# Quantum detection for on-off keyed mixed-state signals with a small amount of thermal noise

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The detection strategy proposed previously by the authors for binary pure-state signals, which consists of a unitary transformation and photon counting afterward [Phys. Rev. A **54**, 2728 (1996)], can also be applied to the case of on-off keyed signals of coherent states with a small amount of thermal noise. It yields an error level almost the same as the quantum minimum bound in practical parameter region. [S1050-2947(97)03904-8]

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The optimal decision problem for nonorthogonal quantum states, i.e., how to discriminate between them with minimum error probability, is one of the fundamental problems in quantum physics [1–3]. This problem is also important in applications of quantum communication. Much work has been done to understand how to deal with pure states, but comprehensive treatment for mixed states is still lacking. Most work relevant to mixed states has been concerned with theoretical predictions of the minimum error bound. The corresponding detection operators have never been given explicitly. Only several practical methods have been proposed which achieve performance close to the optimum [4–9].

In this paper, we will show that the strategy proposed by the authors for decision between binary pure-coherent-state signals can also give the optimum for a signal with a small amount of thermal noise. The strategy consists of a unitary transformation of the signal states and photon counting afterwards. This transformation can be effected by a multiphoton nonlinear optical process [10–12]. It will be shown that its direct application to the mixed-state signals can achieve the optimum performance in a practical parameter region.

We consider the case in which the signals are on-off keyed by an imperfect laser and only the on-state signal has thermal noise, while the off-state signal is the pure vacuum state. The impure coherent state with thermal noise $\hat{\rho}_{\text{coh-th}}$ is expressed as

$$\hat{\rho}_{\text{coh-th}}=\sum_{m=0}^{\infty} w_m|\psi_m\rangle\langle\psi_m|, \tag{1a}$$

with the weight

$$w_m=\frac{1}{1+n_{\text{th}}}\left(\frac{n_{\text{th}}}{1+n_{\text{th}}}\right)^m, \tag{1b}$$

and the states

$$|\psi_m\rangle=\hat{D}(\alpha)|m\rangle, \tag{1c}$$

where $n_{\text{th}}$ is the mean number of the thermal noisy photon, $\hat{D}(\alpha)=e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}$ is the displacement operator, $\alpha$ being the complex amplitude of the coherent component, $\hat{a}$ and $\hat{a}^\dagger$ being the annihilation and creation operators, correspondingly, and $|m\rangle$ being the $m$-photon Fock state. In the practical case of communication using optical frequency, $n_{\text{th}}$ is very small, less than, at most, 0.01 leading to a rapid decrease of the weight factor $w_m$, as $m$ becomes larger. So in numerical estimation of the error probability, the expression of Eq. (1a) can be replaced by a sum of a finite number of basis vectors, $\hat{\rho}_2$ as,

$$\hat{\rho}_2=\sum_{m=0}^{M} w'_m|\psi_m\rangle\langle\psi_m|, \tag{2a}$$

where the new weight factors are

$$w'_m=\frac{1-\varepsilon}{1-\varepsilon^{M+1}}\varepsilon^m, \tag{2b}$$

defining $\varepsilon=(n_{\text{th}}/1+n_{\text{th}})\sim n_{\text{th}}$. The number $M$ is taken to be large enough to satisfy

$$\sum_{m=M+1}^{\infty} w_m \ll P_e(\text{opt}), \tag{3}$$

where $w_m$ is the exact weight in Eq. (1b) and $P_e(\text{opt})$ represents the minimum error probability.

Hereafter we assume that $\hat{\rho}_2$ represents the on-state signals from an imperfect laser. The off-state signal is represented by the vacuum state

$$\hat{\rho}_1=|0\rangle\langle 0|. \tag{4}$$

We shall start with the binary signals $\{\hat{\rho}_1,\hat{\rho}_2\}$ with the respective prior probabilities $\{\xi_1,\xi_2\}$ $(\xi_1+\xi_2=1)$. The Hilbert space $\mathcal{H}_s$, describing these signal states, is of $(M+2)$ dimension corresponding to the number of linearly independent basis vectors involved in $\hat{\rho}_1$ and $\hat{\rho}_2$.

Knowing the above kinds of *a priori* knowledge about the transmitted signal states, the optimal decision strategy , i.e., a set of POM $\{\hat{\Pi}_1,\hat{\Pi}_2\}$ $(\hat{\Pi}_1+\hat{\Pi}_2=\hat{I}_s$, where $\hat{I}_s$ is the identity operator on $\mathcal{H}_s)$, is determined so as to minimize the error probability

 

$$P_e = \xi_2 \mathrm{Tr}(\hat{\rho}_2 \hat{\Pi}_1) + \xi_1 \mathrm{Tr}(\hat{\rho}_1 \hat{\Pi}_2) = \xi_1 + \xi_2 \mathrm{Tr}[(\hat{\rho}_2 - \lambda \hat{\rho}_1)\hat{\Pi}_1], \tag{5}$$

where $\lambda = \xi_1/\xi_2$. That is, we choose $\hat{\Pi}_1$ minimizing the term $\mathrm{Tr}[(\hat{\rho}_2 - \lambda\hat{\rho}_1)\hat{\Pi}_1]$. It can easily be made by taking

$$\hat{\Pi}_1 = \sum_{\omega_i < 0} |\omega_i\rangle\langle\omega_i|, \tag{6}$$

with the eigenvectors $|\omega_i\rangle$ of the Hermite operator $\hat{\rho}_2 - \lambda\hat{\rho}_1$ and the corresponding eigenvalues $\omega_i$. To find the negative eigenvalue is easy in this case, as done by Helstrom [13]. Actually, reflecting the fact that the state $\hat{\rho}_1$ is the pure state, it can be obtained as a single solution by solving the following equation:

$$\omega = -\lambda + \lambda \sum_{m=0}^{M} \frac{w'_m c_m^2}{w'_m - \omega}, \tag{7}$$

where $c_m^2 = |\langle 0|\psi_m\rangle|^2$. Denoting it as $\omega_-$, the minimum error probability is written as

$$P_e(\mathrm{opt}) = \xi_1 + \xi_2 \omega_- . \tag{8}$$

The corresponding eigenvector $|\omega_-\rangle$ can also be derived somehow, but to connect it to a physical detection method is not trivial.

To cope with the problem, let us set the suitable complete orthonormal set $\{|\eta_k\rangle | k = 1,2, \ldots, M+2\}$ on $\mathcal{H}_s$. It is constructed from the constituent vectors of $\hat{\rho}_1$ and $\hat{\rho}_2$, i.e., $\{|0\rangle, |\psi_0\rangle, \ldots, |\psi_M\rangle\}$, by the Schmidt orthogonalization as

$$|\eta_1\rangle = |0\rangle,$$

$$|\eta_2\rangle = \frac{|\psi_0\rangle - c_0|0\rangle}{\sqrt{1-c_0^2}}, \quad (c_0 \equiv \langle 0|\psi_0\rangle = e^{-|\alpha|^2/2}),$$

$$\vdots \tag{9}$$

$$|\eta_{M+2}\rangle = \frac{|\psi_M\rangle - \Sigma_{k=1}^{M+1} |\eta_k\rangle\langle\eta_k|\psi_M\rangle}{\sqrt{1 - \Sigma_{k=1}^{M+1} |\langle\eta_k|\psi_M\rangle|^2}}.$$

Since $\hat{\rho}_2 - \lambda\hat{\rho}_1$ is an Hermite operator, there exists a unitary operator on $\mathcal{H}_s$ which diagonalizes it as

$$\hat{U}(\hat{\rho}_2 - \lambda\hat{\rho}_1)\hat{U}^\dagger = \sum_{k=1}^{M+2} \omega_k |\eta_k\rangle\langle\eta_k|. \tag{10}$$

The eigenvectors $|\omega_k\rangle$ with the eigenvalues $\omega_k$ are then expressed as

$$|\omega_k\rangle = \hat{U}^\dagger |\eta_k\rangle. \tag{11}$$

In the pure-state limit $\varepsilon \to 0$, Eq. (10) is written as

$$\hat{U}^{(0)}(\hat{\rho}_2 - \lambda\hat{\rho}_1)\hat{U}^{(0)\dagger} = \sum_{k=1}^{2} \omega_k^{(0)} |\eta_k\rangle\langle\eta_k|, \tag{12}$$

where

$$\hat{U}^{(0)} = \exp\gamma(|\eta_1\rangle\langle\eta_2| - |\eta_2\rangle\langle\eta_1|), \tag{13a}$$

with the interaction parameter $\gamma$ chosen as

$$\gamma = -\tan^{-1}\left(\frac{\sqrt{1-4\xi_1\xi_2 c_0^2} - 1 + 2\xi_2 c_0^2}{\sqrt{1-4\xi_1\xi_2 c_0^2} + 1 - 2\xi_2 c_0^2}\right)^{1/2} \tag{13b}$$

and

$$\omega_1^{(0)} = \tfrac{1}{2}\{1 - \lambda - \sqrt{(1+\lambda)^2 - 4\lambda c_0^2}\} \quad (<0), \tag{13c}$$

$$\omega_2^{(0)} = \tfrac{1}{2}\{1 - \lambda + \sqrt{(1+\lambda)^2 - 4\lambda c_0^2}\} \quad (>0). \tag{13d}$$

Obviously the detection operators are

$$\hat{\Pi}_i^{(0)} = \hat{U}^{(0)\dagger} |\eta_i\rangle\langle\eta_i| \hat{U}^{(0)}, \quad (i = 1,2). \tag{14}$$

As $\varepsilon$ increases gradually, the negative eigenvalue continuously varies from $\omega_1^{(0)}$ to $\omega_1 (= \omega_-)$ in the extent of $\varepsilon < P_e^{(0)} = [\xi_1 + \xi_2\omega_1^{(0)} = 1/2\,(1 - \sqrt{1 - 4\xi_1\xi_2 c_0^2})]$. So $\hat{\Pi}_1 = |\omega_-\rangle\langle\omega_-|$ should be given as $|\omega_1\rangle\langle\omega_1|$. Namely, the decision strategy is the following:

$$\hat{\Pi}_1 = \hat{U}^\dagger |\eta_1\rangle\langle\eta_1| \hat{U}, \quad \text{for } \hat{\rho}_1, \tag{15a}$$

$$\hat{\Pi}_2 = \hat{U}^\dagger \sum_{k=2}^{M+2} |\eta_k\rangle\langle\eta_k| \hat{U}, \quad \text{for } \hat{\rho}_2, \tag{15b}$$

By substituting Eq. (15a) into Eq. (5), the minimum error probability is written as

$$P_e(\mathrm{opt}) = \xi_1 + \xi_2 \langle\eta_1|\hat{U}(\hat{\rho}_2 - \lambda\hat{\rho}_1)\hat{U}^\dagger|\eta_1\rangle. \tag{16}$$

This structure means that transforming the signal states by the unitary process $\hat{U}$ and then detecting by the projectors $\{|\eta_1\rangle\langle\eta_1|, \Sigma_{k=2}^{M+2}|\eta_k\rangle\langle\eta_k|\}$, corresponds to the optimal detection. Physically, the projectors can be replaced by the photon counting distinguishing whether photon is registered or not, $\{|0\rangle\langle 0|, \Sigma_{n=1}^{\infty}|n\rangle\langle n|\}$, since $|\eta_1\rangle$ is the vacuum state while $|\eta_k\rangle$ $(k>2)$ only include the Fock states with a finite photon number. On the other hand, how to implement the process $\hat{U}$ is not trivial. So we apply $\hat{U}^{(0)}$, Eq. (13a), to it. $\hat{U}^{(0)}$ can be generated by a Hamiltonian describing a multiphoton nonlinear process, as shown in our previous work [12]. Consequently we consider the following strategy:

$$\hat{\Pi}_1^* = \hat{U}^{(0)\dagger} |0\rangle\langle 0| \hat{U}^{(0)}, \tag{17a}$$

$$\hat{\Pi}_2^* = \hat{U}^{(0)\dagger} \sum_{n=1}^{\infty} |n\rangle\langle n| \hat{U}^{(0)}, \tag{17b}$$

Here one should note that the operators $\hat{\Pi}_i^*$ are defined on the whole Fock space. And if the unitary process is constructed correctly instead of $\hat{U}^{(0)}$, the optimum bound in the error probability can be available. But in the practical parameter region, the strategy $\{\hat{\Pi}_i^*\}$ is enough. Actually this strategy yields the error probability that coincides with the optimum bound up to the first order of $\varepsilon$, as shown below.

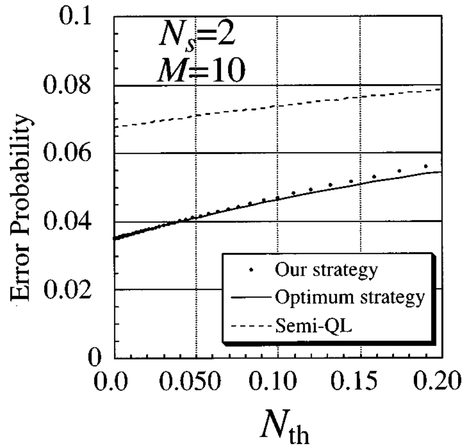


FIG. 1. The error probabilities in several decision strategies vs the mean photon number of the thermal noisy component $n_{th}$ in the case of $N_s=2$. It shows how the strategy $\{\hat{\Pi}_i^*\}$ (dotted line) deviates from the optimal one (solid line). The broken line indicates the SQL.

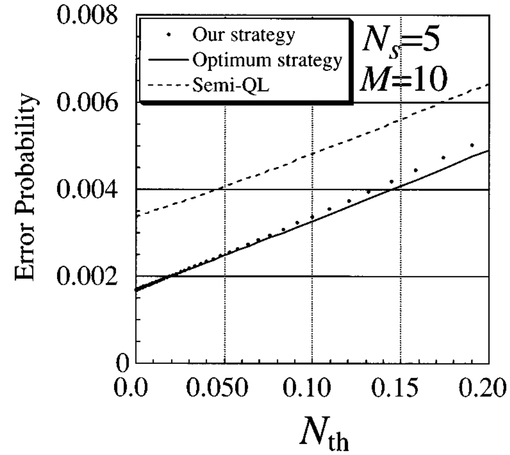


FIG. 2. The error probabilities vs $n_{th}$ as in Fig. 1 but in the case of $N_s=5$.

The error probability is expressed as

$$P_e^* = \xi_1 + \xi_2 \langle \eta_1 | \hat{U}^{(0)} (\hat{\rho}_2 - \lambda \hat{\rho}_1) \hat{U}^{(0)\dagger} | \eta_1 \rangle. \tag{18}$$

Deviation from the minimum bound $\Delta P_e = P_e^* - P_e(\text{opt})$ can be expanded in a power of $\varepsilon$. At first,

$$\hat{\rho}_2 - \lambda \hat{\rho}_1 = |\psi_0\rangle\langle\psi_0| - \lambda |0\rangle\langle 0| + \varepsilon(|\psi_1\rangle\langle\psi_1| - |\psi_0\rangle\langle\psi_0|) + O(\varepsilon^2). \tag{19}$$

Next the unitary operator $\hat{U}$ can be expanded in the following form:

$$\hat{U} = \hat{U}^{(0)} [\hat{I}_s + \varepsilon \hat{Q} + O(\varepsilon^2)]. \tag{20}$$

Up to the first order in $\varepsilon$, the unitarity condition is written as

$$\hat{Q} + \hat{Q}^\dagger = 0. \tag{21}$$

Substitute Eqs. (19) and (20) into Eqs. (16) and (18), and use the unitarity condition Eq. (21) and the fact that $\hat{U}^{(0)}$ diagonalizes the term $|\psi_0\rangle\langle\psi_0| - \lambda|0\rangle\langle 0|$ in $\{|\eta_1\rangle, |\eta_2\rangle\}$ representation. Then one can see that the first-order term in $\varepsilon$ vanishes in the deviation $\Delta P_e$ and nonzero values arises only from the second-order terms, i.e., $\Delta P_e = O(\varepsilon^2)$. Therefore, the strategy $\{\hat{\Pi}_i^*\}$ coincides with the optimal one to the extent of the first-order perturbation in $\varepsilon$ for the error probability.

So there exists a parameter region of a small value of $\varepsilon$, where the strategy $\{\hat{\Pi}_i^*\}$ works as well as the optimal one. This is demonstrated in Figs. 1 and 2 for the case of $\xi_1 = \xi_2 = 1/2$. The parameter $\varepsilon$ is converted to the thermal noisy photon by $\varepsilon = n_{th}/(1+n_{th})$. $P_e^*$ (dotted lines) can be calculated by the following formulas:

$$P_e^* = \frac{1}{4}(1+w_0')(1-\sqrt{1-c_0^2}) + \frac{1+\sqrt{1-c_0^2}}{4(1-c_0^2)} \sum_{m=1}^{M} w_m' c_m^2, \tag{22}$$

which is derived by substituting Eqs. (2a), (4), and (13a) into Eq. (18). $P_e(\text{opt})$ (solid lines) can be evaluated by solving Eq. (7) numerically and by substituting the negative solution $\omega_-$ into Eq. (8). As a referential quantity, the error probability obtained by the conventional method, which consists of direct photon counting and the decision of setting an appropriate threshold, is also plotted (dashed lines). Figure 1 corresponds to the case of $N_s=2$ where $N_s$ is the mean photon number of the coherent component $|\alpha|^2$ in the on-state signal, while Fig. 2 is the case $N_s=5$. For both cases, the number $M$ appearing in the assumed on-state signal Eq. (2a) is taken as 10. Compared to the gap between the optimum bound (solid lines) and the conventional error level (dashed lines), the deviation $\Delta P_e$ is very small for the region $n_{th}<0.05$ in both figures. As $N_s$ becomes larger under fixing $\varepsilon \sim n_{th}$, the error probability itself becomes smaller and, therefore, $0(\varepsilon^2)$ term starts to contribute as a meaningful quantity. The strategy $\{\hat{\Pi}_i^*\}$ is then inapplicable. On the other hand, as $n_{th}$ increases, the gap between the optimum bound and the conventional error level decreases, meaning the quantum nature causing the performance improvement from the conventional one becomes weaker.

Before concluding this paper, we shall comment on the origin of the error reduction from the conventional level (direct photon counting). Let the signal observable be $\hat{X}$ and its spectral decomposition be $\hat{X} = \int x|x\rangle\langle x| dx$. It keys the binary signals $\{\hat{\rho}_1, \hat{\rho}_2\}$. The conventional detection process consists of the standard measurement described by $|x\rangle\langle x|$ and the optimal division of the decision region $\{R_1, R_2\}$ in the set of measurement results $\{x\}$. The obtained error probability

$$P_e(\text{SQL}) = \xi_1 \int_{R_2} dx \langle x|\hat{\rho}_1|x\rangle + \xi_2 \int_{R_1} dx \langle x|\hat{\rho}_2|x\rangle, \tag{23}$$

corresponds to the bound in the classical detection theory and is especially called the *semiquantum limit* (semi-QL). One of the most remarkable features in quantum measurement is the quantum interference appearing in the above probability amplitudes $\langle x|\hat{\rho}_i|x\rangle$. Due to it, the probability for each outcome $x$ is *not*, in general, the sum of the separate probabilities pertaining to possible paths. The unitary transformation used in our scheme induces the quantum interference to reduce the error probability, Eq. (23). By writing this transformation as $\hat{U}=e^{i\gamma F(\hat{X},\hat{Y})}$, where the operator $\hat{Y}$ represents the conjugate observable with $\hat{X}$, satisfying $[\hat{X},\hat{Y}]=i/2$, the generating function $F(\hat{X},\hat{Y})$ should be at least a higher-order Hermite form than $\lambda_1(\hat{X}\hat{Y}+\hat{Y}\hat{X})+\lambda_2\hat{X}+\lambda_3\hat{Y}$, as the necessary condition for the error reduction below the Semi-QL [14]. In fact, the corresponding $F(\hat{X},\hat{Y})$ to our strategy $\{\hat{\Pi}_i^*\}$ includes highly nonlinear process of photons and satisfies this condition [12]. Unlike the case of binary pure-state signals, the quantum interference should be optimized for every pair between $|\psi_m\rangle$ and $|0\rangle$, depending on the weight $w_m$ in this case. And the available quantum interference for error reduction is strongly limited by the decoherence character of the classical noise, i.e., the summation of the projectors in $|\rho_2\rangle$. Our strategy optimizes the most significant pair $\{|0\rangle,|\psi_0\rangle\}$ to the error-reducing quantum interference. Consequently, the obtained error performance coincides with the optimal one in the linear expansion of the noise power for the error probability. The detection scheme is simply realized by installing the nonlinear optical process specified by $\hat{U}^{(0)}$ in front of the photon counter. Concerning a physical implementation of $\hat{U}^{(0)}$, we point out that the cavity QED scheme with a two channel Raman transition [15] is a possible candidate. It allows the vacuum state to evolve to an arbitrarily prescribed superposition of Fock states, which is exactly the same function as $U^{(0)}$.

In summary, a physical detection scheme was proposed for on-off keyed signals of a coherent state with thermal noise. It consists of the unitary transformation caused by the nonlinear optical process and the photon counting. The numerical simulation showed that this scheme provides an almost optimal decision bound in the error probability in a practical parameter region.

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