Solutions without preacceleration to the one-dimensional Lorentz-Dirac equation

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An approach to the problem of violation of causality in classical electrodynamics is proposed. This approach is based on the construction of exact analytic solutions without preacceleration to the Lorentz-Dirac equation in an electrostatic field that vanishes identically outside a certain region. Exact solutions are given for the potential well and the linear potential wall, cases for which Plass [Rev. Mod. Phys. **33**, 37 (1961)] already found the corresponding preaccelerative solutions. In addition, an exact solution which differs from the one found by Plass is given for a thin infinite charged plate. Finally, an exact solution is constructed for a special electrostatic field. All these nonpreaccelerative solutions have a jump in the acceleration at points where the electrostatic field has a jump; this implies that they cannot be obtained as solutions to the usual integro-differential equation associated to the Lorentz-Dirac equation. [S1050-2947(97)02404-9]

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I. INTRODUCTION

Early in this century, Abraham and Lorentz built the equation of motion for a charged particle taking into account the radiation reaction. Lorentz's work can be found in Jackson's book [1], and leads to the following equation of motion for a charged particle of mass m and charge e:

$$m\left(\frac{d\mathbf{v}}{dt}-\tau_0\frac{d^2\mathbf{v}}{dt^2}\right)=\mathbf{F}, \tag{1}$$

where $\tau_0=2e^2/3mc^3$, c is the velocity of light, $\mathbf{v}$ denotes the velocity of the charge, and $\mathbf{F}$ is the external force; in addition, factor $\frac{4}{3}$ in front of m has been changed to 1, which is the correct value.

In Lorentz's time, relativity theory was incipient, and a fully relativistic formulation of the equation of motion was only achieved in 1938 by Dirac in his classic paper [2]. The Lorentz-Dirac equation reads

$$ma^{\mu}=(e/c)F^{\mu v}u_v+\Gamma^{\mu}, \tag{2}$$

with

$$\Gamma^{\mu}\equiv(2e^2/3c^3)(\dot{a}^{\mu}-a^{\lambda}a_{\lambda}u^{\mu}/c^2), \tag{3}$$

where the charge world line $z_{\mu}(\tau)$ is parametrized by its proper time τ, and $u_{\mu}=dz_{\mu}/d\tau$, $a_{\mu}=du_{\mu}/d\tau$, and $\dot{a}_{\mu}=da_{\mu}/d\tau$. Moreover, Greek indices range from 0 to 3, and the diagonal metric of Minkowski space is $(-1,1,1,1)$. Equation (2), as well as the main basic research associated with the classical theory of a point charge, is considered in detail in Rohrlich's book [3]. The term $(e/c)F^{\mu v}u_v$ in Eq. (2) is the well-known Lorentz force due to the external field $F^{\mu v}$. In addition, Γ^{μ} represents the effect of radiation, and consists of the Schott term given by the proper-time derivative of the four-acceleration a_{μ} and the Larmor nonlinear term $a^{\lambda}a_{\lambda}u^{\mu}/c^2$.

In his paper, Dirac highlights undesired aspects of Eq. (2), namely, the self-acceleration and preacceleration effects. Self-acceleration refers to solutions where the charge is under acceleration even in the absence of an external field. Preacceleration means that the charge begins to accelerate before the force is actually applied. Dirac [2] illustrated the existence of preacceleration by studying Eq. (1) in the case of an electron disturbed by a force represented by a δ function. The simplicity of the nonrelativistic equation (1) is such that it can be integrated immediately with the help of the integrating factor e^{t/τ_0} for a rather general time-dependent force $\mathbf{F}(t)$. The result is [1,3–5]

$$m\frac{d\mathbf{v}}{dt}=\int_0^{\infty}e^{-s}\mathbf{F}(t+\tau_0 s)ds. \tag{4}$$

This procedure allows a rather natural incorporation of Dirac's asymptotic condition on the vanishing of the acceleration for an asymptotically free particle. In fact, if $\mathbf{F}(t)$ vanishes identically for a large value of t, then Eq. (4) shows that the acceleration also vanishes for a large value of t and therefore solution (4) is not self-accelerating. In the fully relativistic case, this method leads to an integro-differential equation [3], but, as Rohrlich pointed out (see Ref. [3], p. 147), in this case the asymptotic condition is not ensured by the integrodifferential equation.

The existence of preacceleration can be easily seen from Eq. (4). For example, if we consider a time-dependent force $\mathbf{F}(t)$ that vanishes identically for $t<0$ and has a constant value for $t>0$, then it is evident that the acceleration is different from zero for $t<0$. Following Dirac's work, the problem of preacceleration has received a great deal of attention in the literature [1–6]. The violation of causality implied by preacceleration is particularly disappointing since the Lorentz-Dirac equation (2) can be derived by using only retarded fields [7]. Mainly due to this problem, some authors have proposed alternative equations of motion without the time derivative of the acceleration (see Ref. [5]), because it has been considered that the presence of this term in Eqs. (1) and (2) is what leads to preacceleration. As we will clearly show here, the existence of preacceleration is not a consequence of the presence of the third derivative, but of the method through which the solution has been obtained. The standard procedure that leads to Eq. (4), or to the integro-

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'differential equation in the relativistic case, does not allow for a solution without preacceleration.

Recently, Valentini [8] and Comay [9] have provided insights into preacceleration and self-acceleration effects. Valentini noted that there is no justification for imposing the validity of the Lorentz-Dirac equation in nonanalytical points of the charge world line $z_\mu(\tau)$. On the other hand, Comay studied the asymptotic properties of the Lorentz-Dirac equation for a one-dimensional motion in an electrostatic field that behaves as x^{-n}, with $n \geqslant 2$, for large values of x. He concluded that Eq. (2) admits an asymptotically inertial solution, that is, a solution without self-acceleration when x tends to infinity.

The nonlinear character of Eq. (2) makes the construction of exact analytical solutions a very difficult task. This explains why only very few exact solutions have been obtained, most of them by Plass [4]. This author gives a solution for the rectilinear motion of a charged particle under a time-dependent force, as well as the solutions for the rectilinear motion of a charge under a space-dependent force in the cases of the potential well and the linear potential wall. All Plass's solutions present preacceleration, which led him to conclude that the phenomenon of preacceleration always occurs when radiative reaction is considered [10]. As we will show in this paper, this conclusion is incorrect. In fact, we construct exact analytic solutions without preacceleration for the potential well and the linear potential wall. We also present an exact solution without preacceleration for the motion of a charged particle moving in an electrostatic field given (except for a factor) by $\sinh(x/r_0)\{\frac{3}{2}-\cosh(x/r_0)\}$, with $r_0 = e^2/mc^2$. This last case is of special relevance because the nonlinear radiative reaction term makes a nontrivial contribution to the solution. The present solutions are such that the acceleration experiences a jump at a point where the external force presents a jump. This differs from the solution obtained from the usual integro-differential formulation of Eq. (2), where the acceleration is necessarily continuous around a finite jump in the external force.

II. PREACCELERATION AND SMOOTHNESS OF THE SOLUTION

In order to see in a simple way the difference between the present approach and the integral representation (4) of Eq. (1), it is advisable to consider the example studied in Jackson's book, where a constant force of magnitude F_0 pointing along the X axis is applied to the particle for time $t>0$, while for $t<0$, the force vanishes identically. In this case, from Eq. (4) it is easy to see that for $t \leqslant 0$ we have

$$\frac{dv}{dt} = \frac{F_0}{m} e^{t/\tau_0}, \tag{5}$$

while for $t \geqslant 0$, the acceleration dv/dt has a constant value equal to F_0/m. In particular, the acceleration is continuous everywhere, including $t=0$, where the force has a jump. The continuity of the acceleration at a finite jump in the force is a rather general property of the solution in Eq. (4), because a finite jump in $\mathbf{F}(t)$ is smoothed over by the integration process. The acceleration given by Jackson is certainly a solution of the equation of motion (1) for $t<0$, and also for $t>0$. But this problem makes room for another solution; in fact, $dv/dt \equiv 0$ is a solution of Eq. (1) for $t<0$, and $dv/dt = F_0/m$ is a solution for $t>0$. This solution is, of course, different from Jackson's and free of the preacceleration effect. This is the solution that we are going to choose as *the* physical solution from here on. Unlike Jackson's solution, the solution $dv/dt=0$ for $t<0$, and $dv/dt=F_0/m$ for $t>0$, is certainly discontinuous at $t=0$. Thus we are going to solve the problem of preacceleration at the expense of introducing singularities in the acceleration at points where the force has a jump. At first sight, it would seem that we are simply exchanging one problem for another. However, a solution with preacceleration is of a completely different nature compared to one without preacceleration but with discontinuities.

The introduction of singularities in the acceleration represents no real physical problem. For example, the radiation rate, which is quadratic in acceleration, is not defined at the jump in the force. But, in this case, the relevant physical quantity is the energy radiated during a time interval, to which the contribution of the singularity point does not matter because the acceleration, although undefined at this point, is bounded. Moreover, the introduction of a discontinuous force in Jackson's example, or in the cases we consider here, corresponds to a kind of idealization, whose main usefulness is that it makes it possible to find exact analytic solutions. In reality, though, the forces have no jump and, in this case, we expect a solution without preacceleration and without acceleration discontinuities. However, in this case, Eq. (4) presents preacceleration anyway. This can be easily shown by considering an $F(t)$ that vanishes identically for $t \leqslant 0$, that is positive and continuous in the interval $0<t<t_1$, and vanishes identically for $t \geqslant t_1$. Then, for any $t<0$, the integrand of Eq. (4) is positive in the interval $-t/\tau_0 < s < (t_1 - t)/\tau_0$. Therefore the acceleration dv/dt is positive for all $t<0$, notwithstanding that for these values of t, $F(t)$ is identically zero.

The retarded-field solution of Maxwell's equations is completely causal; for this reason, the existence of preacceleration effects, no matter how small, is intrinsically contradictory in classical electrodynamics. Thus to preserve causality we must consider the introduction of discontinuities in the acceleration at points where the force has a jump. On the other hand, the introduction of singularities in the acceleration is nothing new in physics, because this is exactly what happens in Newton's equation $m d\mathbf{v}/dt = \mathbf{F}$.

The usual procedure where the differential equation is integrated with the help of the integrating factor e^{t/τ_0} is such that self-accelerating solutions are discarded. Unfortunately, and although mathematically correct, this approach leads to preacceleration. In fact, the acceleration (5) in the free-force region arises precisely from this integrating factor. Thus we must abandon this approach if we pretend to find a solution without preacceleration. We will choose an inertial motion in any region where the external force vanishes identically and work directly with the Lorentz-Dirac equation (2) elsewhere. Such a procedure will generate a solution different from the one that follows from the integro-differential equation. The lack of uniqueness of the Lorentz-Dirac equation was found earlier by Baylis and Huschilt [11] in a specific example. This lack of uniqueness is also present in the example considered in Jackson's book as well as in the solutions for the potential well and the linear potential wall.

III. THE POTENTIAL WELL

Here we will consider in some detail the motion of a positive charge along a straight line, the X axis, in the presence of an external electrostatic field that depends only on variable x and points along the X axis, vanishing identically for $x<x_1$ and $x>x_2$. This field automatically satisfies the equation $\boldsymbol{\nabla}\times\mathbf{E}=0$, and can be considered as generated by a set of sheets orthogonal to the X axis, each with a uniform density of charge given by the remaining Maxwell equation $\rho(x)=(1/4\pi)dE/dx$. This type of force is chosen here so that the whole issue of preacceleration becomes particularly transparent. The electric field $E(x)$ and the charge density $\rho(x)$ will be considered simply as mathematical functions, disregarding the practical and experimental aspects.

Now, since by assumption the electric field $E(x)$ vanishes identically for $x<x_1$ and $x>x_2$, the charge density $\rho(x)$ must be such that

$$\int_{x_1}^{x_2}\rho(x)dx=0. \tag{6}$$

In what follows, $\overline{x}$ will denote the position of the charge at time t, while x will denote the coordinate of a general point in the X axis. Let v be the ordinary charge velocity, i.e., $v=d\overline{x}/dt$, and

$$\gamma=(1-v^2/c^2)^{-1/2} \tag{7}$$

be the usual relativistic factor. For the sake of simplicity, the velocity v will be assumed as always positive. In this case, the proper-time derivatives in Eq. (2) can be changed to derivatives with respect to $\overline{x}$. This transformation is quite natural, since we will be working with an electric field which depends on position instead of time. The change of variable is possible since, by assumption, $d\overline{x}/d\tau=v\gamma$ never vanishes, thus making the correspondence between $\overline{x}$ and τ a one-to-one relation. For a motion along the X axis, the four-velocity u^μ in Eq. (2) only has the components

$$u^0=c\gamma, \tag{8}$$

$$u^1=c(\gamma^2-1)^{1/2}. \tag{9}$$

Then, it is easy to see that in terms of the charge position $\overline{x}$, the two radiation reaction terms in Eq. (2) become

$$\dot{a}^1=c^3(\gamma^2-1)^{1/2}\gamma(d^2\gamma/d\overline{x}^2)+c^3(\gamma^2-1)^{1/2}(d\gamma/d\overline{x})^2 \tag{10}$$

and

$$a^\lambda a_\lambda u^1/c^2=c^3(\gamma^2-1)^{1/2}(d\gamma/d\overline{x})^2. \tag{11}$$

Hence Eq. (2) takes the form

$$(\gamma^2-1)^{1/2}\gamma''-(\tfrac{3}{2})\gamma'+\overline{E}=0, \tag{12}$$

where $\gamma''=d^2\gamma/d\xi^2$, $\gamma'=d\gamma/d\xi$, with ξ the dimensionless variable defined by $\xi=\overline{x}/r_0$, $r_0=e^2/mc^2$, and $\overline{E}=(3r_0^2/2e)E(\overline{x})$. Equation (12) was originally derived by Plass [4] as the energy equation, but this author did not use it to build solutions for Eq. (2).

Equation (12) is very appropriate for building solutions by assuming a reasonable $\gamma(\xi)$, and then looking for the field $\overline{E}$ for which this $\gamma(\xi)$ is the solution. For example, if we try $\gamma=a\xi+b$ with a and b constant, then $\gamma''=0$, and Eq. (12) is satisfied by $\overline{E}=(\tfrac{3}{2})a$. We can build the electric field $E(x)=(e/r_0^2)a=E_0>0$, for x in $(-l,l)$, by means of two sheets located at $x=-l$ and $+l$, with uniform surface density charges $+E_0/4\pi$ and $-E_0/4\pi$, respectively. In this case, the electric field for $x<-l$ and $x>l$ vanishes identically, and $\gamma(\xi)=(r_0^2E_0/e)\xi+b$ is not a solution to Eq. (12) in these regions. But, for $\xi<-l/r_0$ and $\xi>l/r_0$, $\gamma=\text{const}$ is a solution, as it comes immediately from Eq. (12). Now, at $x=-l$ the electric field has a jump, but the function $\gamma(\overline{x})$ basically represents the kinetic energy of the charge, which from a physical point of view can vary only continuously. Therefore at $x=-l$ the solution $\gamma(\overline{x})=(r_0E_0/e)\overline{x}+b$ must coincide with the value γ_{in} that $\gamma(\overline{x})$ has in the field-free region $x<-l$. Thus the solution for any x reads

$$\gamma(\overline{x})=\begin{cases}\gamma_{\text{in}} & \text{for } \overline{x}\leqslant -l,\\ \gamma_{\text{in}}+(eE_0/mc^2)(\overline{x}+l) & \text{for } -l\leqslant\overline{x}\leqslant l,\\ \gamma_{\text{out}}=\gamma_{\text{in}}+2elE_0/mc^2 & \text{for } \overline{x}\geqslant l.\end{cases} \tag{13}$$

Solution (13) for the potential well is, as in Plass [4], an exact solution of Eq. (2). However, our solution, as opposed to Plass's, is without preacceleration. According to Eq. (13), all the work done by the external field E_0 goes into the kinetic energy increase. On the other hand, Rohrlich's local radiation criterion [12] implies that the charge emits radiation in the interval $(-l,l)$, and, as this author points out [13], for uniform acceleration all the radiation comes from the Schott term (10). This fact can be directly seen from Eq. (13) because the Schott term (10) is exactly canceled out by the Larmor term (11).

Immediately to the right of $x=-l$, solution (13) fulfills the following boundary conditions:

$$\gamma=\gamma_{\text{in}} \tag{14}$$

and

$$\frac{d\gamma}{d\overline{x}}=\frac{eE_0}{mc^2}. \tag{15}$$

If E_0 tends to zero, the jump in the derivative of γ also tends to zero, and, as expected, Eq. (13) describes an inertial motion in this limit.

On physical grounds, the continuity of the kinetic energy of the charge, as expressed by Eq. (14), is a fundamental requirement. Therefore Eq. (14) is not restricted to the case of a jump in a homogeneous electric field; yet, it holds independently of the functional dependency of $E(x)$. However, as we will see in Secs. V and VI below, unlike Eq. (14), the boundary condition (15) does not hold for a general case. This condition certainly differs from the one associated with the usual preaccelerative solution, a case in which both the acceleration and the velocity are continuous across the jump.

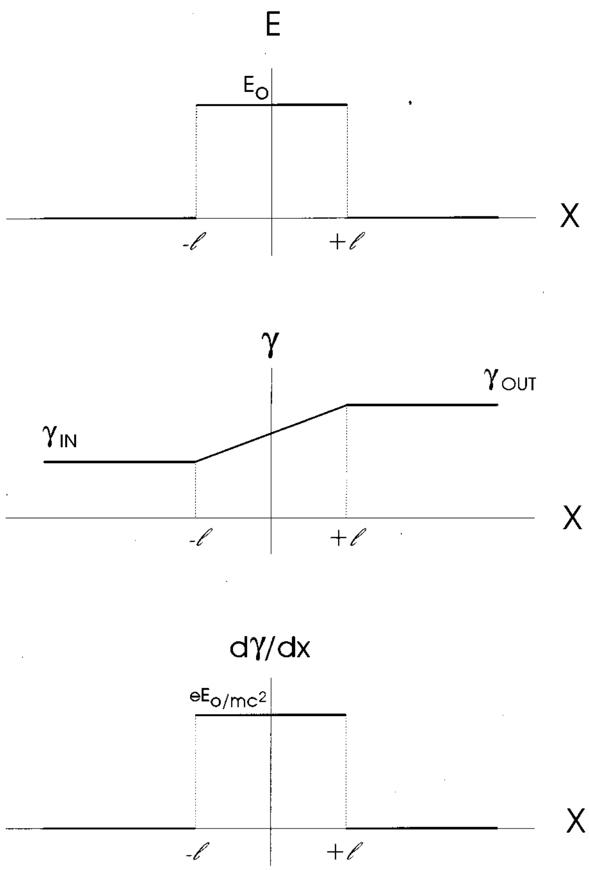

FIG. 1. The graphs depict the electric field $E(x)$, the function $\gamma(\bar{x})$, and $d\gamma/d\bar{x}$ as functions of $\bar{x}$ for the exact solution (13) without preacceleration corresponding to the potential well.

The differences between the two approaches can also be seen by analyzing Eq. (12) in more detail around the jump in E at x_1. If, as usual, we integrate Eq. (12) around the jump and assume that $\gamma(\bar{x})$ and $d\gamma/d\bar{x}$ behave well, we do not obtain any jump for $d\gamma/d\bar{x}$. This is precisely what happens with the preacceleration solution. But our solution (13) is singular at x_1, and formal handling to obtain the jump of $d\gamma/d\bar{x}$ cannot be justified. Thus this procedure fails to reproduce the jump in Eq. (15) in our case. As pointed out by Valentini [8], there is actually no justification for imposing the validity of the Lorentz-Dirac equation at point x_1. The acceleration is not well defined at this point and, consequently, the electromagnetic field of the charge is not defined at any space-time point whose retarded position corresponds to the charge located at x_1. The latter makes the derivations of Eq. (2) invalid at x_1, since they require the existence of derivatives of any order of the charge world line $z_\mu(\tau)$. Figure 1 illustrates the solution (13) for the potential well.

IV. THE LINEAR POTENTIAL WALL

Equation (12) can also be solved for the linear potential wall, that is, for an electrostatic field given by

$$E(x)=\begin{cases}0 & \text{for } x<0\\ E_0 & \text{for } x>0,\end{cases} \tag{16}$$

where E_0 is any positive number. Since for $x>0$ the electrostatic field has a fixed value E_0, the solution of Eq. (12) is such that $(d\gamma/d\bar{x})=eE_0/mc^2$, independently of $\bar{x}$. So in this case, the exact solution without preacceleration reads

$$\gamma=\begin{cases}\gamma_{\rm in} & \text{for } \bar{x}\leqslant 0\\ \gamma_{\rm in}+(eE_0/mc^2)\bar{x} & \text{for } \bar{x}\geqslant 0.\end{cases} \tag{17}$$

According to Eq. (17), when $\bar{x}$ tends to infinity, the charge velocity tends to the velocity of light. This derives from the fact that we are assuming a homogeneous electric field in the region $x>0$. It is easy to see that in the nonrelativistic limit, that is, when c tends to infinity, Eq. (17) reduces to the nonpreaccelerative solution of the example considered by Jackson and discussed in Sec. II of this paper.

The problem of the reflection of a charge due to a constant repulsive force is also immediate. In this case,

$$E(x)=\begin{cases}0 & \text{for } x<0\\ -E_0 & \text{for } x>0,\end{cases} \tag{18}$$

with E_0 an arbitrary positive number. We assume that the charge is initially moving to the right in the free-field region; so, solution (17) holds but with E_0 changed to $-E_0$. Now, since $(d\gamma/d\bar{x})=-(eE_0/mc^2)<0$, the charge stops at point $x_s>0$ when γ equals 1; that is,

$$x_s=\frac{mc^2}{eE_0}(\gamma_{\rm in}-1). \tag{19}$$

At x_s the charge starts moving to the left, with γ increasing at the same rate at which it was decreasing when the charge was moving to the right. Thus when the charge returns to the origin $x=0$, it has exactly the same absolute velocity as the initial velocity at that same point. After the reflection, that is, in the region $x<0$, the value of γ is, of course, equal to $\gamma_{\rm in}$. In other words, the charge loses no energy in the reflection process. At first sight, this result appears somewhat curious since the charge certainly radiates energy in the region $0<x<x_s$. Nevertheless, it is easy to see that this solution presents no conflict with energy conservation. In fact, the idealization of a homogeneous electric field in the region $x>0$ means that the energy stored in this field is infinite; it is precisely this unlimited source of energy which supplies the energy radiated by the charge.

The attractive electric field generated by an infinite plate located in the Y-Z plane is another example for which Lorentz-Dirac equation (2) admits an exact analytic solution. In this case, the external field is

$$E(x)=\begin{cases}+E_0 & \text{for } x<0\\ -E_0 & \text{for } x>0,\end{cases} \tag{20}$$

where $E_0>0$. If we call $\gamma_{\rm in}$ the value of γ at $\bar{x}=0$, the solution reads

$$\gamma(\bar{x})=\begin{cases}\gamma_{\rm in}-(eE_0/mc^2)\bar{x} & \text{for } \bar{x}>0\\ \gamma_{\rm in}+(eE_0/mc^2)\bar{x} & \text{for } \bar{x}<0.\end{cases} \tag{21}$$

Thus the charge oscillates indefinitely around the plate, with a fixed amplitude x_s given by Eq. (19). As in the case of the reflection problem, the energy radiated by the charge is also supplied here by the infinite amount of energy stored in the field of the plate. Solution (21) is different from and simpler than the one found by Plass [4] for the same problem.

V. THE CASE OF AN ANALYTIC $\overline{E}(\xi)$

We are going to consider now a rather general formalism for an electrostatic field $E(x)$ which is analytic inside the interval $x_1<x<x_2$ and vanishes identically for $x<x_1$ and $x>x_2$. By an analytic funtion, we mean here a real-value function that can be represented by means of a convergent power series in the real variable $\xi=x/r_0$. The dimensionless function $\overline{E}(\xi)$ in Eq. (12) is clearly nonanalytic at $\xi_1=x_1/r_0$ and $\xi_2=x_2/r_0$. Let the origin be inside the interval (ξ_1,ξ_2). Then, we can write

$$\overline{E}=\sum_{n=0}^{\infty} a_n\xi^n. \tag{22}$$

We will use a power series to construct the solution of Eq. (12), namely,

$$\gamma(\xi)=\sum_{n=0}^{\infty} b_n\xi^n. \tag{23}$$

It is also convenient to introduce coefficients c_n, defined by

$$(\gamma^2-1)^{1/2}=\sum_{n=0}^{\infty} c_n\xi^n. \tag{24}$$

It is easy to show that each c_n can be expressed in terms of the coefficients $b_0,b_1,\ldots,b_n$ in Eq. (23). In fact, taking successive derivatives of Eqs. (23) and (24) and evaluating them at $\xi=0$, we find that

$$c_0=(b_0^2-1)^{1/2},$$

$$c_1=b_0b_1(b_0^2-1)^{-1/2}, \tag{25}$$

$$c_2=\{-b_0^2b_1^2(b_0^2-1)^{-3/2}+(b_1^2+2b_0b_2)(b_0^2-1)^{-1/2}\}/2,$$

and so on. Since $\gamma>1$, we have $b_0>1$. Substituting Eqs. (22), (23), and (24) in Eq. (12), we obtain

$$b_2=(3b_1/2-a_0)/2c_0,$$

$$b_3=(-2b_2c_1+3b_2-a_1)/6c_0, \tag{26}$$

$$b_4=(-2b_2c_2-6b_3c_1+9b_3/2-a_2)/12c_0,$$

and so on. The constitutive law for the b_n is, in general, very complicated. Nevertheless, Eqs. (25) and (26) show that if we know $b_0=\gamma(0)$, and $b_1=\gamma'(0)$, then all the coefficients b_n in Eq. (23) are uniquely determined in a constructive way.

For a practical application of the above formalism, it is convenient to see the power series (22)–(24) as expansions around $\xi=\xi_1$. For this end, we call $E^*(\xi)$ the analytic continuation of $\overline{E}(\xi)$ for $\xi\leqslant\xi_1$. In order to clarify the meaning of the analytic continuation, let us consider a point $\xi>\xi_1$ and the power-series representation of $\overline{E}(\xi)$ around this point. In general, this power series will converge for values of $\xi\leqslant\xi_1$ and, consequently, will define an analytic function $E^*(\xi)$ that coincides with $\overline{E}(\xi)$, for $\xi>\xi_1$. As opposed to $\overline{E}(\xi)$, function $E^*(\xi)$ is continuous and has continuous derivatives of all orders at ξ_1. So, the expansion (22) around ξ_1 can be applied to $E^*(\xi)$, thus constructing a solution γ to Eq. (12) be means of an appropriate choice of coefficients b_0 and b_1 in Eq. (23). This γ is also a solution to Eq. (12) for $\overline{E}(\xi)$ in $\xi>\xi_1$, but, since $\overline{E}(\xi)$ vanishes identically in this region, it is not a solution in $\xi<\xi_1$. So, the above solution to $E^*(\xi)$ is chosen as the solution without preacceleration to $\overline{E}(\xi)$ for $\xi>\xi_1$, while for $\xi<\xi_1$, the inertial motion solution $\gamma=\gamma_{\rm in}$ is chosen.

The above procedure is now illustrated with our exact solution (13) to the potential well. In this case, the analytical continuation of $\overline{E}=(3r_0^2/2e)E_0$ is, of course, the same constant function for any ξ. Then, except for a_0, all the a_n in Eq. (22) are zero; this implies that, as seen in Eq. (26), coefficients b_n in Eq. (23), for $n\geqslant 3$, become proportional to b_2. Now, the boundary condition $d\gamma/dx=(r_0/e)E_0$ makes $b_1=(2/3)a_0=(r_0^2/e)E_0$, which in turn means that $b_2=0$, thereby reducing Eq. (23) to $\gamma=b_0+(r_0^2/e)E_0\xi$, for all ξ. However, this function represents the solution only for $0\leqslant\xi\leqslant 2l/r_0$, as shown in Eq. (13). This solution is well defined for any value of the initial energy and any value of the jump in the electric field.

For given b_0 and b_1, the above construction allows the finding of a unique solution to the Lorentz-Dirac equation. Therefore coefficients b_0 and b_1 must be completely determined by the electric field $E(x)$ and the inertial motion existing for $\xi<\xi_1$. As we already pointed out, the kinetic energy of the charge must be continuous at any ξ, particularly at $\xi=\xi_1$. This boundary condition can be fulfilled only if $b_0=\gamma_{\rm in}$, where $\gamma_{\rm in}$ is the value of γ for $\xi<\xi_1$. The determination of b_1 is not as direct as in the case of b_0, since there seems to be no simple closed formula for b_1 in terms of $\gamma_{\rm in}$ and $E^*(\xi)$. However, an approximate evaluation of b_1 can be obtained by the method of Baylis and Huschilt [14]. By iterating Eq. (12), these authors found the following formula for γ' in terms of γ and the electric field $E(x)$ with its derivatives.

$$\gamma'=E+\varepsilon uE'+\varepsilon^2(u^2E''+\gamma EE')+\varepsilon^3u(2\gamma E'^2+3\gamma EE'' + u^2E'''+E^2E')+O(\varepsilon^4), \tag{27}$$

where we use the notation of Ref. [14], that is, the primes indicate derivatives with respect to x, $u=(\gamma^2-1)^{1/2}$, $\varepsilon=\frac{2}{3}$, and the units are such that $e=m=c=1$. For our purpose, the electric field and the derivatives in Eq. (27) must be evaluated at $x=x_1$ using the analytic continuation $E^*(x)$ of the electric field $E(x)$. Note that for a homogeneous electric field, Eq. (27) reduces exactly to the boundary condition for $d\gamma/dx$ in Eq. (15).

Since we have not proven that the coefficients in Eq. (26) lead to a convergent power series in Eq. (23), nor that this power series is a solution of Eq. (12), our treatment for a general $\overline{E}(\xi)$ has a rather formal nature. Our conjecture is that the power series (23) is convergent and a solution to Eq.

(12) when $\overline{E(\xi)}$ is an analytical function of ξ, but this complex problem will not be discussed here. This is a delicate matter, given that the nonlinear nature of Eq. (12) leads to a complex structure for the coefficients b_n in Eq. (26). Hence to deal with this problem it seems advisable first to try to study some specific form for $\overline{E(\xi)}$ and then attack the general problem of the convergence of the power series (23). However, given the fruitfulness of the power-series method for generating solutions in different areas of mathematical physics, its use implies no real restrictions. Thus our method is a constructive approach for building up a solution without self-acceleration and without preacceleration. Therefore the solution is the following: for $\overline{x} \leqslant x_1$, we set $\gamma = \gamma_{\rm in}$, where $\gamma_{\rm in}$ is defined by the initial velocity of the charge. For $x_1 \leqslant \overline{x} \leqslant x_2$, $\gamma(\overline{x})$ is determined by Eq. (23), where the coefficients are given by Eq. (26), with b_0 and b_1 determined as we already explained. For $\overline{x} \geqslant x_2$, we simply set $\gamma = \gamma_{\rm out} = \gamma(x_2)$.

VI. AN EXACT SOLUTION WITH A NONLINEAR $\gamma(\overline{x})$

Since for the potential well and for the linear potential wall $\gamma(\overline{x})$ is a linear function of $\overline{x}$, the nonlinear radiative reaction term of Eq. (12) vanishes identically. In particular, this simplification allows the finding of exact solutions for these cases for any initial energy of the charge. On the other hand, when the nonlinear term is different from zero, we expect to find exact solutions only for very special electrostatic fields and for exceptional values of the initial energy of the charge. Let us consider the motion of a charge $e > 0$ in the following electrostatic field:

$$E(x) = \left(\frac{2e}{3r_0^2}\right)\sinh\left(\frac{x}{r_0}\right)\left\{\frac{3}{2} - \cosh\left(\frac{x}{r_0}\right)\right\} \quad \text{for} \quad 0 < x_1 < x < x_2, \tag{28}$$

where x_1 is a certain positive number less than x_m, the point at which $E(x)$ reaches its maximum, and x_2 is defined through x_1 by means of $E(x_2) = E(x_1) \equiv E_0$. In the interval (x_1, x_2), the charge density $\rho(x)$ generating the electric field (28) is given by

$$\rho(x) = \left(\frac{e}{6\pi r_0^3}\right)\left\{\frac{3}{2}\cosh\left(\frac{x}{r_0}\right) - \cosh^2\left(\frac{x}{r_0}\right) - \sinh^2\left(\frac{x}{r_0}\right)\right\}, \tag{29}$$

with $\rho(x) \equiv 0$, for $x < x_1$ and $x > x_2$. Figure 2 shows the graph of $\overline{E} = (3r_0^2/2e)E(x)$ in terms of the dimensionless variable $\xi = x/r_0$, where $\xi_1 = x_1/r_0$, $\xi_2 = x_2/r_0$, and $\xi_m = x_m/r_0$.

The analytic continuation $\overline{E^*}(\xi)$ of $E(\xi)$ defined in the preceding section is given by

$$\overline{E^*}(\xi) = \sinh\xi\left(\tfrac{3}{2} - \cosh\xi\right) \tag{30}$$

for any ξ. If we apply the formalism of the preceding section around $\xi = \xi_1$, we find that the coefficients a_n of Eq. (22) are given by

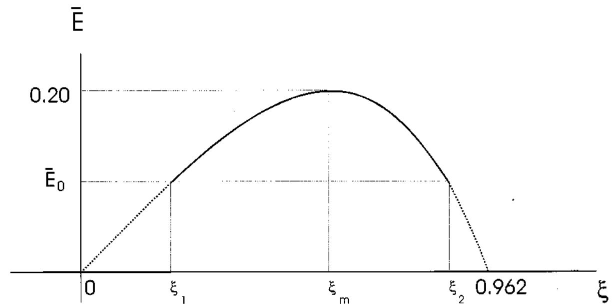

FIG. 2. The full line represents the function $\overline{E(\xi)}$ that vanishes identically for $\xi < \xi_1$ and $\xi > \xi_2$, and is given by Eq. (30) in the interval $\xi_1 < \xi < \xi_2$. The dotted line represents the analytic continuation $\overline{E^*}(\xi)$ of $\overline{E(\xi)}$.

$$a_0 = \left(\tfrac{3}{2}\right)\sinh\xi_1 - \sinh\xi_1\cosh\xi_1,$$

$$a_1 = \left(\tfrac{3}{2}\right)\cosh\xi_1 - \cosh^2\xi_1 - \sinh^2\xi_1, \tag{31}$$

$$a_2 = \left(\tfrac{3}{4}\right)\sinh\xi_1 - 2\sinh\xi_1\cosh\xi_1,$$

and so on. Now, in order to build an exact analytic solution, we choose the following boundary conditions at $\xi = \xi_1$:

$$b_0 = \gamma(\xi_1) = \cosh\xi_1, \qquad b_1 = \gamma'(\xi_1) = \sinh\xi_1. \tag{32}$$

Then, from Eqs. (31), (32), (25), and (26) we obtain

$$b_2 = \left(\tfrac{1}{2}\right)\cosh\xi_1,$$

$$b_3 = \left(\tfrac{1}{6}\right)\sinh\xi_1, \tag{33}$$

$$b_4 = \left(\tfrac{1}{24}\right)\cosh\xi_1,$$

and so on. Coefficients (32) and (33) correspond to the expansion of the function

$$\gamma(\xi) = \cosh\xi \tag{34}$$

around $\xi = \xi_1$. It is easy to verify that Eq. (34) is indeed an exact solution to Eq. (12) for $\overline{E^*}(\xi)$ given by Eq. (30). Then, the exact solution without preacceleration in the electric field (28) reads

$$\gamma(\overline{x}) = \begin{cases} \cosh\left(\dfrac{x_1}{r_0}\right) & \text{for } \overline{x} \leqslant x_1 \\ \cosh\left(\dfrac{\overline{x}}{r_0}\right) & \text{for } x_1 \leqslant \overline{x} \leqslant x_2 \\ \cosh\left(\dfrac{x_2}{r_0}\right) & \text{for } \overline{x} \geqslant x_2. \end{cases} \tag{35}$$

This solution is such that at $x = x_1$, $d\gamma/d\overline{x}$ has a jump given by

$$\frac{1}{r_0}\sinh\left(\frac{x_1}{r_0}\right). \tag{36}$$

If we apply Eq. (15) to obtain the jump at $x=x_1$, instead of Eq. (36) we would obtain

$$\frac{1}{r_0}\sinh\left(\frac{x_1}{r_0}\right)-\frac{2}{3r_0}\sinh\left(\frac{x_1}{r_0}\right)\cosh\left(\frac{x_1}{r_0}\right). \tag{37}$$

The difference between Eqs. (36) and (37) shows that the value $\overline{E_0}$ of $\overline{E^*}(\xi)$, determining the jump of the electric field at $\xi=\xi_1$, is not enough to calculate coefficient b_1, a fact that is also clear from Eq. (27). The simplification which is present in the cases of the potential well and the linear potential wall —where coefficient b_1 is entirely determined by the jump of the electric field— is due to the homogeneous character of the electric field.

Solution (35) is such that the jump (36) can be made arbitrarily small if x_1 is chosen very close to zero. So, in this limit, Eq. (35) gives an analytic solution with an acceleration that vanishes identically for $x<x_1$ and which is very small immediately to the right of x_1. In other words, for very small values of x_1, the acceleration in Eq. (35) changes in an almost continuous way around x_1. This kind of behavior—where both the velocity and the acceleration of the charge change continuously—is what is generally expected for the motion of the charge in a spatially localized electric field that vanishes smoothly in the borders.

VII. SOME REMARKS

In the absence of an electromagnetic field, the coefficients a_n in Eq. (22) are equal to zero; thus the boundary conditions in Eqs. (14) and (15), along with Eq. (26), imply that all the b_n in Eq. (23), except for b_0, are equal to zero. That is, we obtain $\gamma=\gamma_{\text{in}}$, for any x, as the only solution. Thus in the absence of an external force an inertial motion cannot become self-accelerating.

The existence of preacceleration is illustrated in Dirac's paper with the motion of an electron disturbed by a pulse. If we choose appropriate parameters for l and E_0 in the exact solution (13), we may approximate a Dirac δ function with the potential well; however, Eq. (13) does not present preacceleration for any choice of the parameters l and E_0. The nonexistence of preacceleration can also be seen in Fig. 1. Moreover, the solution (13) has a perfectly well-defined nonrelativistic limit, which certainly does not present preacceleration.

For a motion along a straight line, the usual nonrelativistic procedure of neglecting the Larmor term, compared to the Schott term, needs careful examination. In fact, for the sake of consistency, neglecting term (11) would also imply neglecting the second term on the right-hand side of Eq. (10) leading thus to Eq. (12), which is the exact equation and not the approximation we are looking for. A similar finding, although within a somewhat different context, is reported by Rohrlich [15]. The above analysis seems to indicate that the Larmor non-linear term is an essential element of the equation of motion that may not be neglected. This would also explain why the solution of the truncated linearized version (1) of Eq. (2) presents some pathologies. We plan to study this issue further in the future. Let us remark though, that our exact solutions for a homogeneous electrostatic field in Secs. III and IV are such that the sum of the Larmor and Schott terms vanishes identically, while the Larmor term itself is different from zero.

Important simplifications in the structure of the b_n in Eq. (26) occur in the ultrarelativistic case, since in this case the c_n in Eq. (24) become equal to the b_n in Eq. (23). But Eq. (26) is not wholly appropriate for studying the nonrelativistic limit, as may be seen from the form of the coefficients in Eqs. (25) and (26).

If the electric field always points in a positive direction on the X axis, then a positive charge coming from the left will have a velocity that is positive for any $\overline{x}$, and the formulation developed in the preceding section will need no changes. However, if the electric field changes direction, the charge may stop at a point, and then turn back. In this case, our formalism is not appropriate, because the power series (23) cannot be written around the turning point, since $\gamma(\overline{x})$ will be defined only to the left of the turning point. The same difficulty arises in Eq. (24), given that $(\gamma^2-1)^{1/2}$ vanishes at the turning point. This problem is of a purely technical nature, having nothing to do with preacceleration, since inertial motion can be imposed anyway in the free-field region. A way to deal with this complication seems to be to write a power series for $\overline{x(t)}$ in terms of the laboratory time, and work directly with the Lorentz-Dirac equation (2), instead of Eq. (12).

The electrostatic field of the solution considered in this paper is generated by a set of sheets, each one with a uniform density of charge. For this reason, the total amount of energy stored in the electric field is necessarily infinite, a fact which does not allow a quantitative discussion of energy conservation. This trouble can be avoided by considering the electrostatic field generated by a set of spherical shells with a uniform charge each, such that the electric field is spatially localized. In this case, a one-dimensional motion still is possible along a straight line across the center of the charge distribution, and the change in the kinetic energy of the charge can be directly compared to the energy radiated away by the charge. This methodology has already been employed by Comay [16], who solved the old controversy around Eliezer's problem [17] regarding the motion of a particle with charge e and mass m, attracted towards an infinitely massive charge of opposite sign. Eliezer and several other authors [14,18,19] claimed that, in this case, all solutions to the Lorentz-Dirac equation violate energy conservation. Nevertheless, by replacing the field of the infinitely massive point charge by the field of a uniformly charged spherical shell, Comay was able to construct the correct solution and show that it is in full agreement with energy conservation. The idea of replacing the point charge by a charge distribution of finite extension was first suggested by Rohrlich [20].

In order to avoid self-accelerating solutions to Eq. (2), Dirac [2] suggested the idea that besides the initial condition regarding the position and velocity of the charge for $t=0$, the vanishing of the acceleration for a large value of t must be imposed as a boundary condition as well. The development of this idea led to the integro-differential formulation of the Lorentz-Dirac equation. Unfortunately, however, as is well known, this formulation implies the existence of preacceleration. On the other hand, the exact solutions discussed in this paper are without preacceleration and without self-acceleration, and cannot be obtained from the integro-differential equation. These results privilege the differential form of the Lorentz-Dirac equation over its integro-differential formulation.

Our focus on the Lorentz-Dirac equation is motivated by the wide attention it has received in the literature, and by the fact that this equation follows quite naturally from the formalisms of classical electrodynamics. Furthermore, the exact solutions without preacceleration presented in this paper give insight into the issue of the violation of causality so far considered inherent to the Lorentz-Dirac equation. In connection with this, other alternative equations of motion of second order have been proven to be inconsistent with fundamental physical requirements [21–24].

Let us also point out that the integro-differential equation has been of little help to the construction of solutions to the Lorentz-Dirac equation. The only known exact solutions to the integro-differential equation are those found by Plass for the potential well and the linear potential wall. Plass was also able to obtain the solution for a one-dimensional motion along a straight line for a rather general time-dependent force. As a consequence of this, a great number of exact solutions to the Lorentz-Dirac equation seem to be known in this case. Yet, if we consider that the natural forces on charged particles are those due to electric and magnetic fields, Plass's formula represents just a restriction on the motion rather than a solution. This point becomes clear by considering one of the simplest possible cases, that is, the motion in a straight line in an electrostatic field like the one discussed in Sec. V. In fact, if we substitute the trajectory $\overline{x(t)}$ in the electrostatic field $E(x)$, the latter can be seen as a time-dependent force $E(\overline{x(t)})$, and, consequently, Plass's formula applies. But the problem is that we do not know $\overline{x(t)}$, since this function is precisely the solution we are looking for.

Although our approach has been developed for motions in only one dimension, its generalization to the cases of more than one dimension seems natural considering that its essential features could be formulated for such cases. These features are the following: (1) The charge has inertial motion in any region where the external electromagnetic field vanishes identically; (2) the Cartesian components of the charge position vector are analytic functions of time in the regions where the external electromagnetic field is analytic; and (3) the inertial motion present in a free-field region matches with the solution of the non-free-field region by imposing the continuity of the velocity and an eventual jump in the acceleration. This jump is determinated by the electromagnetic field and by the velocity present in the free-field region. A further treatment in order to find the explicit formalism for the determination of the jump is still pending; however, a generalization of the Baylis-Huschilt expansion (27) seems to be a good starting point to deal with this problem.

Thus it would seem that, when properly formulated, the Lorentz-Dirac equation has a solution that is free of preacceleration and self-acceleration for rather general external electromagnetic fields. The most immediate way of constructing the solution is by means of power series in laboratory time. We hope that this procedure will be useful, at least for some particular space-time-dependent electromagnetic fields.

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