Operational theory of homodyne detection

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We discuss a balanced homodyne detection scheme with imperfect detectors in the framework of the operational approach to quantum measurement. We show that a realistic homodyne measurement is described by a family of operational observables that depends on the experimental setup, rather than a single field quadrature operator. We find an explicit form of this family, which fully characterizes the experimental device and is independent of a specific state of the measured system. We also derive operational homodyne observables for the setup with a random phase, which has been recently applied in an ultrafast measurement of the photon statistics of a pulsed diode laser. The operational formulation directly gives the relation between the detected noise and the intrinsic quantum fluctuations of the measured field. We demonstrate this with two examples: the operational uncertainty relation for the field quadratures, and the homodyne detection of suppressed fluctuations in photon statistics. [S1050-2947(97)00904-9]

PACS number(s): 42.50.Dv, 03.65.Bz

I. INTRODUCTION

Homodyne detection is a well-known technique in detecting phase-dependent properties of optical radiation. In quantum optics it has been widely used in studies and applications of squeezed light [1]. A statistical distribution of the outcomes of a homodyne detector has recently found novel applications in the measurement of the quantum state of light via optical homodyne tomography [2] and the direct probing of quantum phase space by photon counting [3]. The phase sensitivity of homodyne detection is achieved by performing a superposition of the signal field with a coherent local oscillator by means of a beam splitter [4]. It was an important observation [5] that in a balanced scheme, with a 50%:50% beam splitter, the local oscillator noise can be canceled by subtracting the photocurrents of the detectors facing two outgoing beams. Then, in the limit of a classical local oscillator, the statistics of difference photocounts is simply a signal quadrature distribution rescaled by the amplitude of the local oscillator [6,7]. Therefore, balanced homodyne detection is an optical realization of an abstract quantum-mechanical measurement of the field quadratures described by a quantum observable $\hat{x}_\theta$. The statistical outcomes of an ideal measurement of $\hat{x}_\theta|x_\theta\rangle = x_\theta|x_\theta\rangle$, are described by the spectral measure

$$p(x_\theta) = \langle |x_\theta\rangle\langle x_\theta| \rangle. \tag{1}$$

Although the spectral measure contains all the relevant statistical information about the homodyne measurement, it corresponds to a quantity that is measured by an ideal noise-free detector. Due to this property $\hat{x}_\theta$ will be called an *intrinsic homodyne quantum observable*.

However, analysis of the homodyne setup with imperfect detectors [7] shows that the relation between the statistics of the difference counts and the quadrature spectral distribution is, in fact, more complicated. The distribution measured in a real experiment is smoothed by a convolution with a Gaussian function of width dependent on the detector efficiency. Consequently, realistic homodyne detection cannot be straightforwardly interpreted as a measurement of the intrinsic field quadratures $\hat{x}_\theta$.

A recent experimental application of homodyne detection to the reconstruction of the photon-number distribution of a weak field from a pulsed diode laser [8] has shown that a homodyne setup with the fluctuating phase θ is a powerful tool in measuring phase-insensitive properties of light. However, it is not possible to associate with this setup any spectral measure, even in the case of perfect detectors. Therefore, homodyne detection with the random phase cannot be described in terms of measuring any intrinsic quantum observable.

It is the purpose of this paper to show that homodyne detection provides an interesting and nontrivial example of a realistic quantum measurement leading to operational quantum observables, i.e., to quantum operators that depend on properties of a specific experimental setup used in homodyne detection. In particular, these operational observables will depend on the detector losses described by a quantum efficiency η and on the phase θ of the local oscillator used to probe the signal field. Such operational observables provide a natural link between the quantum formalism and raw data recorded in a realistic homodyne experiment.

General features of the operational approach, with references to earlier literature are given in [9]. The main conclusions of this approach, if applied to the homodyne measurement, can be summarized as follows. A quantity delivered by the homodyne experiment is a propensity density $\Pr(a)$ of a certain classical variable a. This density is given by an expectation value of an a-dependent positive operator valued

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measure (POVM), denoted by $\hat{\mathcal{H}}(a)$

$$\Pr(a) = \langle \hat{\mathcal{H}}(a) \rangle. \qquad (2)$$

Thus, the POVM given by $\hat{\mathcal{H}}(a)$ corresponds to a realistic homodyne detection and is the mathematical representation of the device dependent measurement. In one way of looking at quantum measurements, the emphasis is put on the construction and properties of such POVMs. In such an approach, in realistic homodyne detection, the spectral decomposition $dx_\theta |x_\theta\rangle\langle x_\theta|$ of the intrinsic observable $\hat{x}_\theta$, is effectively replaced by the POVM $da\hat{\mathcal{H}}(a)$. Consequently the moments of $\Pr(a)$ can be represented as

$$\overline{a^n} = \int da\, a^n \Pr(a) = \langle \hat{x}_{\theta_{\mathcal{H}}}^{(n)} \rangle, \qquad (3)$$

defining in this way a family of *operational homodyne quantum observables*

$$\hat{x}_{\theta_{\mathcal{H}}}^{(n)} = \int da\, a^n \hat{\mathcal{H}}(a), \qquad (4)$$

where the index $\mathcal{H}$ stands for the homodyne detection scheme associated with the given POVM. This family characterizes the experimental device and is independent on a specific state of the measured system.

In this paper we derive and discuss the family of operational observables for balanced homodyne detection with imperfect photodetectors. We show that for balanced homodyne detection an exact reconstruction of the POVM $\hat{\mathcal{H}}(a)$ and of the corresponding operational quantum quadratures $\hat{x}_{\theta_{\mathcal{H}}}^{(n)}$ can be performed. Thus, homodyne detection provides a nontrivial measurement scheme for which an exact derivation of the corresponding POVM and the operational observables is possible. The interest in construction of this operational algebra is due to the fact that the number of physical examples, where the operational description can be found explicitly, is very limited [10]. We show that the algebraic properties of the $\hat{x}_{\theta_{\mathcal{H}}}^{(n)}$ differ significantly from those of the powers of $\hat{x}_\theta$. In particular, $\hat{x}_{\theta_{\mathcal{H}}}^{(2)} \neq (\hat{x}_{\theta_{\mathcal{H}}}^{(1)})^2$. This property will have immediate consequences in the discussion of the operational uncertainty relation for dispersions measured in a realistic setup.

This paper has the following structure. First, in Sec. II, we derive the POVM and the generating operator for the operational observables. Their explicit form is found in the limit of a classical local oscillator in Sec. III. Given this result, we discuss the operational uncertainty relation in Sec. IV. In Sec. V we derive the family of operational homodyne observables for the homodyne detector with a random phase between the signal and the local oscillator fields, and relate them to the intrinsic photon-number operator. These calculations link the homodyne noise with fluctuations of the photon statistics, and can be useful in the time-resolved measurement of the properties of pulsed diode lasers. Finally, in Sec. VI we summarize the results.

II. GENERATING OPERATOR FOR HOMODYNE DETECTION

The family of the operational homodyne quantum observables defined in Eq. (4) can be written conveniently with the help of the generating operator

$$\hat{Z}_{\mathcal{H}}(\lambda) = \int da\, e^{i\lambda a} \hat{\mathcal{H}}(a). \qquad (5)$$

Operational quantum observables are given by derivatives of the generating operator at $\lambda = 0$

$$\hat{x}_{\mathcal{H}}^{(n)} = \frac{1}{i^n} \frac{d^n}{d\lambda^n} \hat{Z}_{\mathcal{H}}(\lambda) \bigg|_{\lambda=0}. \qquad (6)$$

This compact representation will noticeably simplify further calculations.

We will start the calculations by finding the generating operator for the homodyne detector. In a balanced setup, the signal field described by an annihilation operator $\hat{a}$, is superimposed on a local oscillator $\hat{b}$ by means of a 50%:50% beam splitter. The annihilation operators of the outgoing modes are given, up to the irrelevant phase factors, by the relation

$$\begin{pmatrix} \hat{c} \\ \hat{d} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}. \qquad (7)$$

We will assume that the local oscillator is in a coherent state $|\beta\rangle_{LO}$. If another state of the local oscillator is considered, our formulas can be generalized in a straightforward manner by averaging the results over an appropriate Glauber's P representation.

A quantity recorded in the experiment is the statistics of the difference counts between photodetectors facing the modes $\hat{c}$ and $\hat{d}$. The difference of the counts ΔN corresponds to the classical variable, denoted before as a, recorded in a homodyne detection experiment. The POVM $\hat{\mathcal{H}}(\Delta N)$ describing this detection scheme can be easily derived. It is clear that this POVM is an operator acting in the Hilbert space of the signal mode. Its explicit form can be found with the help of standard theory of photodetection [11]

$$\hat{\mathcal{H}}(\Delta N) = \sum_{n_1 - n_2 = \Delta N} \text{Tr}_{LO} \Bigg\{ |\beta\rangle\langle\beta|_{LO} : e^{-\eta\hat{c}^\dagger\hat{c}} \frac{(\eta\hat{c}^\dagger\hat{c})^{n_1}}{n_1!}$$
$$\times e^{-\eta\hat{d}^\dagger\hat{d}} \frac{(\eta\hat{d}^\dagger\hat{d})^{n_2}}{n_2!} : \Bigg\}, \qquad (8)$$

where η is the quantum efficiency, assumed to be identical for both the detectors. In this formula the partial trace is over the local oscillator mode and a marginal average with a fixed value of ΔN is performed. We will now convert this POVM into the generating operator according to Eq. (5). As we will discuss later, the POVM $\hat{\mathcal{H}}$ and, consequently, the generating operator $\hat{Z}_{\mathcal{H}}$ have their natural parametrization, independent of the LO intensity. Before we find this scaling, we will use ξ instead of λ as a parameter of the generating operator

$$\hat{Z}_{\mathcal{H}}(\xi) = \sum_{\Delta N = -\infty}^{\infty} e^{i\xi \Delta N} \hat{\mathcal{H}}(\Delta N)$$

$$= \mathrm{Tr}_{LO}\{|\beta\rangle\langle\beta|_{LO}:\exp[\,\eta(e^{i\xi}-1)\hat{c}^\dagger\hat{c}$$

$$+ \eta(e^{-i\xi}-1)\hat{d}^\dagger\hat{d}]:\}. \qquad (9)$$

Let us transform this expression to the form which does not contain the normal ordering symbol. For this purpose we will use the technique developed by Yuen and Shapiro [4] consisting of extending the Hilbert space by two additional modes $\hat{c}_v$ and $\hat{d}_v$ and constructing the field annihilation operators

$$\hat{c}_d = \sqrt{\eta}\,\hat{c} + \sqrt{1-\eta}\,\hat{c}_v, \quad \hat{d}_d = \sqrt{\eta}\,\hat{d} + \sqrt{1-\eta}\,\hat{d}_v. \quad (10)$$

The generating operator can be written in the extended four-mode space using these operators as

$$\hat{Z}_{\mathcal{H}}(\xi) = \mathrm{Tr}_{LO,v}\{|\beta\rangle\langle\beta|_{LO}$$

$$\otimes |0\rangle\langle 0|_v:\exp[(e^{i\xi}-1)\hat{c}_d^\dagger\hat{c}_d + (e^{-i\xi}-1)\hat{d}_d^\dagger\hat{d}_d]:\},$$

$$(11)$$

where Tr_v denotes the trace over both the vacuum modes $\hat{c}_v$ and $\hat{d}_v$. We can now apply the relation [12]

$$:\exp[(e^{i\xi}-1)\hat{v}^\dagger\hat{v}]: = \exp(i\xi\hat{v}^\dagger\hat{v}) \qquad (12)$$

valid for an arbitrary bosonic annihilation operator $\hat{v}$, which finally gives

$$\hat{Z}_{\mathcal{H}}(\xi) = \mathrm{Tr}_{LO,v}\{|\beta\rangle\langle\beta|_{LO}\otimes|0\rangle\langle 0|_v\exp[i\xi(\hat{c}_d^\dagger\hat{c}_d - \hat{d}_d^\dagger\hat{d}_d)]\}. \qquad (13)$$

This expression contains the most compact form of the homodyne POVM. The exponent in Eq. (11) resembles the one from Eq. (9), with $\hat{c},\hat{d}$ replaced by $\hat{c}_d,\hat{d}_d$ and the detector efficiency equal to one. It is known [4], that there is a physical picture behind this similarity. An imperfect photodetector can be equivalently described by an ideal detector preceded by a beam splitter with the power transmissivity equal to the quantum efficiency of the real detector, assuming that the vacuum state enters through the unused port of the beam splitter. Mathematically, this construction corresponds to the so-called Naimark extension of the POVM into a projective measure on a larger Hilbert space [10].

III. APPROXIMATION OF A CLASSICAL LOCAL OSCILLATOR

When the local oscillator is in a strong coherent state, the bosonic operators $\hat{b},\hat{b}^\dagger$ can be replaced by c numbers β,β^*. However, this approximation violates the bosonic commutation relations for the pairs $\hat{c},\hat{c}^\dagger$ and $\hat{d},\hat{d}^\dagger$, which have been used implicitly several times in the manipulations involving $\hat{Z}_{\mathcal{H}}(\xi)$. Therefore, some care should be taken when considering the classical limit of the local oscillator.

We will perform the approximation on the exponent of Eq. (13). We will replace the quantum average over the state

$|\beta\rangle$ by inserting β,β^* in place of $\hat{b},\hat{b}^\dagger$ and keep only the terms linear in β. This gives

$$\hat{c}_d^\dagger\hat{c}_d - \hat{d}_d^\dagger\hat{d}_d = \sqrt{\eta}\beta^*\left(\sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\frac{\hat{c}_v + \hat{d}_v}{\sqrt{2}}\right) + \mathrm{H.c.}$$

$$(14)$$

The operator in the parenthesis has a form analogous to Eq. (10) with the combination $(\hat{c}_v + \hat{d}_v)/\sqrt{2}$ as a vacuum mode. Consequently, the imperfectness of the photodetectors in the balanced homodyne detection can be modeled by superposing the signal on a fictitious vacuum mode before superposing it with the local oscillator and attenuating the amplitude of the local oscillator field by $\sqrt{\eta}$. This observation was originally made by Leonhardt and Paul [13], and is an example of the Naimark extension involving a nonquantized local oscillator.

Under the approximation of a classical local oscillator, it is now easy to perform the trace over the vacuum modes with the help of the Baker-Campbell-Hausdorff formula. This yields

$$\hat{Z}_{\mathcal{H}}(\xi) = \exp[-\xi^2\eta(1-\eta)|\beta|^2/2]\exp[i\xi\eta(\beta\hat{a}^\dagger + \beta^*\hat{a})]. \qquad (15)$$

The exponent $\exp[-\xi^2\eta(1-\eta)|\beta|^2/2]$ introduces a specific ordering of the creation and annihilation operators in the generating operator. Therefore, the detector efficiency η can be related to the ordering of the operational observables. For example, for $\eta = 1/2$ we get $\hat{Z}_{\mathcal{H}}(\xi) = \exp(i\xi\beta^*\hat{a}/2)\exp(i\xi\beta\hat{a}^\dagger/2)$, i.e., the generating operator is ordered antinormally.

The expansion of the generating operator into a power series of $i\xi$ gives

$$\frac{1}{i^n}\frac{d^n\hat{Z}_{\mathcal{H}}}{d\xi^n}\bigg|_{\xi=0} = \left[\frac{1}{i}\left(\frac{\eta(1-\eta)}{2}\right)^{1/2}|\beta|\right]^n H_n\left[i\left(\frac{\eta}{1-\eta}\right)^{1/2}\hat{x}_\theta\right], \qquad (16)$$

where H_n denotes the nth Hermite polynomial and $\hat{x}_\theta$ is the standard quadrature operator

$$\hat{x}_\theta = \frac{e^{i\theta}\hat{a}^\dagger + e^{-i\theta}\hat{a}}{\sqrt{2}} \qquad (17)$$

expressed in terms of the creation and annihilation operators of the signal field, and dependent on the local oscillator phase θ defined as $\beta = |\beta|e^{i\theta}$. In the terminology of the operational approach to quantum measurement, $\hat{x}_\theta$ is called an *intrinsic quantum observable*, since it refers to internal properties of the system independent of the measuring device [9].

It will be convenient to change the parameter of the generating operator in order to make the derivatives (16) independent of the amplitude of the local oscillator. A scaling factor which can be directly obtained from an experiment is the square root of the intensity of the local oscillator field measured by the photodetector. We will multiply it by $\sqrt{2}$ in order to get the intrinsic quadrature operator (17) in the limit

$\eta \rightarrow 1$. Thus substituting $\xi = \lambda / \sqrt{2 \eta |\beta|^2}$ yields the generating operator independent of the amplitude of the local oscillator

$$\hat{Z}_{\mathcal{H}}(\lambda, \theta) = \exp[-\lambda^2(1-\eta)/4 + i\lambda \sqrt{\eta/2}(e^{i\theta}\hat{a}^\dagger + e^{-i\theta}\hat{a})]. \tag{18}$$

The derivatives of $\hat{Z}_{\mathcal{H}}(\lambda, \theta)$ give the final form of the family of the operational observables $\hat{x}_{\theta_{\mathcal{H}}}^{(n)}$ for the homodyne detector

$$\hat{x}_{\theta_{\mathcal{H}}}^{(n)} = \left(\frac{\sqrt{1-\eta}}{2i}\right)^n H_n\left[i\left(\frac{\eta}{1-\eta}\right)^{1/2}\hat{x}_\theta\right]. \tag{19}$$

The algebraic properties of the operational observables are quite complicated, since $\hat{x}_{\theta_{\mathcal{H}}}^{(n)}$ is not simply an nth power of $\hat{x}_{\theta_{\mathcal{H}}}^{(1)}$. Thus a single operator does not suffice to describe the homodyne detection with imperfect detectors. Complete characterization of the setup is provided by the whole family of operational observables. In fact the operators $\hat{x}_{\theta_{\mathcal{H}}}^{(n)}$ define an infinite algebra of operational homodyne observables for an arbitrary state of the signal mode. As mentioned above, for $\eta = 50\%$, the general formula reduces to antinormally ordered powers of the intrinsic quadrature operators

$$\hat{x}_{\theta_{\mathcal{H}}}^{(n)} = \frac{1}{2^{n/2}} \vdots (\hat{x}_\theta)^n \vdots. \tag{20}$$

This expression shows that the operational operators are, in some sense, equivalent to a prescription of ordering of the intrinsic operators. This prescription is dynamical in character, i.e., it depends on the efficiency η of the detectors used in the homodyne detection. In fact the homodyne operational algebra is defined by a one-parameter family of dynamical orderings defined by the generating operator derived in this section.

IV. OPERATIONAL UNCERTAINTY RELATION

With explicit forms of operational observables in hand, we can now analyze their relation to the intrinsic quadrature operator. For this purpose, let us look at the first lowest-order operational quadrature observables

$$\hat{x}_{\theta_{\mathcal{H}}}^{(1)} = \eta^{1/2}\hat{x}_\theta, \quad \hat{x}_{\theta_{\mathcal{H}}}^{(2)} = \eta\left(\hat{x}_\theta^2 + \frac{1-\eta}{2\eta}\right),$$

$$\hat{x}_{\theta_{\mathcal{H}}}^{(3)} = \eta^{3/2}\left(\hat{x}_\theta^3 + \frac{3}{2}\frac{1-\eta}{\eta}\hat{x}_\theta\right). \tag{21}$$

The imperfectness of photodetectors influences the operational observables in two ways. The first one is a trivial rescaling of the observables by the powers of $\sqrt{\eta}$, the second way is a contribution of the lower-order terms to the operational counterparts of $\hat{x}_\theta^n$. In order to see its consequences let us investigate the rescaled operational variance $\overline{(\Delta N)^2} - (\overline{\Delta N})^2$

$$\delta x_{\theta_{\mathcal{H}}}^2 = \frac{1}{2\eta|\beta|^2}(\overline{(\Delta N)^2} - (\overline{\Delta N})^2). \tag{22}$$

From the definitions of the operational operators it is clear that this operational variance involves $\hat{x}_{\theta_{\mathcal{H}}}^{(2)}$ and $\hat{x}_{\theta_{\mathcal{H}}}^{(1)}$. The combination of these two operators is in general different from the intrinsic variance. Because of this the operational dispersion of x_θ is

$$\delta x_{\theta_{\mathcal{H}}}^2 = \langle \hat{x}_{\theta_{\mathcal{H}}}^{(2)}\rangle - \langle \hat{x}_{\theta_{\mathcal{H}}}^{(1)}\rangle^2 = \eta\left(\Delta x_\theta^2 + \frac{1-\eta}{2\eta}\right), \tag{23}$$

where $\Delta x_\theta = \sqrt{\langle \hat{x}_\theta^2\rangle - \langle \hat{x}_\theta\rangle^2}$ is the intrinsic quantum dispersion of the quadrature x_θ. This intrinsic dispersion is enhanced by a term coming from the imperfectness of the detectors. Thus the imperfectness of the photodetectors introduces an additional noise to the measurement and deteriorates its resolution.

Using the above result we can derive the operational uncertainty relation for the quadratures related to the angles θ and θ'

$$\delta x_{\theta_{\mathcal{H}}}\delta x_{\theta'_{\mathcal{H}}} \geq \eta\left(\Delta x_\theta \Delta x_{\theta'} + \frac{1-\eta}{2\eta}\right). \tag{24}$$

Again, an additional term is added to the intrinsic uncertainty product. This situation is similar to that in Ref. [14], where it was argued that taking into account the measuring device raises the minimum limit for the uncertainty product. However, that discussion concerned a *simultaneous* measurement of canonically conjugate variables, which is not the case in homodyne detection. Using the intrinsic uncertainty relation $\Delta x_\theta \Delta x_{\theta'} \geq |\sin(\theta-\theta')|/2$ we get the result that the right-hand side in the operational relation (24) is not smaller than $[\eta|\sin(\theta-\theta')| + 1 - \eta]/2$.

One may wonder if the definition of squeezing is affected by the operational operators. Let us consider the two quadratures $\delta x_{\theta_{\mathcal{H}}}$ and $\delta x_{\theta+(\pi/2)_{\mathcal{H}}}$. In this case the operational uncertainty,

$$\delta x_{\theta_{\mathcal{H}}}\delta x_{\theta+\pi/2_{\mathcal{H}}} \geq \tfrac{1}{2}, \tag{25}$$

is independent of η. However, it has to be kept in mind that only a part of the operational dispersion comes from the field fluctuations. The easiest way to discuss this is to rewrite Eq. (23) in the form of

$$\delta x_{\theta_{\mathcal{H}}} = \left[\eta(\Delta x_\theta)^2 + (1-\eta)\left(\frac{1}{\sqrt{2}}\right)^2\right]^{1/2}, \tag{26}$$

which shows that the operational dispersion is a quadratic average of the intrinsic field dispersion Δx_θ and the detector noise $1/\sqrt{2}$ that corresponds to the vacuum fluctuation level. These contributions enter with the weights η and $1-\eta$, respectively. Therefore, if a squeezed quadrature is measured with imperfect detectors, the observed dispersion is larger than the intrinsic one.

V. HOMODYNE DETECTION WITH RANDOM PHASE

Homodyne detection is used primarily to detect phase-dependent properties of light. However, it has been recently shown that even a setup with a random phase between the signal and local oscillator fields can be a useful tool in optical experiments [8,15]. Although in this case the phase sensitivity is lost, the homodyne detector can be applied to measure phase-independent quantities and such a setup presents some advantages over a single photodetector. First, the information on the statistics of the field is carried by the photocurrent difference between the two rather intense fields. Within existing detector technology, this quantity can be measured with a significantly better efficiency than the weak field itself. Secondly, the spatiotemporal mode that is actually measured by the homodyne detector is defined by the shape of the local oscillator field. Consequently, application of the local oscillator in the form of a short pulse allows the measurement to be performed with an ultrafast sampling time. This technique has been used in Ref. [8] to measure the time resolved photon-number statistics from a diode laser operating below threshold. The achieved sampling time was significantly shorter than that of previously used methods.

The photon-number distribution and other phase-independent quantities are reconstructed from the average of the random-phase homodyne statistics calculated with the so-called *pattern functions* [16,17]. For commonly used quantities, such as the diagonal elements of the density matrix in the Fock basis, these pattern functions take a quite complicated form. In this section we will consider observables that are related to the experimental data in the most direct way, the moments of the homodyne statistics with randomized phase. We will derive the family of operational observables and relate them to the powers of the photon-number operator $\hat{n} = \hat{a}^\dagger \hat{a}$.

The generating operator for homodyne detection with random phase $\hat{Z}_{\mathcal{R}}$ ($\mathcal{R}$ stands for the random phase) is obtained readily from $\hat{Z}_{\mathcal{H}}$ by averaging it over the phase θ. This gives

$$\hat{Z}_{\mathcal{R}}(\lambda) = \int_0^{2\pi} \frac{d\theta}{2\pi} \hat{Z}_{\mathcal{H}}(\lambda, \theta) = e^{-\lambda^2/4} : J_0(\lambda \sqrt{2\eta \hat{a}^\dagger \hat{a}}) :,$$
(27)

where J_0 is the Bessel function of the 0th order. With the help of the result derived in the Appendix, the normally ordered form of the Bessel function can be transformed into the following expression:

$$\hat{Z}_{\mathcal{R}}(\lambda) = e^{-\lambda^2/4} L_{\hat{n}}(\eta \lambda^2/2),$$
(28)

where the index of the Laguerre polynomial is the the photon-number operator. The Laguerre polynomials with a operator valued index is defined by the decomposition in the Fock basis.

The family of operational observables is given by the derivatives of the generating operator

$$\hat{x}_{\mathcal{R}}^{(n)} = \frac{1}{i^n} \frac{d^n}{d\lambda^n} \hat{Z}_{\mathcal{R}}(\lambda) \Big|_{\lambda=0}.$$
(29)

Since the homodyne statistics averaged over the phase is even, the odd derivatives disappear. A straightforward calculation yields the operators for even $n = 2m$

$$\hat{x}_{\mathcal{R}}^{(2m)} = \frac{(2m-1)!!}{2^m} : L_m(-2\eta \hat{a}^\dagger \hat{a}) :$$

$$= \frac{(2m-1)!!}{2^m} \sum_{k=0}^{m} \binom{m}{k} \frac{(2\eta)^k}{k!} \hat{n}(\hat{n}-1) \cdots (\hat{n}-k+1).$$
(30)

This formula shows that $\hat{x}_{\mathcal{R}}^{(2m)}$ is a polynomial of $\hat{n}$ of the order of m. Therefore, the first m moments of the photon-number distribution can be computed from $\langle \hat{x}_{\mathcal{R}}^{(2)} \rangle, \ldots, \langle \hat{x}_{\mathcal{R}}^{(2m)} \rangle$. The two lowest-order observables are given explicitly by

$$\hat{x}_{\mathcal{R}}^{(2)} = \eta \hat{n} + \tfrac{1}{2}$$

$$\hat{x}_{\mathcal{R}}^{(4)} = \tfrac{3}{2} [\eta^2 \hat{n}^2 + \eta(2-\eta)\hat{n} + \tfrac{1}{2}].$$
(31)

It is seen that even in the case of ideal noise-free detectors $\hat{x}_{\mathcal{R}}^{(4)} \neq (\hat{x}_{\mathcal{R}}^{(2)})^2$ and the family of the operational observables has nontrivial algebraic properties. Equations (30) can be inverted using the method presented in the Appendix, which yields an expression for the normally ordered moments of the photon statistics in terms of the family of the operational observables

$$:(\eta \hat{a}^\dagger \hat{a})^n: = \frac{(n!)^2}{2^n} \sum_{m=0}^{n} \frac{(-1)^{n-m} 4^m}{(2m)!(n-m)!} \hat{x}_{\mathcal{R}}^{(2m)}.$$
(32)

For example, the first two moments are given by:

$$\hat{a}^\dagger \hat{a} = \frac{1}{\eta} \left(\hat{x}_{\mathcal{R}}^{(2)} - \frac{1}{2} \right),$$

$$:(\hat{a}^\dagger \hat{a})^2: = \frac{1}{\eta^2} \left(\frac{2}{3} \hat{x}_{\mathcal{R}}^{(4)} - 2\hat{x}_{\mathcal{R}}^{(2)} + \frac{1}{2} \right).$$
(33)

As an illustration, let us express the normalized photon-number variance $Q = (\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 - \langle \hat{n} \rangle)/\langle \hat{n} \rangle$ [18] in terms of the expectation values of $\hat{x}_{\mathcal{R}}^{(2)}$ and $\hat{x}_{\mathcal{R}}^{(4)}$. This variance is used to characterize the sub-Poissonian statistics of light. After some simple algebra we arrive at

$$Q = \frac{1}{\eta} \frac{\frac{2}{3}\langle \hat{x}_{\mathcal{R}}^{(4)} \rangle - \langle \hat{x}_{\mathcal{R}}^{(2)} \rangle^2 - \langle \hat{x}_{\mathcal{R}}^{(2)} \rangle + \frac{1}{4}}{\langle \hat{x}_{\mathcal{R}}^{(2)} \rangle - \frac{1}{2}}.$$
(34)

Thus, the variance Q can be read out from the two lowest moments of the homodyne statistics with the randomized phase. The photodetector efficiency η enters into the above formula only as an overall scaling factor. This result is analogous to that obtained for the setup with a single imperfect detector, and is due to the fact that Q describes normally ordered field fluctuations.

We have shown in this section, that there is a one-to-one correspondence between the powers of the photon-number operator and the operational observables describing the ho-

modyne setup with the random phase. However, the POVM representing this scheme does not have properties of a spectral measure due to missing information about the phase of the local oscillator. Consequently, the intrinsic quantum observable cannot be identified. In contrast, the family of the operational observables is well defined, and reflects the nature of the experimental data collected in a realistic setup.

VI. CONCLUSIONS

We have presented the operational description of the balanced homodyne detection scheme with imperfect photodetectors. For homodyne detection it is possible to derive exact expressions for the POVM and the corresponding algebra of operational operators. The result of these calculations shows that a whole family of operational observables rather than a single operator should to be used to discuss a realistic setup. This family allows one to relate easily the experimentally observed fluctuations to the intrinsic properties of the measured system.

APPENDIX

In this Appendix we present mathematical details of manipulations with the generating operator $\hat{Z}_{\mathcal{R}}(\lambda)$ for homodyne detection with the random phase. First we will show how to convert it into the form that does not contain the normal ordering symbol. Let us start by rewriting Eq. (12) to the form $:\exp(\varepsilon \hat{a}^\dagger \hat{a}):=(1+\varepsilon)^{\hat{n}}$ and decomposing its right-hand side in the Fock basis $\{|n\rangle\}$

$$
\begin{aligned}
:e^{\varepsilon \hat{a}^\dagger \hat{a}}: &= \sum_{n=0}^{\infty} (1+\varepsilon)^n |n\rangle\langle n| \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \varepsilon^k |n\rangle\langle n| \\
&= \sum_{k=0}^{\infty} \varepsilon^k \sum_{n=k}^{\infty} \binom{n}{k} |n\rangle\langle n|.
\end{aligned}
\tag{A1}
$$

If we now assume the convention that $\binom{n}{k}=0$ for $n<k$, the lower bound of the second sum in the last expression can be changed to $n=0$. It is then natural to denote this sum as a binomial coefficient of the operator $\hat{n}$. Comparing the equal powers of ε in Eq. (A1) yields a very compact representation of the normally ordered powers of $\hat{a}^\dagger \hat{a}$ in terms of $\hat{n}$

$$
(\hat{a}^\dagger)^k \hat{a}^k = k! \binom{\hat{n}}{k}.
\tag{A2}
$$

Expanding the normally ordered Bessel function in Eq. (27) and applying the above identity gives

$$
\begin{aligned}
:J_0(\lambda \sqrt{2\eta \hat{a}^\dagger \hat{a}}): &= \sum_{k=0}^{\infty} \frac{(-\eta\lambda^2/2)^k}{(k!)^2} (\hat{a}^\dagger)^k \hat{a}^k \\
&= \sum_{k=0}^{\infty} \binom{\hat{n}}{k} \frac{(-\eta\lambda^2/2)^k}{k!} \\
&= L_{\hat{n}}(\eta\lambda^2/2),
\end{aligned}
\tag{A3}
$$

where the Laguerre polynomial with the photon-number operator index

$$
L_{\hat{n}}(x) = \sum_{n=0}^{\infty} L_n(x)|n\rangle\langle n|
\tag{A4}
$$

is defined analogously to the binomial coefficient via decomposition in the Fock basis.

We will now express the normally ordered moments of the photon-number operator in terms of the operational observables $\hat{x}_{\mathcal{R}}^{(2m)}$. We start from the integral formula for the Bessel function [19]

$$
\int_0^{\infty} :J_0(\lambda \sqrt{2\eta \hat{a}^\dagger \hat{a}}):\exp(-q\lambda^2)\lambda\, d\lambda = \frac{1}{2q}:\exp\left(-\frac{\eta \hat{a}^\dagger \hat{a}}{2q}\right):.
\tag{A5}
$$

Substituting $q=1/2\zeta$ and $:J_0(\lambda \sqrt{2\eta \hat{a}^\dagger \hat{a}}):=\hat{Z}_{\mathcal{R}}(\lambda)\exp(\lambda^2/4)$ gives

$$
:\exp(-\zeta \eta \hat{a}^\dagger \hat{a}): = \frac{1}{\zeta}\int_0^{\infty} \hat{Z}_{\mathcal{R}}(\lambda)\exp\left[-\left(\frac{1}{2\zeta}-\frac{1}{4}\right)\lambda^2\right]\lambda\, d\lambda.
\tag{A6}
$$

We now insert the expansion of the generating operator in λ

$$
\hat{Z}_{\mathcal{R}}(\lambda) = \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^{2m}}{(2m)!} \hat{x}_{\mathcal{R}}^{(2m)}
\tag{A7}
$$

and perform the integrals over λ, which gives

$$
:\exp(-\zeta \eta \hat{a}^\dagger \hat{a}): = \sum_{m=0}^{\infty} \frac{(-1)^m m!}{(2m)!} \frac{(2\zeta)^m}{(1-\zeta/2)^{m+1}} \hat{x}_{\mathcal{R}}^{(2m)}.
\tag{A8}
$$

Expanding both sides of this equation into the power series in ζ, and comparing coefficients multiplying equal powers of ζ yields Eq. (32).

[1] See, for example, Special issues of J. Opt. Soc. Am. B **4** (10), (1987); J. Mod. Opt. **34** (6/7), (1987).

[2] K. Vogel and H. Risken, Phys. Rev. A **40**, R2847 (1989); D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993); D. T. Smithey, M. Beck, J. Cooper, M. G. Raymer, and A. Faridani, Phys. Scr. **T48**, 35 (1993); G. Breitenbach, T. Müller, S. F. Pereira, J.-Ph. Poizat, S. Schiller, and J. Mlynek, J. Opt. Soc. Am. B **12**, 2304 (1995).

[3] S. Wallentowitz and W. Vogel, Phys. Rev. A **53**, 4528 (1996); K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **76**, 4344 (1996).

[4] H. P. Yuen and J. H. Shapiro, in *Coherence and Quantum Optics IV*, edited by L. Mandel and E. Wolf (Plenum, New York, 1978), p. 719; IEEE Trans. Inf. Theory **IT-26**, 78 (1980).

[5] H. P. Yuen and V. W. S. Chan, Opt. Lett. **8**, 177, (1983); **8**, 345(E) (1983); B. L. Schumaker, *ibid.* **9**, 189 (1984).

[6] S. L. Braunstein, Phys. Rev. A **42**, 474 (1990).

[7] W. Vogel and J. Grabow, Phys. Rev. A **47**, 4227 (1993); M. Munroe, D. Boggavarapu, M. E. Anderson, U. Leonhart, and M. G. Raymer, in *Coherence and Quantum Optics VII*, edited by J. H. Eberly, L. Mandel, and E. Wolf (Plenum, New York, 1996).

[8] M. Munroe, D. Boggavarapu, M. E. Anderson, and M. G. Raymer, Phys. Rev. A **52**, R924 (1995).

[9] B.-G. Englert and K. Wódkiewicz, Phys. Rev. A **51**, R2661 (1995).

[10] See, for example, P. Busch, P. J. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement* (Springer-Verlag, Berlin, 1991).

[11] See, for example, L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, England, 1995), Chap. 14.

[12] R. M. Wilcox, J. Math. Phys. **8**, 962 (1967).

[13] U. Leonhardt and H. Paul, Phys. Rev. A **48**, 4598 (1993).

[14] K. Wódkiewicz, Phys. Lett. A **124**, 207 (1987).

[15] U. Leonhardt and I. Jex, Phys. Rev. A **49**, R1555 (1994).

[16] G. M. D'Ariano, C. Macchiavello, and M. G. A. Paris, Phys. Rev. A **50**, 4298 (1994); U. Leonhardt, H. Paul, and G. M. D'Ariano, *ibid.* **52**, 4899 (1995).

[17] Th. Richter, Phys. Lett. A **211**, 327 (1996); U. Leonhardt, M. Munroe, T. Kiss, Th. Richter, and M. G. Raymer, Opt. Comm. **127**, 144 (1996).

[18] L. Mandel, Opt. Lett. **4**, 205 (1979).

[19] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. (Cambridge University Press, Cambridge, England, 1944), Sec. 13.3.