\rangle &= \int dz'' dz''' G_0(z'-z'')V(z'') \\ &\quad\times G_0(z''-z''')V(z''')G_0(z'''-z) \\ &= \frac{-i}{2v}\theta(z'-z)\,\delta^2(\mathbf{b}'-\mathbf{b})\exp[iq(z'-z)] \\ &\quad\times\left[-\frac{i}{v}\int_z^{z'} d\zeta V(\mathbf{b},\zeta\hat{\mathbf{z}})\right]^2\end{aligned} \tag{11}$$

(making explicit only the dependence of coordinates along $\hat{\mathbf{z}}$ in the first equality) is a consequence of the commutativity of $V(z)$ and $V(z')$, at any z, z'. Considering now G_{eik} in Eq. (8), rather than repeating a similar Born-like expansion

$$G_{\text{eik}} = G_0 + G_0\mathbf{V}_{NN}G_0 + G_0\mathbf{V}_{NN}G_0\mathbf{V}_{NN}G_0 + \cdots, \tag{12}$$

progress can be made if we partially resum Eq. (12) by expanding Eq. (8) as

$$G_{\text{eik}} = G_0' + G_0'\mathbf{V}_SG_0' + G_0'\mathbf{V}_SG_0'\mathbf{V}_SG_0' + \cdots. \tag{13}$$

Here G_{eik} is written in terms of

$$G_0' = \frac{1}{v}\,\frac{1}{q - \hat{\mathbf{q}}\cdot\mathbf{p} - (V_c + \mathbf{V}_{\boldsymbol{\sigma}})/v + i\epsilon}, \tag{14}$$

which contains the commuting parts of the potential $[\mathbf{V}_\sigma, \mathbf{V}_{NN}] = 0$, where $\mathbf{V}_\sigma(r) = V_\sigma(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, $\mathbf{V}_S(\mathbf{r}) = V_S(r)\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2$. The tensor components $\mathbf{V}_S$ do not commute among themselves at different points; in fact, using the formula for the product of Pauli matrices $\boldsymbol{\sigma}\cdot\mathbf{a}\boldsymbol{\sigma}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + i\mathbf{a}\times\mathbf{b}\cdot\boldsymbol{\sigma}$, with $\mathbf{a},\mathbf{b}$ any vectors, one shows that

$$\begin{aligned}[\mathbf{V}_S(\mathbf{r}_a), \mathbf{V}_S(\mathbf{r}_b)] &= \frac{1}{9}V_S(r_a)V_S(r_b)[\mathbf{S}_{12}(\hat{\mathbf{r}}_a), \mathbf{S}_{12}(\hat{\mathbf{r}}_b)] \\ &= 2iV_S(r_a)V_S(r_b)\hat{\mathbf{r}}_a\cdot\hat{\mathbf{r}}_b\hat{\mathbf{r}}_a\times\hat{\mathbf{r}}_b\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2).\end{aligned} \tag{15}$$

However, while the above expression is nonzero in general, it can give us a clue as to finding an initial approximation for the eikonal tensor solution. Equation (15) implies $\mathbf{V}_S(\mathbf{r}_a)$ and $\mathbf{V}_S(\mathbf{r}_b)$ commute if we choose $\hat{\mathbf{r}}_a, \hat{\mathbf{r}}_b$ either parallel or orthogonal among themselves. Now, our assumption for values of the momentum with $\Delta \ll q$ and intermediate values $|\mathbf{p}-\mathbf{q}| \ll q$ translates in r space to relevant values $b \gg |z|$, $b \gg |z'|$, corresponding to a mostly parallel range of configurations. This suggests commutators as in Eq. (15) can be neglected in a first approximation to Eq. (8). Under this assumption we sum only over terms such as those in Eq. (11), and the resulting term $G_{\rm eik}^{(0)}$ in coordinate space is

$$\langle \mathbf{r}'|G_{\rm eik}^{(0)}|\mathbf{r}\rangle = -\frac{i}{v}\theta_{z',z}\delta_{\mathbf{b}',\mathbf{b}}e^{iq(z'-z)}e^{i(\chi_c+\chi_\sigma \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)}e^{i\boldsymbol{\tau}_S}, \tag{16}$$

where $\theta_{z',z}=\theta(z'-z)$,

$$\chi_k(\mathbf{b},z',z) = -\frac{1}{v}\int_z^{z'} d\zeta V_k(\mathbf{b},\zeta\hat{\mathbf{z}}), \quad k=c,\sigma, \tag{17}$$

we use the definitions of the potential in Eq. (7), and we define

$$\boldsymbol{\tau}_S = -\frac{1}{v}\int_z^{z'} d\zeta \mathbf{V}_S(\mathbf{b},\zeta\hat{\mathbf{z}}). \tag{18}$$

Curiously, we note that for $\mathbf{b}=\mathbf{0}$, $G_{\rm eik}^{(0)}$ in Eq. (16) becomes a representation of the exact solution of $G_{\rm eik}$ in Eq. (8), which contains $\mathbf{V_{NN}}$. We also note that contributions to the linear spin operators as in Eq. (15) start with at least an $O(1/v)$ lower as compared to bilinear terms, which we shortly show enter Eq. (16). In addition, terms linear in the spin operators give a vanishing contribution to the forward amplitude and hence to the total cross section [13]. We expect then that $G_{\rm eik}^{(0)}$ should give a very good account of the relevant components in the Green's function and the scattering amplitude, at least at small angles, and perhaps even at wider angles.

III. COORDINATE CHANGE

The contribution of the tensor part of the potential in Eq. (18) is

$$\begin{aligned}\boldsymbol{\tau}_S = {} & \tau_{xx}\sigma_{1x}\sigma_{2x} + \tau_{yy}\sigma_{1y}\sigma_{2y} + \tau_{zz}\sigma_{1z}\sigma_{2z} \\ & + \tau_{xy}(\sigma_{1x}\sigma_{2y}+\sigma_{1y}\sigma_{2x}) + \tau_{xz}(\sigma_{1x}\sigma_{2z}+\sigma_{1z}\sigma_{2x}) \\ & + \tau_{yz}(\sigma_{1y}\sigma_{2z}+\sigma_{1z}\sigma_{2y}),\end{aligned} \tag{19}$$

where the τ terms are given by

$$\begin{aligned}\tau_{xx} &= c_\phi^2 \chi_{S,0}(\mathbf{b},z',z),\\ \tau_{yy} &= s_\phi^2 \chi_{S,0}(\mathbf{b},z',z),\\ \tau_{zz} &= \chi_{S,2}(\mathbf{b},z',z),\\ \tau_{xy} &= c_\phi s_\phi \chi_{S,0}(\mathbf{b},z',z),\\ \tau_{xz} &= c_\phi \chi_{S,1}(\mathbf{b},z',z),\\ \tau_{yz} &= s_\phi \chi_{S,1}(\mathbf{b},z',z),\end{aligned} \tag{20}$$

with

$$\chi_{S,l}(\mathbf{b},z',z) = -\frac{b^{2-l}}{v}\int_z^{z'} d\zeta \frac{\zeta^l}{b^2+\zeta^2}V_S(\mathbf{b},\zeta\hat{\mathbf{z}}), \quad l=0,1,2, \tag{21}$$

and $c_x=\cos(x)$, $s_x=\sin(x)$, $\mathbf{b}=b(c_\phi,s_\phi)$.

An inspection of the tensor contribution $\boldsymbol{\tau}_S$ in Eqs. (19) and (20) shows that it contains new terms beyond those appearing in $\mathbf{V}_{NN}(r)$ in Eqs. (6) and (7). Although noncommuting terms appear to make the expression for $G_{\rm eik}^{(0)}$ in Eq. (16) intractable, we now show that $\boldsymbol{\tau}_S$ in Eq. (19) can in fact be written in terms of rotated Pauli matrices (or similarly, one can rotate the coordinates) in such a way that it can be brought into the form

$$\boldsymbol{\tau}'_S = \mathbf{m}\cdot\boldsymbol{\sigma}_1\mathbf{m}\cdot\boldsymbol{\sigma}_2 + \mathbf{n}\cdot\boldsymbol{\sigma}_1\mathbf{n}\cdot\boldsymbol{\sigma}_2 + \mathbf{k}\cdot\boldsymbol{\sigma}_1\mathbf{k}\cdot\boldsymbol{\sigma}_2, \tag{22}$$

where the vectors $\mathbf{v}=\mathbf{m}$, $\mathbf{n}$, and $\mathbf{k}$ are taken orthogonal, so that the different operators $\mathbf{v}\cdot\boldsymbol{\sigma}_1\mathbf{v}\cdot\boldsymbol{\sigma}_2$ commute among themselves. By obtaining such vectors we will be able to write $G_{\rm eik}^{(0)}$ in terms of scalar quantities multiplying matrix operators.

The orthogonality condition among the Pauli matrix components of $\boldsymbol{\sigma}$ is kept if we apply a rotation transformation to each of the Pauli matrices operating on particles 1 and 2, namely

$$\mathbf{m}\cdot\boldsymbol{\sigma} = C_m\mathbf{R}^{-1}\sigma_x\mathbf{R}, \quad \mathbf{n}\cdot\boldsymbol{\sigma} = C_n\mathbf{R}^{-1}\sigma_y\mathbf{R},$$
$$\mathbf{k}\cdot\boldsymbol{\sigma} = C_k\mathbf{R}^{-1}\sigma_z\mathbf{R}, \tag{23}$$

where C_m, C_n, C_k are, respectively, the lengths of the vectors $\mathbf{m}$, $\mathbf{n}$, and $\mathbf{k}$. Writing the components $\mathbf{v}=(v_x,v_y,v_z)$ for each of the vectors $\mathbf{v}$ we demand $\boldsymbol{\tau}_S$ in Eq. (19) to be equal to $\boldsymbol{\tau}'_S$ in Eq. (22), using Eq. (23). We arrive at a system of six equations for the six unknowns: the three angles in the rotation matrix $\mathbf{R}$ and the lengths C_m, C_n, C_k,

$$C_m^2\hat{m}_i^2 + C_n^2\hat{n}_i^2 + C_k^2\hat{k}_i^2 = \tau_{ii}, \quad i=x,y,z \tag{24}$$

and

$$C_m^2\hat{m}_x\hat{m}_y + C_n^2\hat{n}_x\hat{n}_y + C_k^2\hat{k}_x\hat{k}_y = \tau_{xy}, \tag{25}$$

and the two other similar equations for xz and yz. These equations can be solved analytically, for at least five of the six unknowns, three angles in $\mathbf{R}$, and the magnitudes C_m, C_n, C_k.

Further simplification can be attained when we realize that the tensor term gives contributions to $\boldsymbol{\tau}_S$ with projections of $\boldsymbol{\sigma}$'s along the plane spanned by $\hat{\mathbf{b}}$ and $\hat{\mathbf{z}}$. Explicitly, $\boldsymbol{\tau}_S$ can be written

$$\begin{aligned}\boldsymbol{\tau}_S = {} & \chi_{S,0}\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_2 + \chi_{S,2}\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}_2 \\ & + \chi_{S,1}(\sigma_{1\hat{\mathbf{b}}}\sigma_{2\hat{\mathbf{z}}} + \sigma_{1\hat{\mathbf{z}}}\sigma_{2\hat{\mathbf{b}}}).\end{aligned} \tag{26}$$

Our rotation then simplifies to trying to write $\boldsymbol{\tau}_S$ in the form

$$\boldsymbol{\tau}''_S = s^2\hat{\mathbf{s}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{s}}\cdot\boldsymbol{\sigma}_2 + t^2\hat{\mathbf{t}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{t}}\cdot\boldsymbol{\sigma}_2, \tag{27}$$

where the new orthogonal vectors $\mathbf{s}$, $\mathbf{t}$ have components

$$\mathbf{s} = (\sqrt{\rho}\hat{\mathbf{b}}, \sqrt{\eta}\hat{\mathbf{z}}), \quad \mathbf{t} = \lambda(-\sqrt{\eta}\hat{\mathbf{b}}, \sqrt{\rho}\hat{\mathbf{z}}). \tag{28}$$

Vectors $\mathbf{s}$ and $\mathbf{t}$ are determined by matching $\boldsymbol{\tau}''_S$ in Eq. (27), with $\boldsymbol{\tau}_S$ in Eq. (26), which leads to the equations

$$\rho+\lambda^2\eta=\chi_{S,0},$$

$$\lambda^2\rho+\eta=\chi_{S,2},$$

$$\sqrt{\rho\eta}(1-\lambda^2)=\chi_{S,1}. \qquad (29)$$

These have the solution

$$\rho=\frac{\chi_{S,0}\sqrt{\mu}+\chi_{S,0}^2+2\,\chi_{S,1}^2-\chi_{S,0}\,\chi_{S,2}}{2\sqrt{\mu}},$$

$$\eta=\frac{\chi_{S,2}\sqrt{\mu}+\chi_{S,2}^2+2\chi_{S,1}^2-\chi_{S,0}\chi_{S,2}}{2\sqrt{\mu}},$$

$$\lambda^2=\frac{\chi_{S,0}+\chi_{S,2}-\sqrt{\mu}}{\chi_{S,0}+\chi_{S,2}+\sqrt{\mu}}, \qquad (30)$$

where

$$\mu=\chi_{S,0}^2+4\chi_{S,1}^2-2\,\chi_{S,0}\,\chi_{S,2}+\chi_{S,2}^2, \qquad (31)$$

and we have the dependence $\rho=\rho(\mathbf{b},z',z)$, $\eta=\eta(\mathbf{b},z',z)$, $\lambda=\lambda(\mathbf{b},z',z)$. Any of the other three solutions of these equations can be equally used.

An explicit expression for the spin components for $G_{\rm eik}^{(0)}$ in Eq. (16) can now be obtained by considering

$$\langle\mathbf{r}'|G_{\rm eik}^{(0)}|\mathbf{r}\rangle=-\frac{i}{v}\theta_{z',z}\delta_{\mathbf{b}',\mathbf{b}}e^{iq(z'-z)}e^{i\boldsymbol{\tau}^{(0)}}, \qquad (32)$$

with

$$\boldsymbol{\tau}^{(0)}=\chi_c+\chi_s\hat{\mathbf{s}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{s}}\cdot\boldsymbol{\sigma}_2+\chi_t\hat{\mathbf{t}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{t}}\cdot\boldsymbol{\sigma}_2+\chi_u\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_2, \qquad (33)$$

where $\hat{\mathbf{u}}=\hat{\mathbf{b}}\times\hat{\mathbf{z}}$ ($\hat{\mathbf{s}}\times\hat{\mathbf{t}}$ is collinear to vector $\hat{\mathbf{u}}$, and the direction here is immaterial),

$$\chi_s=\chi_\sigma+\rho+\eta=\chi_\sigma+\frac{\chi_{S,0}+\chi_{S,2}+\sqrt{\mu}}{2},$$

$$\chi_t=\chi_\sigma+\lambda^2(\rho+\eta)=\chi_\sigma+\frac{\chi_{S,0}+\chi_{S,2}-\sqrt{\mu}}{2},$$

$$\chi_u=\chi_\sigma, \qquad (34)$$

and we have omitted in the notation the fact that χ_s, χ_t, and χ_u depend on $\mathbf{b}$, z', and z through the functions χ_i. Since $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{u}$ are chosen orthogonal, the components of $\boldsymbol{\tau}^{(0)}$ in Eq. (33) commute among themselves. Using this fact, the equation

$$\exp(iy\hat{\mathbf{l}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{l}}\cdot\boldsymbol{\sigma}_2)=\cos(y)+i\ \sin(y)\hat{\mathbf{l}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{l}}\cdot\boldsymbol{\sigma}_2 \qquad (35)$$

for y a c number and $\hat{\mathbf{l}}$ any unit vector, and Eq. (A2) in the Appendix, we obtain for the exponential term in our approximate Green's function in Eq. (32)

$$e^{\boldsymbol{\tau}^{(0)}}=\Gamma_{0c}+\Gamma_{0s}\hat{\mathbf{s}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{s}}\cdot\boldsymbol{\sigma}_2+\Gamma_{0t}\hat{\mathbf{t}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{t}}\cdot\boldsymbol{\sigma}_2+\Gamma_{0u}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_2 \qquad (36)$$

and

$$\Gamma_{0c}=e^{i\chi_c}(c_sc_tc_u+is_ss_ts_u),$$

$$\Gamma_{0s}=e^{i\chi_c}(c_ss_ts_u+is_sc_tc_u),$$

$$\Gamma_{0t}=e^{i\chi_c}(s_sc_ts_u+ic_ss_tc_u),$$

$$\Gamma_{0u}=e^{i\chi_c}(s_ss_tc_u+ic_sc_ts_u), \qquad (37)$$

where we use the notation $c_s=\cos(\chi_s)$, $s_s=\sin(\chi_s)$, etc.

IV. FURTHER CORRECTIONS

A. Corrections to order V_S^2

Equation (36) implies $G_{\rm eik}$ contains only four of the expected five terms which are needed in the description of the scattering amplitude of a system of two spin one-half particles (we show below how this information is obtained from the Green's function). In the following we improve the approximation by looking at the lowest order correcting terms that we have so far discarded.

Indeed, the first term not accounted for in $G_{\rm eik}^{(0)}$ in Eq. (16) is given by the commutator of the potential $\mathbf{V}_S$ with itself at different points, in the third term of the expansion of $G_{\rm eik}$ in Eq. (13),

$$\langle\mathbf{r}'|[G_0'\mathbf{V_S}G_0'\mathbf{V_S}G_0']_{\rm com}|\mathbf{r}\rangle$$
$$=-\frac{i}{v}\theta_{z',z}\delta_{\mathbf{b}',\mathbf{b}}e^{iq(z'-z)}e^{i(\chi_c+\chi_\sigma\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)}\boldsymbol{\tau}_r, \qquad (38)$$

where

$$\boldsymbol{\tau}_r=-\frac{1}{2v^2}\int dz''dz'''\,\theta(z'-z'')\,\theta(z''-z''')\,\theta(z'''-z)$$
$$\times[\mathbf{V_S}(z''),\mathbf{V_S}(z''')]. \qquad (39)$$

$\boldsymbol{\tau}_r$ contributes to the fifth type of term in the amplitude as, by using Eq. (15), one shows it is of the form

$$\boldsymbol{\tau}_r=i\chi_r\hat{\mathbf{u}}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2), \qquad (40)$$

where

$$\chi_r=-\frac{2b}{2v^2}\int dz''dz'''\,\theta_{z',z''}\theta_{z'',z'''}\theta_{z''',z}(z'''-z'')(b^2+z''z''')$$
$$\times\frac{V_S(z'')V_S(z''')}{(b^2+z''^2)(b^2+z'''^2)}. \qquad (41)$$

B. Analysis of series

Further analysis of the series allows us to find the different operators participating in it, several of its simplifying properties, and the next correction terms in our approximation.

A simple extension of the analysis performed so far for the third term in expansion (13) shows that each term of given order in $\mathbf{V_S}$ can be separated into

$$(\text{permutations of } \mathbf{V_S}) + (\text{commutators of } \mathbf{V_S}), \quad (42)$$

and whereas we have taken in our first approximation in Eq. (16) all permutation terms, Eqs. (38)–(41) describe the first contribution to the second type of term.

In the Appendix we calculate the products among all the nonorthogonalized operators appearing in Eqs. (33) and (40), which are derived from products of the potential $\mathbf{V}_{NN}$ in Eq. (6) at different points. We deduce that the multiplication of all terms lead to the same ones, that is, the algebra is closed. In particular, $\boldsymbol{\tau}_r$ in Eq. (40) anticommutes with the first two non-trivial components in $\boldsymbol{\tau}^{(0)}$ in Eq. (33) and commutes with the others. From this information and following the example of $G^{(0)}_{\text{eik}}$ we shall assume the conjecture (valid locally) that the general solution to the tensor eikonal formula can be expressed in terms of an exponential,

$$\langle \mathbf{r}' | G^{(T)}_{\text{eik}} | \mathbf{r} \rangle = -\frac{i}{v}\theta_{z',z}\delta_{\mathbf{b}',\mathbf{b}} e^{iq(z'-z)} e^{i\boldsymbol{\tau}^{(T)}}, \quad (43)$$

with $\boldsymbol{\tau}^{(T)}$ of the form

$$\boldsymbol{\tau}^{(T)} = \chi_{c'} + \chi_{u'}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_2 + \chi_{s'}\hat{\mathbf{s}}'\cdot\boldsymbol{\sigma}_1\hat{\mathbf{s}}'\cdot\boldsymbol{\sigma}_2 + \chi_{t'}\hat{\mathbf{t}}'\cdot\boldsymbol{\sigma}_1\hat{\mathbf{t}}'\cdot\boldsymbol{\sigma}_2 + \chi_{r'}\hat{\mathbf{u}}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2), \quad (44)$$

that is, $\boldsymbol{\tau}^{(T)}$ contains the set of specific operators spanned by those contained in $\boldsymbol{\tau}^{(0)}$ in Eq. (33) and $\boldsymbol{\tau}_r$ in Eq. (40) (for some new orthogonalized vectors $\mathbf{s}'$, $\mathbf{t}'$ and coefficients $\chi_{c'}$, $\chi_{u'}$, $\chi_{s'}$, $\chi_{t'}$, $\chi_{r'}$, which are functionals of the potential).

To propose an improved approximation to second order in $\mathbf{V_S}$ we recall that a first useful separation was made when we realized that the first two terms in Eq. (6) commute with the tensor part, and the tensor contribution was partially resumed in terms of an exponential, which led to $G^{(0)}_{\text{eik}}$ in Eq. (16). We now suggest

$$\langle \mathbf{r}' | G^{(1)}_{\text{eik}} | \mathbf{r} \rangle = -\frac{i}{v}\theta_{z',z}\delta_{\mathbf{b}',\mathbf{b}} e^{iq(z'-z)} e^{i\boldsymbol{\tau}^{(1)}}, \quad (45)$$

with all five terms included in exponential $e^{i\boldsymbol{\tau}^{(1)}}$, where

$$\boldsymbol{\tau}^{(1)} = \boldsymbol{\tau}^{(0)} + \chi_r\hat{\mathbf{u}}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2). \quad (46)$$

In this construction we have improved the previous approximation $G^{(0)}_{\text{eik}}$ by taking into account the contribution $\boldsymbol{\tau}_r$ in Eqs. (40), (41), and our approximation is valid to order $\mathbf{V_S}^2$. The exponential form in Eq. (45) is suggested since it contains $G^{(0)}_{\text{eik}}$, it simulates the conjecture $G^{(T)}_{\text{eik}}$, and it also reproduces some aspects of the exact solution as it sums over some terms in the series. For example, to order $(\mathbf{V_S})^3$ we have cancellations of some commutators of $\mathbf{V_S}$ in Eq. (42), some of which are reproduced by the putative anticommutative property of the components of $\boldsymbol{\tau}^{(1)}$.

To obtain an explicit expression for $G^{(1)}_{\text{eik}}$ in terms of scalar functions we separate the arguments of $\boldsymbol{\tau}^{(1)}$ in Eq. (46), in such a way that we divide the exponential in Eq. (45) into one part containing the commuting terms with coefficients χ_c and χ_u, and another one with the others [$\boldsymbol{\tau}^{(0)}$ is given in Eq. (33)]. The latter exponential can be expanded using the anticommutativity of the χ_r term with the χ_s, χ_t terms, using the product relations in the Appendix, and collecting terms around the two projection operators

$$\mathbf{P}_{u+} = \frac{1}{2}(1+\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_2), \quad \mathbf{P}_{u-} = \frac{1}{2}(1-\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_2). \quad (47)$$

We get

$$e^{\boldsymbol{\tau}^{(1)}} = \Gamma_c + \Gamma_u\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_2 + \Gamma_s\hat{\mathbf{s}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{s}}\cdot\boldsymbol{\sigma}_2 + \Gamma_t\hat{\mathbf{t}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{t}}\cdot\boldsymbol{\sigma}_2 + \Gamma_r\hat{\mathbf{u}}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2), \quad (48)$$

where

$$\Gamma_c = \frac{e^{i\chi_c}}{2}(e^{i\chi_u}c_d + e^{-i\chi_u}c_f),$$

$$\Gamma_u = \frac{e^{i\chi_c}}{2}(e^{i\chi_u}c_d - e^{-i\chi_u}c_f),$$

$$\Gamma_r = i e^{i(\chi_c+\chi_u)} s_d \chi_r / d,$$

$$\Gamma_s = \frac{i e^{i\chi_c}}{2}[e^{i\chi_u}s_d(\chi_s-\chi_t)/d + e^{-i\chi_u}s_f(\chi_s+\chi_t)/f],$$

$$\Gamma_t = \frac{i e^{i\chi_c}}{2}[-e^{i\chi_u}s_d(\chi_s-\chi_t)/d + e^{-i\chi_u}s_f(\chi_s+\chi_t)/f], \quad (49)$$

and

$$d = (4\chi_r^2 + \chi_s^2 + \chi_t^2 - 2\chi_t\chi_s)^{1/2},$$

$$f = (\chi_s^2 + \chi_t^2 + 2\chi_t\chi_s)^{1/2}. \quad (50)$$

We observe that this procedure can similarly be applied to obtain an explicit expression for $G^{(T)}_{\text{eik}}$ in Eq. (44), or any other expression of this form. In the $G^{(T)}_{\text{eik}}$ case, the five independent functions of Eq. (48) prove the eikonal solution can be expressed in the form of Eq. (43), at least locally.

C. Spin-orbit term

The spin-orbit (SO) term is also an important contribution in the nuclear force, beyond the six local parts already considered in Eq. (6). This has the form

$$\mathbf{V}_{\text{SO}}(\mathbf{r}) = V_{\text{SO}}(r)\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) = V_{\text{SO}}(r)\mathbf{r}\times\mathbf{p}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2). \quad (51)$$

In the context of the eikonal formula the spin-orbit part of the potential can be approximated by dropping the momentum dependence when, after writing $\mathbf{p}=\mathbf{q}+(\mathbf{p}-\mathbf{q})$, we eliminate the lower order contribution in q. The eikonal formula that includes both parts of the potential can then be written

$$G'_{\rm eik}=\frac{1}{v}\,\frac{1}{q-\hat{\mathbf{q}}\cdot\mathbf{p}-[\mathbf{V}_{NN}+V_{\rm SO}(r)bq\hat{\mathbf{u}}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)]/v+i\epsilon}, \tag{52}$$

where we have used $\mathbf{r}\times\mathbf{q}=\mathbf{b}\times\mathbf{q}=bq\hat{\mathbf{u}}$.

The same procedure can be followed as to arrive to Eq. (16) by expanding the Green's function in Eq. (52) in terms of G'_0. Similarly, we need to examine the commutator between the expanded potential operators which enter the expansion series as in Eq. (13):

$$[\mathbf{V}_S(\mathbf{r}_a)+\mathbf{V}^q_{\rm SO}(\mathbf{r}_a),\mathbf{V}_S(\mathbf{r}_b)+\mathbf{V}^q_{\rm SO}(\mathbf{r}_b)], \tag{53}$$

with $\mathbf{V}^q_{\rm SO}=V_{\rm SO}(\mathbf{r})bq\hat{\mathbf{u}}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$. This commutator is composed of the contribution in Eq. (15) and the remaining terms

$$\begin{aligned}[\mathbf{V}_S(\mathbf{r}_a),\mathbf{V}^q_{\rm SO}(\mathbf{r}_b)]+[\mathbf{V}^q_{\rm SO}(\mathbf{r}_a),\mathbf{V}_S(\mathbf{r}_b)]\\ =2iqb\Bigg[2b\Bigg(\frac{V_S(r_a)V_{\rm SO}(r_b)}{b^2+z_a^2}z_a\\ -\frac{V_S(r_b)V_{\rm SO}(r_a)}{b^2+z_b^2}z_b\Bigg)(\mathbf{t}_{bb}-\mathbf{t}_{zz})\\ +\Bigg(\frac{V_S(r_a)V_{\rm SO}(r_b)}{b^2+z_a^2}(z_a^2-b^2)\\ -\frac{V_S(r_b)V_{\rm SO}(r_a)}{b^2+z_b^2}(z_b^2-b^2)\Bigg)\mathbf{t}_{bz}\Bigg],\end{aligned} \tag{54}$$

where we have expressed the spin operators with the help of the definitions in Eq. (A1) of the Appendix. Recalling our argument that relevant values of the coordinates are $b\gg|z|$, $b\gg|z'|$, so that $r_a\sim r_b$, Eq. (54) suggests that it can be discarded in a first approximation. An analytic approximated expression is consequently obtained for the Green's function $G'_{\rm eik}$ in Eq. (52) by summing permutation contributions which now include terms with $V_{\rm SO}(r)bq\hat{\mathbf{u}}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$. Using the previously developed formalism, an expression for $G'_{\rm eik}$ is obtained by taking now for the last term in Eq. (46)

$$\chi_r\to\chi_{r''}=\chi_r-\frac{bq}{v}\int_z^{z'}d\zeta V_{\rm SO}(\mathbf{b},\zeta\hat{\mathbf{z}}), \tag{55}$$

and using the corresponding expressions in Eqs. (45)–(50).

V. ON-SHELL SCATTERING AMPLITUDES

The scattering amplitude is related to the t matrix by the equation $f=-(m/2\pi)t$, and its on-shell elements contain direct information on the differential cross section. It is possible to use a simple prescription to obtain the t matrix corresponding to the G approximations that we have proposed. To find this prescription we consider the expansion of the eikonal t, similarly to that of $G_{\rm eik}$ in Eq. (12), we evaluate it at on-shell momenta, and we compare it with an expression that can be derived for $G_{\rm eik}$.

We first use Eq. (2), which gives the t matrix in terms of G and we evaluate in momentum space

$$\begin{aligned}\langle\mathbf{k}_f|t_{\rm eik}|\mathbf{k}_i\rangle=\int d^2b\exp(-i\boldsymbol{\Delta}\cdot\mathbf{b})\int dz\mathbf{V}_{NN}(z)\\ \times\left[1+\int dz'G_{\rm eik}(z,z')\mathbf{V}_{NN}(z')\right]\end{aligned} \tag{56}$$

(in its generalization to the spin case), where the definitions for the momenta in Eq. (3) were used, and only the z part of the coordinate arguments were written. When we substitute Eq. (12) into Eq. (56), a Born-like expansion of t is obtained, of the form

$$\begin{aligned}\langle\mathbf{k}_f|t_{\rm eik}|\mathbf{k}_i\rangle=\langle\mathbf{k}_f|\mathbf{V}_{NN}+\mathbf{V}_{NN}G_0\mathbf{V}_{NN}\\ +\mathbf{V}_{NN}G_0\mathbf{V}_{NN}G_0\mathbf{V}_{NN}+\cdots|\mathbf{k}_i\rangle.\end{aligned} \tag{57}$$

We now realize that series (57) in momentum space is reproduced when in the formal expression for $G_{\rm eik}$ in Eq. (12) in coordinate space, one sets $z'\to\infty$, $z\to-\infty$ (which cancels the two external theta functions), subtracts the zero order term in $\mathbf{V_{NN}}$, multiplies by $(iv)^2$, and Fourier transforms by integrating over the $\mathbf{b}$ variable, with the transverse momentum $\boldsymbol{\Delta}$ as argument.

Applying this to the expression for $G^{(T)}_{\rm eik}$ in Eq. (43), for example, we get

$$\langle\mathbf{k}_f|t^{(T)}_{\rm eik}|\mathbf{k}_i\rangle=iv\int d^2b\exp(-i\boldsymbol{\Delta}\cdot\mathbf{b})[e^{i\boldsymbol{\tau}^{(T)}}-1]. \tag{58}$$

In fact, similar expressions can be obtained for $t^{(0)}_{\rm eik}$, $t^{(1)}_{\rm eik}$ by choosing $\boldsymbol{\tau}^{(0)}$, $\boldsymbol{\tau}^{(1)}$, respectively, with the prescription for z, z' above, or any other possible $\boldsymbol{\tau}$ term describing an approximation to the solution.

We consider the scattering amplitude from $t^{(1)}_{\rm eik}$, for which we use $\boldsymbol{\tau}^{(1)}$ in Eq. (46). We find that among the different coefficient components of $\boldsymbol{\tau}^{(1)}$ in Eqs. (33) and (34), $\chi_{S,1}$ in Eq. (21) vanishes because of the symmetry in the potential so that we no longer need to rotate the $\boldsymbol{\sigma}$'s, and we have $\hat{\mathbf{s}}=\hat{\mathbf{b}}$, $\hat{\mathbf{t}}=\hat{\mathbf{z}}$. The final expression for the on-shell scattering amplitude corresponding to $t^{(1)}_{\rm eik}$ is

$$\begin{aligned}f^{(1)}_{\rm eik}(\theta,q)=\alpha+\beta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}_2+\gamma\hat{\mathbf{n}}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\\ +\delta\hat{\boldsymbol{\Delta}}\cdot\boldsymbol{\sigma}_1\hat{\boldsymbol{\Delta}}\cdot\boldsymbol{\sigma}_2+\epsilon\hat{\mathbf{q}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{q}}\cdot\boldsymbol{\sigma}_2,\end{aligned} \tag{59}$$

where we use $\hat{\mathbf{n}}=\hat{\boldsymbol{\Delta}}\times\hat{\mathbf{q}}$ and

$$\alpha=-\frac{iq}{2\pi}\int db^2\exp(-i\boldsymbol{\Delta}\cdot\mathbf{b})[\Gamma^{\rm on}_c(\mathbf{b})-1], \tag{60}$$

$$\begin{aligned}\beta=-\frac{iq}{2}\int_0^\infty db\ b\{[J_0(\Delta b)+J_2(\Delta b)]\Gamma^{\rm on}_s(\mathbf{b})\\ +[J_0(\Delta b)-J_2(\Delta b)]\Gamma^{\rm on}_u(\mathbf{b})\},\end{aligned}$$

$$\epsilon=-\frac{iq}{2\pi}\int db^2\exp(-i\boldsymbol{\Delta}\cdot\mathbf{b})\Gamma^{\rm on}_t(\mathbf{b}),$$

$$\delta=-\frac{iq}{2}\int_0^\infty db\ b[J_0(\Delta b)-J_2(\Delta b)]\Gamma_s^{\rm on}(\mathbf{b})$$

$$+[J_0(\Delta b)+J_2(\Delta b)]\Gamma_u^{\rm on}(\mathbf{b}),$$

$$\gamma=-q\int_0^\infty db\ bJ_1(\Delta b)\Gamma_r^{\rm on}(\mathbf{b}).$$

In the last integrals we have taken into account the dependence of $\hat{\mathbf{u}}$ on $\hat{\mathbf{b}}$ and we have used the Bessel functions

$$J_n(z)=\frac{i^{-n}}{\pi}\int_0^\pi d\theta\ e^{iz\cos(\theta)}\cos(n\theta), \tag{61}$$

and now

$$\Gamma_c^{\rm on}=\frac{e^{i\chi_c}}{2}(e^{i\chi_\sigma}c_{d'}+e^{-i\chi_\sigma}c_{f'}),$$

$$\Gamma_u^{\rm on}=\frac{e^{i\chi_c}}{2}(e^{i\chi_\sigma}c_{d'}-e^{-i\chi_\sigma}c_{f'}),$$

$$\Gamma_r^{\rm on}=ie^{i(\chi_c+\chi_\sigma)}s_{d'}\chi_r/d',$$

$$\Gamma_s^{\rm on}=\frac{ie^{i\chi_c}}{2}[e^{i\chi_\sigma}s_{d'}(\chi_{S,0}-\chi_{S,2})/d'$$

$$+e^{-i\chi_\sigma}s_{f'}(\chi_{S,0}+\chi_{S,2}+2\chi_\sigma)/f'],$$

$$\Gamma_t^{\rm on}=\frac{ie^{i\chi_c}}{2}[-e^{i\chi_\sigma}s_{d'}(\chi_{S,0}-\chi_{S,2})/d'$$

$$+e^{-i\chi_\sigma}s_{f'}(\chi_{S,0}+\chi_{S,2}+2\chi_\sigma)/f'], \tag{62}$$

with

$$d'=(4\chi_r^2+\chi_{S,0}^2+\chi_{S,2}^2-2\chi_{S,2}\chi_{S,0})^{1/2},$$

$$f'=[\chi_{S,0}^2+\chi_{S,2}^2+2\chi_{S,2}\chi_{S,0}+4\chi_\sigma(\chi_{S,0}+\chi_{S,2})+4\chi_\sigma^2]^{1/2}. \tag{63}$$

For the on-shell $\Gamma_i^{\rm on}$ we take $\chi_{S,l}=\chi_{S,l}(\mathbf{b},\infty,-\infty)$, with the notation of Eq. (21), and similarly for χ_σ, χ_c, and χ_r [or its generalization to the spin-orbit potential $\chi_{r''}$ in Eq. (55)].

VI. SUMMARY

In this work we have investigated the structure of the eikonal solution for tensor forces, and this has led us to construct approximated solutions suited to the high energy regime. We have used the fact that the relevant configurations in this regime are found at values of the coordinates $(\mathbf{b},\zeta\hat{\mathbf{z}})$ with $|\zeta|\ll b$, implying a nearly parallel set of configurations, and the resulting solution leads to analytical expressions in terms of the potential $\mathbf{V}_{NN}$ in Eq. (6), valid in the forward direction and nearby angles. The main result of this paper is a closed-form eikonal approximated formula for the scattering amplitude, which is given in Eqs. (59)–(63), with the coefficients χ defined in Eqs. (17), (21), and (41). The expression contains information for each of the five expected spin amplitudes as a function of the three local potential components in $\mathbf{V}_{NN}$ and an additional spin-orbit contribution. From the formalism developed it should be possible to obtain further corrections systematically.

While we have argued in this paper that the approximations should be valid in the high energy, small angle regime, it would be interesting to consider the outcome of this approximation for a given potential with an exact solution with realistic parameters and try to compare them with data as, for example, the case of high energy nucleon-nucleon interaction, for which one can establish the threshold and ranges of validity of this approximation. This work is currently in progress.

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APPENDIX

We present here the operators that are generated from a local tensor potential, which appear in the eikonal formula. By displaying their products, we show that they form a closed algebra. The list of the relevant operators is

$$\mathbf{t}_{bb}=\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}_2,$$

$$\mathbf{t}_{zz}=\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}_2,$$

$$\mathbf{t}_{bz}=\sigma_{1\hat{\mathbf{b}}}\sigma_{2\hat{\mathbf{z}}}+\sigma_{1\hat{\mathbf{z}}}\sigma_{2\hat{\mathbf{b}}},$$

$$\mathbf{t}_{uu}=\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}_2,$$

$$\mathbf{S}_u=\hat{\mathbf{u}}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2). \tag{A1}$$

These operators appear, for example, in Eqs. (26) and (44) and they commute or anticommute among themselves. The products among commuting operators are

$$\mathbf{t}_{bb}\mathbf{t}_{zz}=\mathbf{t}_{zz}\mathbf{t}_{bb}=-\mathbf{t}_{uu},$$

$$\mathbf{t}_{uu}\mathbf{t}_{bb}=\mathbf{t}_{bb}\mathbf{t}_{uu}=-\mathbf{t}_{zz},$$

$$\mathbf{t}_{zz}\mathbf{t}_{uu}=\mathbf{t}_{uu}\mathbf{t}_{zz}=-\mathbf{t}_{bb},$$

$$\mathbf{t}_{uu}\mathbf{S}_u=\mathbf{S}_u\mathbf{t}_{uu}=\mathbf{S}_u,$$

$$\mathbf{t}_{bz}\mathbf{t}_{uu}=\mathbf{t}_{uu}\mathbf{t}_{bz}=\mathbf{t}_{bz}, \tag{A2}$$

and those for anticommuting ones are

$$-\mathbf{t}_{bb}\mathbf{S}_u=\mathbf{S}_u\mathbf{t}_{bb}=i\mathbf{t}_{bz},$$

$$-\mathbf{t}_{zz}\mathbf{S}_u=\mathbf{S}_u\mathbf{t}_{zz}=-i\mathbf{t}_{bz},$$

$$-\mathbf{S}_u\mathbf{t}_{bz}=\mathbf{t}_{bz}\mathbf{S}_u=2i(\mathbf{t}_{bb}-\mathbf{t}_{zz}),$$

$$-\mathbf{t}_{bz}\mathbf{t}_{bb}=\mathbf{t}_{bb}\mathbf{t}_{bz}=i\mathbf{S}_u,$$

$$-\mathbf{t}_{bz}\mathbf{t}_{bz}=\mathbf{t}_{zz}\mathbf{t}_{bz}=-i\mathbf{S}_u. \tag{A3}$$

The operators described above generate a $u(1)\times u(1)\times su(2)$ algebra, as can be seen by considering the combinations $\mathbf{t}_{bb}+\mathbf{t}_{zz}$, $\mathbf{t}_{uu}$, which commute with all, and $\mathbf{t}_{bz}$, $\mathbf{S}_u$, $\mathbf{t}_{bb}-\mathbf{t}_{zz}$, which form an su(2) algebra.

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