# Collective effects in the microlaser

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(Received 2 August 1996)

In a microlaser experiment performed by An *et al.* with collaborators [Phys. Rev. Lett. **73**, 3375 (1994)] an atomic beam interacts with a microcavity field under such conditions that the average number of atoms in the cavity is less than unity. However, due to the random statistics of atoms in the beam, the probability of two atoms being in the cavity simultaneously in time is not negligible. This must lead to collective two-atom effects that are not included in the standard microlaser model. We have developed a theory that incorporates such collective effects. This theory allows a better understanding of the observed experimental results and predicts some different features of the microlaser. [S1050-2947(97)02003-9]

PACS number(s): 42.55.−f, 42.50.Ct, 42.50.Dv

## I. INTRODUCTION

The micromaser and its optical counterpart, the microlaser, belong to the simplest systems in quantum optics. Both consist of just one atom interacting with a single mode of the quantized electromagnetic field. Neglecting all irreversible processes such as spontaneous emission of the atom or damping of the field mode, Jaynes and Cummings have solved this problem exactly [1]. A characteristic feature of their solution is a periodic exchange of energy between the atom and the field mode.

In the experimental realization of the micromaser or microlaser an atomic beam of excited two-level atoms is injected into a microwave or optical cavity at such a low rate that the mean number of atoms inside the cavity is less than unity. Since the atoms are injected in their excited state, they are capable of supplying energy to the cavity mode and thus produce a gain. However, in contrast to the conservative Jaynes-Cummings model, in a real laser energy is dissipated due to cavity losses and atomic spontaneous emission. The competition between gain and dissipation leads to an equilibrium photon number in the cavity.

While the micromaser was realized experimentally more than ten years ago [2], success in the optical domain was reported only recently [3]. In the experiment of Ref. [3] the mean number of atoms in the cavity was ranging from 0.1 to 1.0, resulting in a mean number of photons between 0.14 and 11. Standard micromaser theory is based on the assumption that at any time only one atom is present inside the cavity [4]. Comparing that theory with their data the experimenters found a discrepancy once the number of atoms was in excess of $\sim 0.6$. Intuitively one would expect that due to the random statistics of the atoms in the atomic beam the one-atom model begins to break down when the mean number of atoms approaches unity. For such injection rates one has to take into account events with two, three, and more atoms inside the cavity. It was the desire to explain the discrepancy between the experiment [3] and the one-atom micromaser theory that motivated us to take on the problem of including collective atomic effects.

The question of the two-atom collective effects has been addressed theoretically in several papers [5–7]. In Refs. [5,6] the authors have considered the case of the Poissonian atomic injection and of an ideal cavity, i.e., neglected the damping of the field mode. Our approach is closer to that of Ref. [7] where cavity loss was taken into account. However, the analysis of Ref. [7] allows only for such two-atom events for which both atoms enter and leave the resonator simultaneously. We shall consider a Poissonian beam of atoms. Pairs of atoms can then have partial overlaps of their sojourns in the cavity.

## II. STATISTICS OF THE ATOMIC BEAM AND MASTER EQUATION FOR THE CAVITY MODE

As in the one-atom micromaser theory developed by Filipowicz, Javanainen, and Meystre in Ref. [4], we consider a beam of excited two-level atoms injected into a single-mode optical cavity. We assume that the velocity of the atoms in the beam is selected such that all atoms interact with the cavity mode for the same time $\tau$, and that time is much shorter than the cavity decay time $1/\kappa$. We can then neglect the mode damping during the flight of an atom through the cavity and consider the atom-field interaction and the damping process separately.

We also assume that the atoms are injected at so low a rate that the mean number of atoms in the cavity is smaller than unity. However, due to the random sequence of atoms in the beam several atoms may be in the cavity simultaneously. We confine our discussion to two-atom effects.

The master equation for the density matrix $\rho$ of the cavity mode to be derived will involve one- and two-atom generators $u_1$ and $u_2$. To introduce these we consider the density matrix $\rho(t+\tau)$ at the moment $t+\tau$ when an atom leaves the cavity without having met another one there. We can relate this density matrix to the one valid at the time $t$ at which the atom entered the cavity,

$$\rho(t+\tau)=(1+u_1)\rho(t), \tag{2.1}$$

thus defining the one-atom generator $u_1$. We take the increment $\rho(t+\tau)-\rho(t)$ due to a single atom to be ''small.''

 

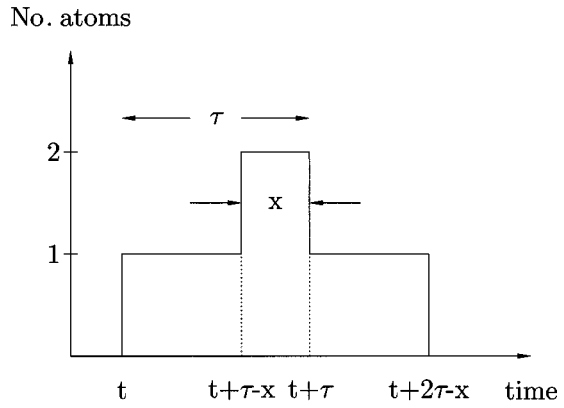


FIG. 1. Temporal structure of the two-atom event.

Similarly, we introduce a two-atom generator $u_2(x)$ for the small increment of the field density matrix brought about by two atoms that have jointly interacted with the cavity mode during an interval $x$, $0<x<\tau$ (see Fig. 1),

$$\rho(t+2\tau-x)=[1+u_2(x)]\rho(t). \tag{2.2}$$

Clearly, the two-atom generator $u_2(x)$ depends on the overlap interval $x$. For instance, in the limit of no overlap, $x=0$, we would have

$$\rho(t+2\tau)=(1+u_1)^2\rho(t)\approx(1+2u_1)\rho(t), \tag{2.3}$$

where we have used the assumed smallness of $u_1$. From Eq. (2.3) we observe that $u_2(0)=2u_1$. In fact, it is the difference between $u_2(x)$ and $2u_1$ that manifests the presence of collective two-atom effects. Another limiting case would be that of full overlap, $x=\tau$, for all pairs of atoms. We would then have the ideal ''two-atom microlaser'' of Ref. [7].

For an atomic beam like the one in the experiment of [3] the arrivals of atoms in the cavity may be considered as the uncorrelated events of a Poissonian process. In particular, the overlap $x$ is random. We proceed to the appropriate statistical considerations, keeping the generators $u_1$ and $u_2(x)$ unspecified for the moment.

Let us consider a time interval $\Delta t\gg\tau$ and denote by $n$ the number of atoms that have entered the cavity at the moments $t_1,\ldots,t_n$ during that interval. For a Poissonian process the probability density of the spacings $s=t_{i+1}-t_i$, $i=1,\ldots,n-1$ between subsequent enterings is the exponential

$$p(s)=re^{-rs}, \quad s>0, \tag{2.4}$$

with $r$ the rate (in atoms/sec) of the atomic flux. We infer the probability density $w(x)$ of the overlap $x$ of two atomic sojourns in the cavity as follows. Let an atom (see Fig. 1) have entered the cavity at $t=0$ and the next one at $t=s>0$. As a function of the spacing $s$ the overlap $x(s)$ reads

$$x(s)=\begin{cases}\tau-s & \text{for } 0\leqslant s\leqslant\tau\\ 0 & \text{for } s>\tau,\end{cases} \tag{2.5}$$

whereupon the distribution $w(x)$ of overlaps may be written as

$$w(x)=\int_0^\infty ds\ p(s)\delta(x-x(s))$$
$$=re^{-r(\tau-x)}\theta(\tau-x)+\delta(x)(1-e^{-r\tau}). \tag{2.6}$$

This density is normalized as $\int_0^\infty dx w(x)=1$.

Next, we introduce a set of probabilities needed for the intended average over the atomic statistics. We first observe

$$\text{Prob}\{\text{one spacing}>\tau\}=\int_\tau^\infty ds\ p(s)=e^{-r\tau}\approx 1-\epsilon. \tag{2.7}$$

The product $r\tau\equiv\epsilon$ determines the mean number of atoms inside the cavity. The smallness of that number, $\epsilon\ll 1$, is almost constitutive of the microlaser and will be taken for granted throughout what follows. Using Eq. (2.7) we proceed to defining the probabilities

$$W_m(n)=\text{Prob}\{(n-m-1)\text{ spacings}>\tau, m\text{ spacings}<\tau\}$$
$$=\binom{n-1}{m}\epsilon^m(1-\epsilon)^{n-m-1}, \tag{2.8}$$

which are obviously binomial in character and normalized as $\Sigma_{m=0}^{n-1}W_m(n)=1$. Actually, we may neglect the $W_m(n)$ with $m>1$ since they scale as $W_m(n)\propto\epsilon^m$. To within corrections of second order in $\epsilon$ the normalization reads

$$W_0(n)+W_1(n)\approx 1. \tag{2.9}$$

To treat the case of a single overlap we shall need a slightly refined tool,

$$w_1(n,x)dx=\text{Prob}\{\text{one interval between } x \text{ and } x+dx, 0<x<\tau, \text{ all the other intervals } >\tau\}$$
$$=(n-1)(1-\epsilon)^{n-2}w(x)\ dx. \tag{2.10}$$

Upon integrating the latter density over the overlap $x$ we recover the probability for precisely two out of the $n$ atoms to jointly sojourn in the resonator for whatever fraction of the passage time $\tau$, $\int_0^\tau dx\ w_1(n,x)=W_1(n)$.

Now let us evaluate separately the contributions to the field density matrix $\rho(t+\Delta t)$ from events when $n$ atoms have passed through the cavity without overlap and, on the other hand, with one overlap of duration $x$. The first of these contributions is obviously given by

$$\rho_0(t+\Delta t)=(1+u_1)^n\rho(t)\approx(1+nu_1)\rho(t). \tag{2.11}$$

The second one equals

$$\rho_1(t+\Delta t)=\frac{1}{(n-1)}\{[1+u_2(x)](1+u_1)^{n-2}$$
$$+(1+u_1)[1+u_2(x)](1+u_1)^{n-3}+\cdots$$
$$+(1+u_1)^{n-2}[1+u_2(x)]\}\rho(t). \tag{2.12}$$

Here we have accounted for all $n-1$ possibilities to place the selected pair of atoms with the overlap interval $x$ among the total of $n$ atoms. Again keeping only terms linear in $u_1$ and $u_2(x)$ we can simplify as

$$\rho_1(t+\Delta t)\approx[1+(n-2)u_1+u_2(x)]\rho(t) \qquad (2.13)$$

and conclude that to first order in $u_1$ and $u_2$ the position of the selected pair inside the group of $n$ atoms is immaterial.

To find the resulting $\rho(t+\Delta t)$ allowing for both possibilities of zero and one overlap, we add the contributions $\rho_0(t+\Delta t)$ and $\rho_1(t+\Delta t)$, weighting them with the corresponding probabilities,

$$\rho(t+\Delta t)=\Big\{ W_0(n)+W_1(n)+[nW_0(n)+(n-2)W_1(n)]u_1 + \int_0^\tau dx\ w_1(n,x)u_2(x)\Big\}\rho(t). \qquad (2.14)$$

To first order in $\epsilon$ this yields

$$\rho(t+\Delta t)=\Big\{1+[n(1-2\epsilon)+2\epsilon]u_1 +(n-1)r\int_0^\tau dx\ u_2(x)\Big\}\rho(t). \qquad (2.15)$$

Finally, we must acknowledge the number $n$ of atoms passed during the time interval $[t,t+\Delta t]$ to be random. Since for a Poissonian beam that number is certainly independent of $\rho(t)$ the average to be performed simply amounts to replacing $n$ by its mean $\overline{n}=r\Delta t$. Choosing the time interval $\Delta t$ so as to accommodate a large number of atoms, $\overline{n}\gg 1$, we can neglect the terms of order unity compared to the terms of order $\overline{n}$ and thus arrive at

$$\frac{\rho(t+\Delta t)-\rho(t)}{\Delta t}\to\dot\rho(t)=\Lambda_p\rho(t),$$

$$\Lambda_p=(1-2\epsilon)u_1+r\int_0^\tau dx\ u_2(x). \qquad (2.16)$$

Here we have introduced a ''pumping'' generator $\Lambda_p$ describing the excitative action of the atomic beam on the field mode.

As a last step towards the desired master equation we add to the time rate of change of the density matrix $\rho(t)$ a term accounting for the damping of the field mode due to leakage of photons through the outcoupling mirror,

$$\Lambda_d\rho(t)=(\kappa/2)(n_{\rm th}+1)(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a) +(\kappa/2)n_{\rm th}(2a^\dagger\rho a-aa^\dagger\rho-\rho aa^\dagger). \qquad (2.17)$$

The cavity decay constant is here denoted by $\kappa$ while $n_{\rm th}$ is the thermal number of photons in the cavity mode. We have thus arrived at our master equation

$$\dot\rho(t)=(\Lambda_p+\Lambda_d)\rho(t). \qquad (2.18)$$

## III. ONE- AND TWO-ATOM GENERATORS

### A. One-atom generator $u_1$

The explicit form of the one-atom generator $u_1$ defined in Eq. (2.1) can be found in Ref. [4]. We sketch here its derivation since we shall use the same technique and some results to construct the two-atom generator $u_2(x)$ below.

Let us consider the interaction between a two-level atom with the upper state $|a\rangle$ and the lower state $|b\rangle$ and a single mode of an optical cavity. The interaction Hamiltonian reads

$$H=-i\hbar g(a^\dagger\sigma-a\sigma^\dagger). \qquad (3.1)$$

Here $a$ and $a^\dagger$ are annihilation and creation operators of photons of the cavity mode. The operators $\sigma$, $\sigma^\dagger$, and $\sigma_z=(1/2)(\sigma^\dagger\sigma-\sigma\sigma^\dagger)$ describe the atomic polarization and inversion; they obey the angular momentum commutation relations $[\sigma^\dagger,\sigma]=2\sigma_z$, $[\sigma,\sigma_z]=\sigma$. The atom-field coupling constant $g$ is normalized so as to have the dimension of a frequency.

We shall use the photon-number representation for the field mode inside the cavity. In this representation only two states of the combined atom-plus-field system become coupled. These are the state $|an\rangle$ with the atom in the upper level and $n$ photons in the cavity mode, and the state $|b,n+1\rangle$ with the atom in the lower level and $n+1$ photons in the resonator. The wave function of the atom-field system can be represented as the sum

$$|\Psi(t)\rangle=\sum_{n=0}^{\infty}[C_{an}(t)|a\ n\rangle+C_{b,n+1}(t)|b,\ n+1\rangle]. \qquad (3.2)$$

Schrödinger's equation $|\dot\Psi(t)\rangle=-(i/\hbar)H|\Psi(t)\rangle$ then takes the form of a system of equations for the coefficients $C_{an}(t)$ and $C_{b,n+1}(t)$. For simplicity, we assume resonance between the atom and the field mode and work in the interaction picture where one easily finds the solutions as

$$C_{an}(t+\tau)=C_{an}(t)\cos[g\tau\sqrt{n+1}] +C_{b,n+1}(t)\sin[g\tau\sqrt{n+1}],$$

$$C_{b,n+1}(t+\tau)=C_{b,n+1}(t)\cos[g\tau\sqrt{n+1}] -C_{an}(t)\sin[g\tau\sqrt{n+1}]. \qquad (3.3)$$

The reduced density matrix $\rho(t)$ of the field mode thus reads $\rho(t)=\Sigma_{\alpha,m,n}C^*_{an}C_{\alpha m}(t)|n\rangle\langle m|$. Confining the discussions of the present paper to the photon statistics we need to consider only diagonal elements $p_n(t)\equiv\langle n|\rho(t)|n\rangle$ of the field density matrix [8]. After the interaction with a single atom the diagonal element $p_n(t+\tau)$ is given by

$$p_n(t+\tau)=|C_{an}(t+\tau)|^2+|C_{bn}(t+\tau)|^2. \qquad (3.4)$$

At the moment $t$ when the atom in its upper state $|a\rangle$ enters the cavity, the density matrix $S(t)$ of the atom-field system represents a tensor product of the field density matrix $\rho(t)$ and the atom density matrix $|a\rangle\langle a|$. At that moment $t$ we therefore have

$$|C_{an}(t)|^2=p_n(t),\quad C_{bn}(t)=0, \qquad (3.5)$$

and thus obtain the following relation between the diagonal elements of the field density matrix before and after interaction with a single atom,

$$p_n(t+\tau)=\sum_m (\delta_{nm}+u_{1,nm})p_m(t)$$
$$=[1-\alpha_{n+1}]p_n(t)+\alpha_n p_{n-1}(t),$$
$$\alpha_n=\sin^2[g\tau\sqrt{n}], \tag{3.6}$$

where $\alpha_n$ is the one-atom coherent gain. The relation (3.6) defines the single-atom generator $u_1$ with respect to the diagonal elements of the density matrix in the Fock representation, i.e., as a matrix $u_{1,nm}$.

### B. Two-atom generator $u_2(x)$

We now proceed to the interaction of two atoms with the cavity mode. To distinguish the atoms we shall denote the atomic polarization and inversion operators as $\sigma_i$, $\sigma_i^\dagger$, and $\sigma_{zi}$ with $i=1,2$. Let us imagine the first atom to enter the cavity in its upper state $|a\rangle$ at the moment $t$ and to interact alone with the cavity field until at the time $t+\tau-x$ the second atom flies in (see Fig. 1). The interaction Hamiltonian for the time interval $[t,t+\tau-x]$ is given by Eq. (3.1) with the substitution $\sigma=\sigma_1,\sigma_z=\sigma_{z1}$. Once the second atom has entered the cavity in its upper state $|a\rangle$, both atoms interact with the cavity mode together during the overlap time $x$, $0<x<\tau$. For this interval the interaction Hamiltonian is

$$H=-i\hbar g[a^\dagger(\sigma_1+\sigma_2)-a(\sigma_1+\sigma_2)^\dagger]. \tag{3.7}$$

At the moment $t+\tau$ the first atom leaves the cavity. For a time span $t-x$ the cavity mode then interacts with the second atom only according to the Hamiltonian (3.1) now with $\sigma=\sigma_2,\sigma_z=\sigma_{z2}$. The two-atom event ends when the second atom leaves the cavity at the time $t+2\tau-x$.

Now four states of the atoms-plus-field system must be reckoned with. These are the state $|aa\ n\rangle$ with both atoms in the upper level and $n$ photons in the cavity; the state $|ba,n+1\rangle$ with the first atom in the lower level, the second one in the upper level, and $n+1$ photons in the cavity; the state $|ab,n+1\rangle$ resulting from the former by exchanging the two atoms; and finally the state $|bb,n+2\rangle$ with both atoms deexcited and two photons delivered to the cavity mode. We write the joint wave function as the superposition

$$|\Psi(t)\rangle=\sum_{n=0}^{\infty}[C_{aan}(t)|aa\ n\rangle+C_{ab,n+1}(t)|ab,n+1\rangle$$
$$+C_{ba,n+1}(t)|ba,n+1\rangle+C_{bb,n+2}(t)|bb,n+2\rangle]. \tag{3.8}$$

The coefficients $C_{\alpha\beta q}(t)$ satisfy the Schrödinger equation

$$\frac{d}{dt}\begin{bmatrix} C_{aan} \\ C_{ab,n+1} \\ C_{ba,n+1} \\ C_{bb,n+2}\end{bmatrix}=\begin{bmatrix} 0 & g\sqrt{n+1} & g\sqrt{n+1} & 0 \\ -g\sqrt{n+1} & 0 & 0 & g\sqrt{n+2} \\ -g\sqrt{n+1} & 0 & 0 & g\sqrt{n+2} \\ 0 & -g\sqrt{n+2} & -g\sqrt{n+2} & 0\end{bmatrix}\begin{bmatrix} C_{aan} \\ C_{ab,n+1} \\ C_{ba,n+1} \\ C_{bb,n+2}\end{bmatrix}. \tag{3.9}$$

We first employ this for the interval from $t$ to $t+\tau-x$ when only the first atom is present. Then only two coefficients, $C_{aan}(t)$ and $C_{ba,n+1}(t)$ are involved. Taking into account that at the time $t$ only the amplitude $C_{\alpha\beta q}(t)$ is nonzero,

$$|C_{aan}(t)|^2=p_n(t),\quad C_{abn}(t)=C_{ban}(t)=C_{bbn}(t)=0, \tag{3.10}$$

we can use the solution (3.3) to find the coefficients $C_{\alpha\beta q}(t+\tau-x)$. These serve as initial conditions for the subsequent interaction of both atoms during the interval $[t+\tau-x,t+\tau]$. Solving the system (3.9) we find the coefficients $C_{\alpha\beta q}(t+\tau)$ in terms of the $C_{\alpha\beta q}(t+\tau-x)$,

$$C_{aan}(t+\tau)=[1-2\epsilon_n s_n^2]C_{aan}(t+\tau-x)+\sqrt{2\epsilon_n}c_n s_n C_{ba,n+1}(t+\tau-x),$$
$$C_{ab,n+1}(t+\tau)=-s_n^2 C_{ab,n+1}(t+\tau-x)-\sqrt{2\epsilon_n}c_n s_n C_{aan}(t+\tau-x), \tag{3.11}$$
$$C_{ba,n+1}(t+\tau)=c_n^2 C_{ba,n+1}(t+\tau-x)-\sqrt{2\epsilon_n}c_n s_n C_{aan}(t+\tau-x),$$
$$C_{bb,n+2}(t+\tau)=2\sqrt{\epsilon_n(1-\epsilon_n)}s_n^2 C_{aan}(t+\tau-x)-\sqrt{2(1-\epsilon_n)}c_n s_n C_{ba,n+1}(t+\tau-x),$$

where we have introduced the shorthand

$$c_n=\cos[gx\sqrt{n+3/2}],\quad s_n=\sin[gx\sqrt{n+3/2}],$$
$$\epsilon_n=\frac{n+1}{2n+3}. \tag{3.12}$$

When at the time $t+\tau$ the first atom leaves the cavity the further evolution proceeds among the field and the second atom only. We can then again employ the results of the previous subsection. Now two coupled pairs of coefficients are to be treated, namely, $C_{aan}(t+\tau)$ with $C_{ab,n+1}(t+\tau)$ and $C_{ba,n+1}(t+\tau)$ with $C_{bb,n+2}(t+\tau)$. Using the single-atom solution (3.3) for both cases we obtain the final coefficients

$C_{\alpha\beta q}(t+2\tau-x)$ after the second atom has left the cavity. Similarly to the one-atom case we find the diagonal element $p_n(t+2\tau-x)$ of the field density matrix by taking the trace over the atomic Hilbert space,

$$p_n(t+2\tau-x)=\sum_{\alpha,\beta=a,b}|C_{\alpha\beta n}(t+2\tau-x)|^2. \quad (3.13)$$

We omit the intermediate algebra and go directly to the final result. However, before we give the explicit expression for the two-atom generator $u_2(x)$, we would like to write its limiting case in which two atoms pass through the cavity with $x=0$, i.e., without overlap. The pertinent result is easily obtained from Eq. (3.6) as

$$p_n(t+2\tau)=R_n+[-\alpha_{n+1}R_n+\alpha_nR_{n-1}], \quad (3.14)$$

where

$$R_n=[1-\alpha_{n+1}]p_n+\alpha_np_{n-1}. \quad (3.15)$$

Quite similar is the structure of $p_n(t+2\tau-x)$ with $x>0$,

$$\begin{aligned}p_n(t+2\tau-x)&=\sum_m[\delta_{nm}+u_{2,nm}(x)]p_m(t)\\&=Q_n+[-\tilde{\alpha}_{n+1}S_n+\tilde{\alpha}_nS_{n-1}],\end{aligned} \quad (3.16)$$

where

$$\begin{aligned}Q_n&=\tilde{R}_n+[-\beta_np_n+\beta_{n-1}p_{n-1}]+[-\gamma_np_n+\gamma_{n-2}p_{n-2}],\\S_n&=\tilde{R}_n+[-\beta_np_n+\beta_{n-1}p_{n-1}]\\&\quad-[(\gamma_n+\delta_n)p_n+(\gamma_{n-1}+\delta_{n-1})p_{n-1}],\\\tilde{R}_n&=[1-\tilde{\alpha}_{n+1}]p_n+\tilde{\alpha}_np_{n-1}.\end{aligned} \quad (3.17)$$

In Eqs. (3.16) and (3.17) we have introduced a single-atom gain coefficient $\tilde{\alpha}_n$ pertaining to the interaction time $\tau-x$ just as $\alpha_n$ as given in Eq. (3.6) pertains to the interaction time $\tau$,

$$\tilde{\alpha}_n=\sin^2[g(\tau-x)\sqrt{n}]. \quad (3.18)$$

The coefficients $\beta_n,\gamma_n$, and $\delta_n$ in Eq. (3.17) are defined as

$$\begin{aligned}\beta_n(x)&=4\epsilon_nc_n^2s_n^2(1-\tilde{\alpha}_{n+1})-2c_n^2s_n^2\tilde{\alpha}_{n+1},\\\gamma_n(x)&=2\epsilon_nc_n^2s_n^2(1-\tilde{\alpha}_{n+1})+s_n^4\tilde{\alpha}_{n+1},\\\delta_n(x)&=4\epsilon_n(1-\epsilon_n)s_n^4(1-\tilde{\alpha}_{n+1})+2(1-\epsilon_n)c_n^2s_n^2\tilde{\alpha}_{n+1},\end{aligned} \quad (3.19)$$

with $c_n,s_n$, and $\epsilon_n$ from Eq. (3.12). We have indicated the dependence on the overlap $x$ here by writing $\beta_n(x)$, etc.; however, in order not to overburden the notation we shall mostly waive such decoration below.

We have thus defined, in Eqs. (3.16), (3.17), and (3.19), the two-atom evolution operator $1+u_2(x)$ with respect to the diagonal elements of the field density matrix, i.e., as a matrix $\delta_{nm}+u_{2,nm}(x)$. It may be worth noting that the expression (3.16) for $p_n(t+2\tau-x)$ readily reveals the generator $u_2(x)$ to reduce to $2u_1$ for $x=0$. Indeed, for $x=0$ we have $\tilde{\alpha}_n=\alpha_n$, i.e., $\tilde{R}_n=R_n$, while all three coefficients $\beta_n,\gamma_n$, and $\delta_n$ vanish.

## IV. MASTER EQUATION AND ITS STATIONARY SOLUTION

Having worked out the single-atom and two-atom generators $u_1,u_2(x)$ as matrices $u_{1,nm},u_{2,nm}(x)$ with respect to the evolution of the diagonal elements of the density matrix in the Fock representation for the field mode, it remains to insert these into the pumping generator $\Lambda_p$ defined in Eq. (2.16),

$$\Lambda_{p,nm}=r\left[(1-2\epsilon)u_{1,nm}+r\int_0^\tau dx u_{2,nm}(x)\right]. \quad (4.1)$$

Similarly, we extract from the damping generator (2.17) a matrix of transition rates $\Lambda_{d,nm}$ [9],

$$\begin{aligned}\sum_m\Lambda_{d,nm}p_m&=\kappa(n_{\rm th}+1)[(n+1)p_{n+1}-np_n]\\&\quad+\kappa n_{\rm th}[np_{n-1}-(n+1)p_n].\end{aligned} \quad (4.2)$$

The master equation for the probabilities $p_n(t)$ can then be written with the help of the following rates of single-step downward and upward transitions and two-step upward transitions (''up'' and ''down'' the ladder of photon numbers),

$$\begin{aligned}&\text{rate}(n\to n-1)\equiv s_n^-=\kappa(n_{\rm th}+1)n,\\&\text{rate}(n\to n+1)\equiv s_n^+=r(\alpha_{n+1}+\epsilon v_n)+\kappa n_{\rm th}(n+1),\\&\text{rate}(n\to n+2)\equiv t_n^+=r\epsilon w_n,\end{aligned} \quad (4.3)$$

with $v_n$ and $w_n$ composed of the rate constants $\alpha_n,\tilde{\alpha}_n(x),\beta_n(x),\gamma_n(x),\delta_n(x)$ introduced above as

$$\begin{aligned}v_n&=\frac{1}{\tau}\int_0^\tau dx\,[(\tilde{\alpha}_{n+1}+\beta_n)(1-\tilde{\alpha}_{n+1}-\tilde{\alpha}_{n+2})\\&\quad-\alpha_{n+1}(1-\alpha_{n+1}-\alpha_{n+2})\\&\quad-(\gamma_n+\delta_n)(\tilde{\alpha}_{n+1}-\tilde{\alpha}_{n+2})],\\w_n&=\frac{1}{\tau}\int_0^\tau dx\,[\gamma_n+\tilde{\alpha}_{n+2}\tilde{\alpha}_{n+1}-\alpha_{n+2}\alpha_{n+1}+\beta_n\tilde{\alpha}_{n+2}\\&\quad-(\gamma_n+\delta_n)\tilde{\alpha}_{n+2}].\end{aligned} \quad (4.4)$$

Clearly, the loss of photons through the outcoupling mirror entails single-step downward transitions as well as (at finite temperatures such that $n_{\rm th}>0$) single-step upward transitions. On the other hand, only upward transitions can be enforced by the atoms since these enter, by our assumption, the cavity in their upper state and can thus deposit but not pick up photons. Single-atom passes can only increase the photon number by one while two-atom events with overlaps may leave behind either one or two added photons. In con-

templating the rates the reader will appreciate that all terms due to two-atom effects are proportional to the small parameter $\epsilon = r\tau$.

In terms of these transition rates the master equation (2.18) reads

$$\dot{p}_n = s_{n+1}^- p_{n+1} - (s_n^+ + s_n^- + t_n^+) p_n + s_{n-1}^+ p_{n-1} + t_{n-2}^+ p_{n-2}. \tag{4.5}$$

The time rate of change $\dot{p}_n$ occurring here can be represented in terms of two probability currents,

$$j_n = s_n^- p_n - s_{n-1}^+ p_{n-1},$$

$$i_n = -t_{n-2}^+ p_{n-2}, \tag{4.6}$$

as

$$\frac{d}{dt} p_n = j_{n+1} - j_n + i_{n+2} - i_n. \tag{4.7}$$

As seen from Eq. (4.3), the first current $j_n$ stems chiefly from the coherent interaction of the field with one atom and the incoherent decay of the field mode, apart from a small two-atom correction in $s_n^+$ proportional to $\epsilon v_n$. Conversely, the second current $i_n$ has its origin entirely in the two-atom collective effects. In the absence of pair effects the current $i_n$ does not arise. The stationary regime is then governed by $j_n = 0$, i.e., detailed balance of upward and downward transitions, $n \leftrightarrow n-1$. The two-step transitions brought about by pair effects destroy detailed balance since they allow probability to flow in loops $n \to n-1 \to n-2 \to n$ [10].

To find the stationary probabilities $\overline{p_n}$ we first formally treat $i_{n+2} - i_n$ as an inhomogeneity in Eq. (4.7) and solve recursively to get $j_n = -i_n - i_{n+1}$ for $n > 2$ after $j_0 = 0$, $j_1 = -i_2$, $j_2 = -i_3$. Upon substituting the definitions (4.6) in that formal solution we arrive at a three-step recursion relation for the $\overline{p_n}$,

$$s_{n+1}^- \overline{p_{n+1}} = (s_n^+ + t_n^+) \overline{p_n} + t_{n-1}^+ \overline{p_{n-1}}. \tag{4.8}$$

We can even further simplify to a two-step recursion relation by introducing the ratio

$$r_n = \overline{p_{n+1}} / \overline{p_n}. \tag{4.9}$$

Indeed, by dividing both sides of Eq. (4.8) by $\overline{p_n}$ we get

$$r_n = a_n + \frac{b_{n-1}}{r_{n-1}}, \tag{4.10}$$

with the coefficients

$$a_n = \frac{s_n^+ + t_n^+}{s_{n+1}^-}, \quad b_n = \frac{t_n^+}{s_{n+2}^-}. \tag{4.11}$$

The ratios $r_n$ are obtained recursively. First, putting $n = 0$ in Eq. (4.8) and requiring that $\overline{p_{-1}} = 0$ we get $r_0 = a_0$. Then, starting with $r_1 = a_1 + b_0 / r_0$ we obtain $r_n$ as the $n$-fold continued fraction,

$$r_n = a_n + \underline{b_{n-1}} \rfloor a_{n-1} + \underline{b_{n-2}} \rfloor a_{n-2} + \cdots + \underline{b_1} \rfloor a_1 + b_0 / r_0, \tag{4.12}$$

where $\rfloor$ demands division by everything that follows. The stationary probability $\overline{p_n}$ is obtained from the $r_k$ as the product

$$\overline{p_n} = \overline{p_0} \prod_{k=0}^{n-1} r_k, \tag{4.13}$$

where $\overline{p_0}$ is determined by normalizing, $\Sigma_{n=0}^{\infty} \overline{p_n} = 1$. Using this stationary probability distribution of the photon number in the cavity we can calculate the mean photon number and its fluctuations.

## V. MEAN PHOTON NUMBER AND PHOTON STATISTICS

In the one-atom theory of the micromaser (see Ref. [4]), the photon statistics are determined by three physical parameters. These are (i) the average number of atoms passed through the cavity during the lifetime of the field, $N_{\mathrm{ex}} = r/\kappa$; (ii) the dimensionless pump parameter, $\theta = N_{\mathrm{ex}}^{1/2} g \tau$; and (iii) the mean number of thermal photons inside the cavity, $n_{\mathrm{th}}$. In addition to these we here incur a fourth parameter, $\epsilon = r\tau$, which gives the mean number of atoms inside the cavity.

Using the stationary distribution function $\overline{p_n}$ found in the previous section we have numerically investigated the mean photon number $\langle n \rangle$,

$$\langle n \rangle = \sum_{k=0}^{\infty} k \overline{p_k}, \tag{5.1}$$

and its normalized standard deviation $\sigma$,

$$\sigma = \left[ \frac{\langle (n - \langle n \rangle)^2 \rangle}{\langle n \rangle} \right]^{1/2}, \tag{5.2}$$

for different values of the four parameters mentioned above.

Figure 2 compares the normalized mean number of photons $\langle n \rangle / N_{\mathrm{ex}}$ as a function of the dimensionless pump parameter $\theta$ without and with collective effects. Both curves have common features such as the threshold value $\theta = 1$, when a finite mean photon number emerges, and the abrupt jumps, which for $N_{\mathrm{ex}} \to \infty$ are customarily associated with phase transitions. Two-atom collective effects shift these transitions towards smaller values of $\theta$. For large values of the pump parameter the jumps become less pronounced and the normalized mean photon number $\langle n \rangle / N_{\mathrm{ex}}$ approaches a constant value. That limiting value is increased by two-atom effects.

Apparently the influence of two-atom collective effects on the mean photon number is opposite to the role of the two-photon absorption studied by Ellinger and Ritsch in Ref. [11]. There it was found that the nonlinear absorption delays the ''phase transitions'' to higher pump parameters. Also the mean photon number for high $\theta$ was found to be smaller with nonlinear absorber than without.

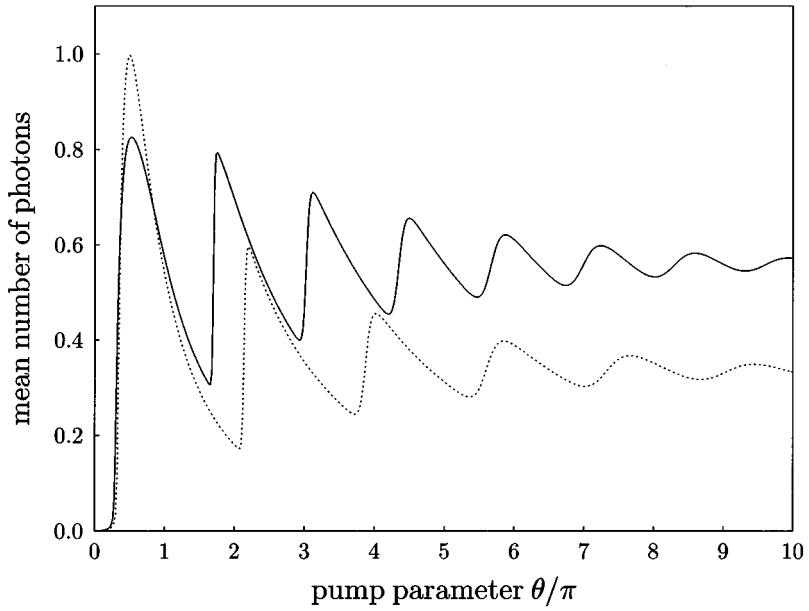


FIG. 2. Stationary mean photon number $\langle n\rangle/N_{\rm ex}$ inside the cavity vs the pump parameter $\theta$ without (dots) and with (solid) collective effects; $N_{\rm ex}=200$, $n_{\rm th}=0.1$, and $\epsilon=0.4$.

In Fig. 3 we have drawn the normalized standard deviation $\sigma$ as a function of the dimensionless pump parameter $\theta$ without and with collective effects. We recall that the value $\sigma=1$ corresponds to Poissonian statistics of photons. Both without and with collective effects we see sharp super-Poissonian peaks at the values of $\theta$ corresponding to the jumps of the mean photon number shown in Fig. 2. For pump parameters between these peaks the variance $\sigma$ takes on values less than unity and thus signals sub-Poissonian statistics. However, the values of $\sigma$ between the peaks on the curve with collective effects are higher then those on the one-atom curve. We conclude, therefore, that collective effects increase the photon fluctuations in sub-Poissonian regions. Again, this is contrary to the case of the two-photon absorption, which was found [11] to decrease $\sigma$.

As in Fig. 2, Fig. 4 compares the normalized mean photon number without and with collective effects, but now for zero temperature of the external reservoir, i.e., for $n_{\rm th}=0$. The one-atom curve then shows a specific fine structure with narrow deep resonances at certain values of $\theta$ (these resonances are not properly resolved on Fig. 4). This fine structure is associated with the so-called trapping states. They were initially predicted in Ref. [12] for a lossless micromaser, i.e., one with ideal mirrors. Later it was understood [13] that the signature of these states persists even when dissipation through the outcoupling mirror is included, provided one sticks to very low temperatures.

For the lossless micromaser the trapping states are the Fock states $|n_q\rangle$ of the cavity field with such number of photons $n_q$ that an atom entering the cavity in its excited state undergoes $q$ full Rabi cycles before leaving the cavity in the excited state,

$$\kappa\tau\sqrt{n_q+1}=\pi q. \tag{5.3}$$

In the absence of dissipation and when the atoms are injected

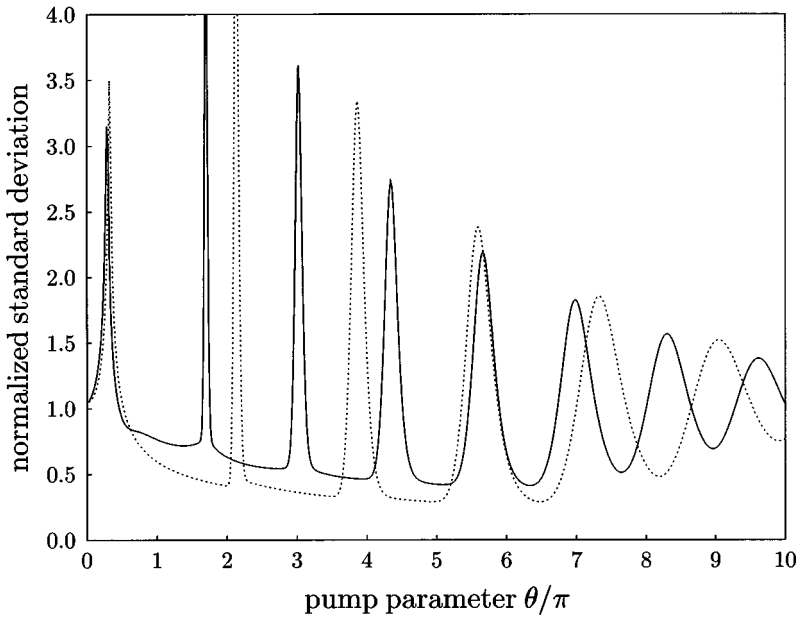


FIG. 3. Normalized standard deviation $\sigma$ of the photon number inside the cavity vs the pump parameter $\theta$ without (dots) and with (solid) collective effects; $N_{\rm ex}=200$, $n_{\rm th}=0.1$, and $\epsilon=0.4$.

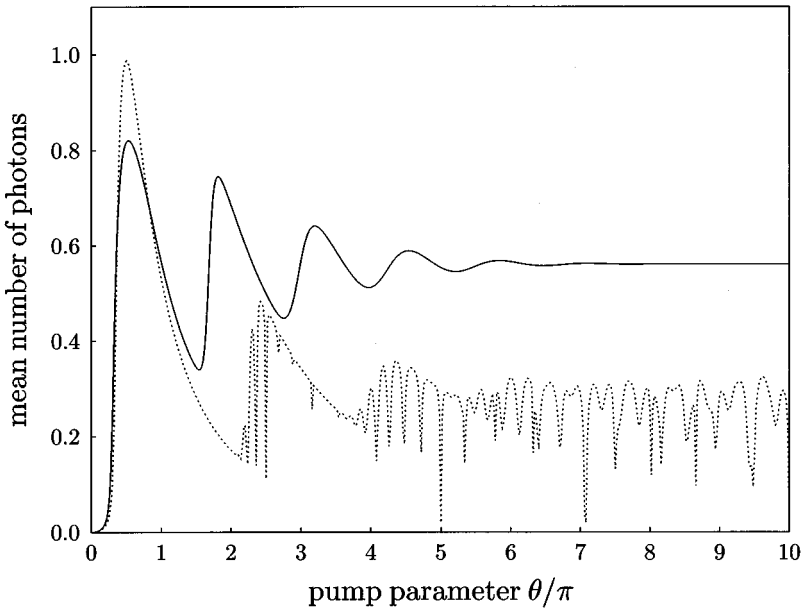


FIG. 4. Stationary mean photon number $\langle n\rangle/N_{\rm ex}$ inside the cavity vs the pump parameter $\theta$ without (dots) and with (solid) collective effects for zero temperature; $N_{\rm ex}=50$ and $\epsilon=0.4$.

in their upper state the stationary state of the micromaser field inside the cavity is a mixture of trapping states. When the dissipation is allowed with finite $n_{\rm th}$ the probability $p_n$ flows between these trapping states according to Eq. (4.3) and the signature of trapping states rapidly becomes washed out. However, for $n_{\rm th}=0$ the probability can flow only downward [see Eq. (4.3)]. Therefore, dissipation at zero temperature does not allow the probability distribution to grow past the trapping state $|n_q\rangle$. If condition (5.3) is fulfilled for some photon number $k$ then the probability distribution $\overline{p_n}=0$ for $n>k$.

The condition (5.3) written in terms of parameters $N_{\rm ex}$ and $\theta$ reads [13]

$$\frac{N_{\rm ex}}{\theta^2}=\frac{n_q+1}{q^2\pi^2}. \tag{5.4}$$

For fixed $N_{\rm ex}$ this gives the resonance conditions for the pump parameter $\theta$ for different values of $n_q$ and $q$. Now we can easily understand why with two-atom effects there is no signature of the trapping states in the $l(\theta)$ curve (solid line in Fig. 4). With two atoms in the cavity at the same time there is a possibility of both one-photon and two-photon transitions. The Rabi frequencies of these transitions are different. To see the remnants of trapping states with collective effects one would have to satisfy two conditions: Eq. (5.4) and an analogous condition for the two-photon transition. It is easy to see that these two conditions cannot be satisfied simultaneously. Therefore, the signature of trapping states disappears in the curve $l(\theta)$ with collective effects.

To compare the prediction of our theory with the results of the experiment [3] we have drawn in Fig. 5 the mean photon number $\langle n\rangle$ as a function of the mean number of atoms $\langle N\rangle$ inside the cavity without and with collective ef-

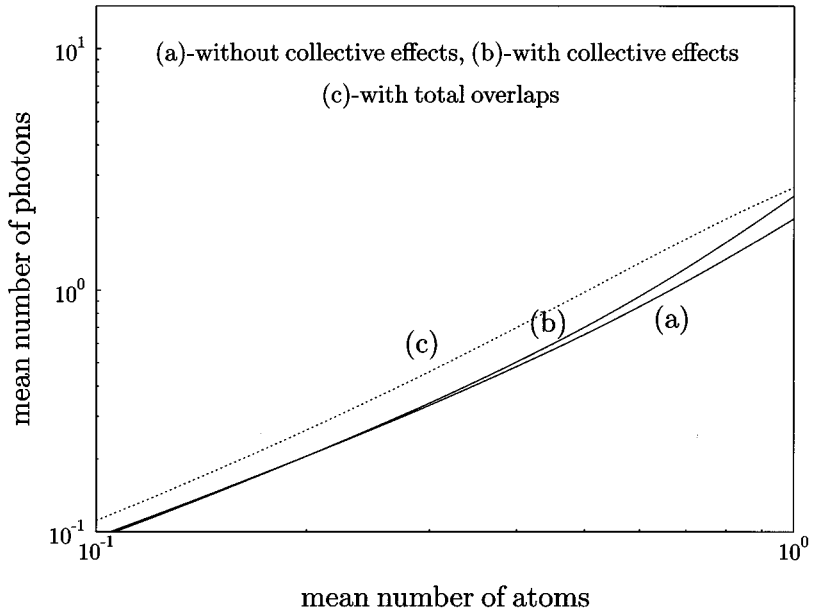


FIG. 5. Mean photon number $\langle n\rangle$ vs mean number of atoms $\langle N\rangle$ inside the cavity without (*a*) and with (*b*) collective effects. The dotted curve (*c*) results in admitting only complete overlaps of pairs of atoms. All parameters are taken from experiment [3], i.e., $g\tau=0.52$, $\kappa\tau=0.28$, and $n_{\rm th}=0$.

fects. The mean number of atoms in the cavity for the Poissonian atomic flux is given by

$$\langle N\rangle = r\tau. \tag{5.5}$$

The values of all other physical parameters in play, namely, $g, \kappa$, and $\tau$, were taken from the experiment [3]. The main feature of the experimental findings [3] was a rapid increase in $\langle n\rangle$ at about $\langle N\rangle \approx 0.6$, a behavior not predicted by the one-atom theory. For $\langle N\rangle$ approaching unity the value of $\langle n\rangle$ was found to be approximately 10, about three times more than predicted by the one-atom theory.

In order to underscore the importance of the statistics of overlaps we have included a third curve (dotted) in Fig. 5. That curve was obtained by accounting only for complete overlaps of pairs of atoms, i.e., by replacing the Poissonian overlap probability (2.6) with $w(x) = \delta(x-\tau)$ while keeping the Poissonian $W_0(n)$, $W_1(n)$ given in Eq. (2.8). Such an approach was advocated in [7]; it considerably overestimates the pair effects.

Our curves in Fig. 5 suggest the following further conclusions: (i) two-atom collective effects tend to increase the mean number of photons with respect to the prediction of the one-atom theory for all values of $\langle N\rangle$; (ii) the slope of the curve $\langle n\rangle$ as a function of $\langle N\rangle$ is steeper with collective effects than without; and (iii) collective effects do not yield any abrupt change of the mean photon number, and in particular none near $\langle N\rangle \approx 0.6$.

It is important to recall that our pumping generator $\Lambda_p$ in Eq. (2.16) was constructed in the limit $\langle N\rangle \ll 1$ and can therefore not be trusted for $\epsilon \approx \langle N\rangle = r\tau$ as large as 0.6 or even beyond. Amusingly, however, the final results as portrayed in Fig. 5 do not become manifestly nonsensical if extrapolated to $\langle N\rangle \approx 1$; rather, they maintain a reasonable trend from the single-atom theory towards the experimental points. To obtain quantitative agreement of theory with experiment we would, of course, have to account for higher orders in $\epsilon$ as well as for larger clusters of atoms in the resonator.

In Ref. [14] it was argued that the standing-wave structure of the field inside the cavity may play an important role when the mean number of photons $\langle n\rangle$ becomes larger than unity. Possibly, a further improved agreement of theory and experiment can be reached by accounting for such spatial inhomogeneities together with the collective two-atom effects discussed here. A recent analysis of normal mode splitting in standing-wave optical cavities [15] further supports our belief that collective effects become even more important when the coupling of the atom with the cavity mode is position dependent.

## ACKNOWLEDGMENT

Financial support by the Sonderforschungsbereich ''Unordnung und gro$\beta$e Fluktuationen'' der Deutschen Forschungsgemeinschaft is gratefully acknowledged.

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