BRIEF REPORTS

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than four printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.

Exact wave functions of a harmonic oscillator with time-dependent mass and frequency

I. A. Pedrosa
Universidade Federal da Paraiba, Departamento de Física, CCEN, Caixa Postal 5008, 58051-970 João Pessoa, PB, Brazil

(Received 16 May 1996)

We use the Lewis and Riesenfeld invariant method [J. Math. Phys. **10**, 1458 (1969)] to obtain the exact Schrödinger wave functions for a harmonic oscillator with time-dependent mass and frequency. Exact coherent states for such system are also constructed. [S1050-2947(97)01703-4]

PACS number(s): 03.65.Ca, 03.65.Ge, 03.65.Fd

I. INTRODUCTION

The study of time-dependent harmonic oscillators has attracted considerable interest in the literature in the past few years [1–8]. The time-dependent oscillator has invoked much attention because it gives a good example of an exactly solved model and has applications in many areas of physics.

In recent papers, some authors have considered the harmonic oscillator with time-dependent mass and frequency [5,7,8]. The wave function obtained in Ref. [5] satisfies the Schrödinger equation only when the mass is constant and in which case the equivalent wave function also appears in Refs. [3,6]. However, for the case where the mass is also time dependent the wave function of Ref. [5] is not correct, i.e., it does not satisfy the Schrödinger equation. On the other hand, the wave function of Ref. [8] satisfies the Schrödinger equation for the case where mass and frequency are both time dependent.

The main purpose of this paper is to obtain an exact Schrödinger wave function for the harmonic oscillator with time-dependent mass and frequency and to correct some results of Ref. [5]. To this end, we use a unitary transformation and the Lewis and Riesenfeld invariant method. The wave function found in this paper is in agreement with those in Refs. [3,5,6] for the case with constant mass and agrees with that in Ref. [8] for the general case where the mass is also time dependent. We also constructed coherent states for the oscillator with time-dependent mass and frequency.

This paper is organized as follows. In Sec. II we briefly review the Lewis and Riesenfeld invariant method for the time-dependent oscillator. In Sec. III we find the wave function for the harmonic oscillator with time-dependent mass and frequency. In Sec. IV we construct exact coherent states for our system and Sec. V summarizes our overall results.

II. EXACT INVARIANTS AND THE SCHRÖDINGER EQUATION

Consider the Hamiltonian of a time-dependent harmonic oscillator

$$H(t)=\frac{1}{2M(t)}p^2+\frac{1}{2}M(t)\omega^2(t)q^2, \tag{1}$$

where q and p are canonically conjugate with $[q,p]=i\hbar$ and $M(t)$ $e\omega(t)$ are, respectively, the mass and frequency of the oscillator. From Eq. (1) we obtain the equation of motion

$$\ddot{q}+\gamma(t)\dot{q}+\omega^2(t)q=0, \tag{2}$$

where

$$\gamma(t)=\frac{d}{dt}[\ln M(t)]. \tag{3}$$

It is known that an exact invariant for Eq. (1) is given by [4,5]

$$I=\frac{1}{2}\left[\left(\frac{q}{\rho}\right)^{1/2}+(p\rho-M\dot{\rho}q)^2\right], \tag{4}$$

where $q(t)$ satisfies Eq. (2) and $\rho(t)$ is a c-number quantity satisfying the auxiliary equation

$$\ddot{\rho}+\gamma\dot{\rho}+\omega^2(t)\rho=\frac{1}{M^2\rho^3}. \tag{5}$$

The invariant $I(t)$ is a constant Hermitian operator and satisfies the equation [1,2]

$$\frac{dI}{dt}=\frac{\partial I}{\partial t}+\frac{1}{i\hbar}[H,I]=0. \tag{6}$$

The eigenfunctions $\phi_n(q,t)$ of $I(t)$ are assumed to form a complete orthonormal set corresponding to the time-independent eigenvalue λ_n. Thus

$$I\phi_n(q,t)=\lambda_n\phi_n(q,t), \tag{7}$$

where $\langle\phi_n'|\phi_n\rangle=\delta_{n',n}$.

Now consider the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(q,t)=H(t)\psi(q,t), \quad (8)$$

with

$$H(t)=-\frac{\hbar^2}{2M(t)}\frac{\partial^2}{\partial q^2}+\frac{1}{2}M(t)\omega^2(t)q^2, \quad (9)$$

where $p=-i\hbar\,\partial/\partial q$ has been used. Lewis and Riesenfeld [1,2] showed that the solution $\psi_n(q,t)$ to the Schrödinger equation (8) is related to $\phi_n(q,t)$ by the relation

$$\psi_n(q,t)=\exp[i\alpha_n(t)]\phi_n(q,t), \quad (10)$$

where the phase functions $\alpha_n(t)$ satisfy the equation

$$\hbar\frac{d\alpha_n(t)}{dt}=\left\langle \phi_n\left| i\hbar\frac{\partial}{\partial t}-H(t)\right|\phi_n\right\rangle. \quad (11)$$

Then, since each $\psi_n(q,t)$ satisfies the Schrödinger equation, the general solution of Eq. (8) may be written as

$$\psi(q,t)=\sum_n C_n\exp[i\alpha_n(t)]\phi_n(q,t), \quad (12)$$

where the C_n are constant.

III. SCHRÖDINGER WAVE FUNCTIONS

To obtain the exact Schrödinger wave function for the time-dependent oscillator (1) we proceed as follows. Consider the unitary transformation

$$\phi_n'(q,t)=\mathcal{U}\phi_n(q,t), \quad (13)$$

with

$$\mathcal{U}=\exp\left(-\frac{iM(t)\dot{\rho}}{2\hbar\rho}q^2\right). \quad (14)$$

Under this unitary transformation the eigenvalue equation (7) is mapped into

$$I'\phi_n'(q,t)=\lambda_n\phi_n'(q,t), \quad (15)$$

with

$$I'=\mathcal{U}I\mathcal{U}^\dagger=-\frac{\hbar^2}{2}\frac{\rho^2\partial^2}{\partial q^2}+\frac{1}{2}\frac{q^2}{\rho^2}. \quad (16)$$

If we now define a new independent variable $\sigma=q/\rho$, we can write the eigenvalue equation in the form

$$\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\sigma^2}+\frac{\sigma^2}{2}\right]\varphi_n(\sigma)=\lambda_n\varphi_n(\sigma) \quad (17)$$

or

$$I'\varphi_n(\sigma)=\lambda_n\varphi_n(\sigma), \quad (18)$$

where

$$\phi_n'(q,t)=\frac{1}{\rho^{1/2}}\varphi_n(\sigma)=\frac{1}{\rho^{1/2}}\varphi_n(q/\rho). \quad (19)$$

The factor $1/\rho^{1/2}$ is introduced into Eq. (19) so that the normalization conditions

$$\int \phi_n^{*\prime}(q,t)\phi_n'(q,t)dq=\int \varphi_n^*(\sigma)\varphi_n(\sigma)d\sigma=1 \quad (20)$$

hold. Now Eq. (17) is an ordinary one-dimensional Schrödinger equation, whose solution is given by

$$\varphi_n(\sigma)=\left[\frac{1}{\pi^{1/2}\hbar^{1/2}n!2^n}\right]^{1/2}\exp[-\sigma^2/2\hbar]\mathcal{H}_n\left[\left(\frac{1}{\hbar}\right)^{1/2}\sigma\right], \quad (21)$$

where

$$\lambda_n=\hbar(n+\tfrac{1}{2}), \quad (22)$$

and $\mathcal{H}_n$ is the usual Hermite polynomial of order n. Thus, by using Eqs. (13), (14), (19), and (21) we find that

$$\phi_n(q,t)=\left[\frac{1}{\pi^{1/2}\hbar^{1/2}n!2^n\rho}\right]^{1/2}\exp\left[\frac{iM(t)}{2\hbar}\left(\frac{\dot{\rho}}{\rho}+\frac{i}{M(t)\rho^2}\right)q^2\right]\times\mathcal{H}_n\left[\left(\frac{1}{\hbar}\right)^{1/2}\frac{q}{\rho}\right]. \quad (23)$$

There remains the problem of finding the phases $\alpha_n(t)$ which satisfy Eq. (11). Carrying out the unitary transformation $\mathcal{U}$ the right-hand side of Eq. (11) becomes

$$\hbar\dot{\alpha}_n(t)=\left\langle \phi_n'\left| i\hbar\frac{\partial}{\partial t}+i\hbar\frac{\dot{\rho}}{\rho}q\frac{\partial}{\partial q}+i\hbar\frac{\dot{\rho}}{2\rho}-\frac{I'}{M\rho^2}\right|\phi_n'\right\rangle, \quad (24)$$

where we have used the auxiliary equation (5) to eliminate $\omega^2(t)$ from $H(t)$. Next substituting Eq. (19) into Eq. (24) we find

$$\hbar\dot{\alpha}_n(t)=\left\langle \varphi_n\left| -\frac{I'}{M\rho^2}\right|\varphi_n\right\rangle. \quad (25)$$

Using Eq. (18) and the normalization of φ_n we have

$$\alpha_n(t)=-\left(n+\frac{1}{2}\right)\int_0^t\frac{1}{M(t')\rho^2}dt'. \quad (26)$$

Finally, using Eq. (10) and (23) we find that the exact solution of the Schrödinger equation (8) is

$$\psi_n(q,t)=\exp[i\alpha_n(t)]\left[\frac{1}{\pi^{1/2}\hbar^{1/2}n!2^n\rho}\right]^{1/2}\times\exp\left[\frac{iM(t)}{2\hbar}\left(\frac{\dot{\rho}}{\rho}+\frac{i}{M(t)\rho^2}\right)q^2\right]\mathcal{H}_n\left[\left(\frac{1}{\hbar}\right)^{1/2}\frac{q}{\rho}\right], \quad (27)$$

where the phase functions $\alpha(t)$ are given by Eq. (26).

When the mass is constant, i.e., $M(t)=m$, our new wave function (27) reduces to those obtained in Refs. [3,5,6]. On the other hand, for the general case where the mass is time dependent the Schrödinger wave function (27) agrees with that of Ref. [8] by setting $\rho^2(t)=g_-(t)/\omega_I$. Also note that the result (27) is different from that obtained in Ref. [5], which as we have already mentioned, is not correct.

IV. COHERENT STATES

Consider the time-dependent creation and annihilation operators defined as

$$a' = \left[\frac{1}{2\hbar}\right]^{1/2}\left[\left(\frac{q}{\rho}\right) + i\rho p\right], \tag{28}$$

$$a'^{\dagger} = \left[\frac{1}{2\hbar}\right]^{1/2}\left[\left(\frac{q}{\rho}\right) - i\rho p\right], \tag{29}$$

where $p = -i\hbar\partial/\partial q$ and $[a', a'^{\dagger}] = 1$. In terms of a' and $a'^{\dagger}$ the invariant I' can be written as

$$I' = \hbar(a'^{\dagger}a + \tfrac{1}{2}). \tag{30}$$

Now coherent states for I' have the form [9,10]

$$\varphi_{\alpha}(\sigma,t) = \exp[-|\alpha|^2/2]\sum_n \frac{\alpha^n}{(n!)^{1/2}}\exp[i\alpha_n(t)]\varphi_n(\sigma), \tag{31}$$

where $\alpha_n(t)$ is given by Eq. (26) and α is an arbitrary complex number.

Now using Eqs. (13), (14), (19), and (31) we obtain that the coherent states for the time-dependent system described by the Hamiltonian (1) are given

$$\phi_{\alpha}(q,t) = \frac{1}{\rho^{1/2}}\exp\left[\frac{iM(t)\dot{\rho}}{2\hbar\rho}q^2\right]\varphi_n(\sigma,t), \tag{32}$$

where $\sigma = q/\rho$. These states satisfy the eigenvalue equation

$$a\phi_{\alpha}(q,t) = \alpha(t)\phi_{\alpha}(q,t), \tag{33}$$

where

$$a = \mathcal{U}^{\dagger}a'\mathcal{U} = \left[\frac{1}{2\hbar}\right]^{1/2}\left[\left(\frac{q}{\rho}\right) + i(\rho p - M\dot{\rho}q)\right], \tag{34}$$

and

$$\alpha(t) = \alpha\exp[2i\alpha_0], \tag{35}$$

$$\alpha_0(t) = -\frac{1}{2}\int_0^t \frac{dt'}{M(t')\rho^2}. \tag{36}$$

Observe that operator (34) factors the invariant (4) as $I = \hbar(a^{\dagger}a + 1/2)$.

From Eqs. (28), (29), and (33) we find that the expectation value of q in the state $\phi_{\alpha}(q,t)$ is given by

$$\langle q\rangle = (2\hbar|\alpha|^2\rho^2)^{1/2}\sin(\Omega(t) + \delta), \tag{37}$$

where δ is the argument of the complex number α and $\Omega(t) = -2\alpha_0(t)$. Also after some calculation we find the uncertainties in q and p in the state $\phi_{\alpha}(q,t)$ are given by

$$(\Delta q)^2 = \frac{\hbar}{2}\rho^2, \tag{38}$$

$$(\Delta p)^2 = \frac{\hbar}{2}\left(\frac{1}{\rho^2} + \mu^2\dot{\rho}^2\right). \tag{39}$$

Thus the uncertainty product is expressed as

$$(\Delta q)(\Delta p) = \frac{\hbar}{2}[1 + M^2(t)\rho^2\dot{\rho}^2]^{1/2}, \tag{40}$$

and, in general, does not obtain its minimun value. However, we have already shown in Ref. [11] that the states $\phi_{\alpha}(q,t)$ are equivalent to well-known squeezed states whose characteristic aspect is the squeezing. On the other hand, note that the results (38) and (39) agreee with those of Ref. [7] by setting $\rho^2 = g_-(t)/\omega_I$ and are different from those of Ref. [5].

V. SUMMARY

In this paper we have used a unitary transformation and the Lewis and Riesenfeld invariant method to obtain a Schrödinger wave function for the harmonic oscillator with time-dependent mass and frequency. Our wave function agrees with that of Ref. [8] which has been obtained by using the Heisenberg picture approach.We have also constructed coherent states for the time-dependent oscillator and found the expectation value of q which reproduces the classical motion. Futhermore, we have calculated the uncertanity product, whose result coincides with that of Ref. [7]. Finally , we would like to mention that our overall results correct the analogous one presented previously in Ref. [5].

[1] H. R. Lewis, Jr., Phys. Rev. Lett. **27**, 510 (1967); J. Math. Phys. **9**, 1976 (1968).

[2] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).

[3] D. C. Khandekar and S.V. Lawande, J. Math. Phys. **16**, 384 (1975).

[4] I. A. Pedrosa, J. Math. Phys. **28**, 2662 (1987).

[5] C. M. A. Dantas, I. A. Pedrosa, and B. Baseia, Phys. Rev. A **45**, 1320 (1992).

[6] K. H. Yean, H. J. Kim, C. I. Um, T. F. George, and L. N. Pandey, Phys. Rev. A **50**, 1035 (1994).

[7] J. Y. Ji, J. K. Kim, and S. P. Kim, Phys. Rev. A **51**, 4268 (1995).

[8] J. Y. Ji, J. K. Kim, S. P. Kim, and K. S. Son, Phys. Rev. A **52**, 3352 (1995).

[9] J. G. Hartley and J. R. Ray, Phys. Rev. D **25**, 382 (1982).

[10] J. R. Ray, Phys. Rev. D **25**, 3417 (1982).

[11] I. A. Pedrosa, Phys. Rev. D **36**, 1279 (1987).