Quasiparticle instabilities in multicomponent atomic condensates

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We extend the Hartree-Bogoliubov theory to the case of a multicomponent Bose condensate and determine the resulting quasiparticle frequency spectrum. We show that interferences resulting from cross coupling between the condensate components can lead to a reversal of the sign of the effective two-body interaction and to the onset of spatial instabilities. [S1050-2947(97)05004-X]

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I. INTRODUCTION

Recent demonstrations of Bose-Einstein condensation in low-density atomic vapors [1–3] have opened up the way for the study of weakly interacting quantum degenerate atomic samples. It is now well established that the Gross-Pitaevskii equation [4] gives an adequate description of the ground-state properties of these systems [5–8]. In addition, the low-lying excitations of the condensate have recently been studied both experimentally [9,10] and theoretically [11–16]. Theoretically, one proceeds by applying a linear-response analysis to the Gross-Pitaevskii equation, and the quasiexcitations spectrum of the condensate is obtained in a Hartree-Bogoliubov approach [11–18] by diagonalizing the resulting set of linear equations via a Bogoliubov transformation [19].

The goal of the present paper is to extend these studies to the case of multicomponent condensates. Such condensates can, in principle, be generated in several ways. For instance, one can use a double-well trap, whereby two condensates are coupled by quantum tunneling and ground-state collisions. Another possibility involves using two different magnetic sublevels of an atomic vapor, in which case the two components of the condensate correspond to the two electronic states involved.[1] Coupling between the two components could result from the near-resonant dipole-dipole interaction via an (adiabatically eliminated) intermediate excited electronic level. Further possibilities involve optically allowed transitions, although in that case the effects of spontaneous emission would certainly need to be included.

Multicomponent condensates are expected to play an important role in future experiments, e.g., in situations where two condensates are made to interfere with each other [20] or where Raman transitions are used to optically study the phase of a condensate [21]. It is, however, known from previous studies of the nonlinear Schrödinger equation, for instance, in the context of nonlinear optics [22,23], that the stability properties of scalar and multicomponent fields can be vastly different due to the occurrence of cross coupling between these various components. Hence it is important and timely to extend such an analysis to the case of a quantum-degenerate atomic system. We find that, like in the optical case, the situation in Bose condensation becomes quite a bit more complex than for scalar fields: the quasiparticle spectrum is now characterized by energy gaps and, more importantly, by regions of complex frequencies, indicating that the condensate state becomes unstable.

The stability of multicomponent Bose fluids has been discussed previously. In particular, Andreev and Bashkin [24] have analyzed the dynamics of ^{3}He in ^{4}He in a three-fluid model including two superfluid components and one normal fluid component. This model leads to the appearance of a cross-fluid density term, which may result in the onset of an instability. A related cross-fluid density was also discussed in the context of a two-component model of Bose-condensed spin-polarized hydrogen [25]. In contrast to these cases, the low-density atomic systems that we have in mind are expected to be almost entirely in their condensed phase at $T=0$. The origin of the instability in that case is not the cross-fluid density, but rather a change in sign of the effective scattering length due to a quantum interference between the two components of the condensate.

This paper is organized as follows. Section II formulates our model and introduces a nonlinear Schrödinger equation that is the generalization of the Gross-Pitaevskii equation for vector fields. Such an equation is familiar, e.g., in the context of nonlinear atom optics [26]. The elementary excitations of this system are described in Sec. III, where we perform the Hartree-Bogoliubov approximation on vector fields and find the quasiparticle dispersion relations from a Bogoliubov transformation. We discuss this spectrum in several simple cases, illustrating the appearance of gaps and instability regions. Finally, Sec. IV summarizes.

II. PHYSICAL MODEL

We consider an ensemble of N atoms subjected to the single-particle Hamiltonian H_0 and a two-body interaction V. The corresponding second-quantized Hamiltonian is

$$\mathcal{H}=\int d1\, d2\langle 1|H_0|2\rangle\Psi^\dagger(1)\Psi(2) + \frac{1}{2}\int d\{l\}\langle 1,2|V|3,4\rangle\Psi^\dagger(1)\Psi^\dagger(2)\Psi(3)\Psi(4), \tag{1}$$

where l denotes a full set of quantum numbers and $\Psi(l)$ and

[1] See *Note added*.

$\Psi^\dagger(l)$ are usual atomic annihilation and creation operators, which are assumed here to satisfy the bosonic commutation relations

$$[\Psi(l),\Psi^\dagger(l')]=\delta(l-l'). \tag{2}$$

The state of the system is given by the N-particle wave function $f(1,2,\ldots,l,\ldots,N)$, which, in the Hartree approximation, is assumed to factorize as a product of normalized single-particle wave functions

$$f(1,\ldots,N;t)=\prod_{l=1}^{N}\phi(l,t). \tag{3}$$

The equation of motion for the effective single-particle wave function results from the Hartree variational principle and takes the form of the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\phi(l,t)}{\partial t}=\int d2\langle l|H_0|2\rangle\phi(2,t)+(N-1)\int d1\,d2\,d3 \times\langle l,1|V|2,3\rangle\phi^\star(1,t)\phi(2,t)\phi(3,t). \tag{4}$$

For scalar particles and in the limit of s-wave scattering, the potential is of the form $V=(4\pi\hbar^2a/m)\delta(\mathbf{r}_{12})$, where m is the particle mass, $\mathbf{r}_{12}$ the relative position of the atoms, and a the s-wave scattering length. Equation (4) reduces then to the Gross-Pitaevskii equation. In the present situation, however, we consider the case of a multicomponent field $l=\{i,\mathbf{r}\}$, where i labels the component of the field (for example an internal degree of freedom of the atoms) and $\mathbf{r}$ is the center-of-mass coordinate. For the specific case of a two-component field that we consider in the following, we have

$$\phi(l)=\begin{pmatrix}\phi_a(x)\\ \phi_b(x)\end{pmatrix}. \tag{5}$$

We assume that in the absence of any interaction and ignoring the effects of kinetic energy, the eigenenergies of the single-particle wave functions $\phi_a(x)$ and $\phi_b(x)$ are $\pm\hbar\delta$, respectively. We further include a single-particle coupling between these two states, with matrix element $\mathcal{R}$, which is taken to be constant in space in the present discussion. Examples of such coupling include the electric-dipole interaction between ground and excited electronic states, Raman transitions between Zeeman sublevels, and tunneling between condenstates in a double-well trap. (In the case of optical couplings, the approximation $\mathcal{R}$=const implies that the condensates are assumed to be small compared to the spatial variations of the light fields.) Finally, the two-body interaction has nonzero matrix elements both between equal and between different components of the single-particle wave function. These are the ''self-phase modulation'' and ''cross-phase modulation'' terms of nonlinear optics, respectively. We assume as in the scalar case that the s-wave scattering approximation is adequate, so that the two-body interaction is proportional to $\delta(\mathbf{r}_{12})$.

The stationary state of the nonlinear Schrödinger equation (4), which describes the condensate state, is the solution of the equation

$$\hbar\omega_0\phi(l)=\int d2\langle l|H_0|2\rangle\phi(2)+(N-1) \times\int d1\,d2\,d3\langle l,1|V|2,3\rangle\phi^\star(1)\phi(2)\phi(3). \tag{6}$$

Since $\mathcal{R}$ is assumed to be a constant, we seek spatially homogeneous solutions. For the model at hand, they are solutions of the coupled algebraic equations

$$\omega_0\phi_a=\frac{1}{2}\delta\phi_a+\mathcal{R}\phi_b+(V_s|\phi_a|^2+V_x|\phi_b|^2)\phi_a,$$

$$\omega_0\phi_b=-\frac{1}{2}\delta\phi_b+\mathcal{R}\phi_a+(V_s|\phi_b|^2+V_x|\phi_a|^2)\phi_b, \tag{7}$$

where V_s and V_x are the self- and cross-phase modulation matrix elements of the two-body potential V. These two equations, together with the condition that the total number of particles is equal to N, are readily solved to yield the spatially homogeneous densities

$$\rho_a\equiv|\phi_a|^2=\frac{\rho}{2}\left(1+\frac{1}{2}\frac{\delta}{\omega_0-V_s\rho}\right),$$

$$\rho_b\equiv|\phi_b|^2=\frac{\rho}{2}\left(1-\frac{1}{2}\frac{\delta}{\omega_0-V_s\rho}\right), \tag{8}$$

as well as ω_0. The total number of atoms in the two components of the condensate is $N_i=\rho_i\mathcal{V}$, $i=a,b$, $\mathcal{V}$ being the quantization volume, and $N\equiv\rho\mathcal{V}=(\rho_a+\rho_b)\mathcal{V}$. The explicit form of ω_0 is rather combersome and we do not reproduce it here since it is not needed in the following.

We note, however, that substituting Eqs. (8) into Eqs. (7) yields the quartic equation for ω_0

$$\left[\omega_0-\frac{1}{2}(V_s+V_x)\rho\right]^2\left[(\omega_0-V_s\rho)^2-\frac{1}{4}\delta^2\right]=\mathcal{R}^2(\omega_0-V_s\rho)^2. \tag{9}$$

In the case $\mathcal{R}\neq0$, it follows from the positivity of the right-hand side of this equation that

$$2|\omega_0-V_s\rho|>|\delta| \tag{10}$$

and hence the densities appearing in Eqs. (8) are positive for all solutions ω_0. For $\mathcal{R}=0$, ω_0 can be given either by

$$|\omega_0-V_s\rho|=|\delta|/2 \tag{11}$$

or by

$$\omega_0=(V_s+V_x)\rho/2. \tag{12}$$

The first solution implies that only one of the two components of the condensate is populated. Specifically, $\rho_a=\rho$ and $\rho_b=0$ for $\omega_0-V_s\rho=\delta/2$, and $\rho_a=0$ and $\rho_b=\rho$ for $\omega_0-V_s\rho=-\delta/2$. In both cases, the condensate is stable, as further discussed later on. For the second solution (12), the positivity of the populations ρ_a and ρ_b imposes that $|(V_s-V_x)\rho|>|\delta|$.

In the next section, we perform a linear-response analysis of this homogeneous two-component condensate. We find that in contrast to the scalar situation, where the condensate solution is always stable for repulsive potentials $(V>0)$, this no longer needs to be the case here due to the combined effects of self- and cross-phase modulation.

III. ELEMENTARY EXCITATIONS

In order to study the spectrum of elementary excitations above the condensate state, we proceed in the usual fashion by expressing the pure condensate state as the sum of a classical condensate wave function ϕ and fluctuating field operators $\delta\psi(l,t)$, which are assumed to be small perturbations,

$$\Psi(l,t)=\phi(l)+\delta\psi(l,t), \tag{13}$$

the fluctuating field operators satisfying the Bose commutation relations [4]

$$[\delta\psi(l,t),\delta\psi^{\dagger}(l',t)]=\delta(l-l'). \tag{14}$$

The second-quantized Hamiltonian of the many-particle system can then be linearized in terms of these operators, leading in a straightforward fashion to the effective Hamiltonian[2]

$$\begin{aligned}\mathcal{H}=&\int d1d2\langle 1|H_0|2\rangle\phi^{\star}(1)\phi(2)+\frac{1}{2}\int d1\,d2d3d4\langle 1,2|V|3,4\rangle\phi^{\star}(1)\phi^{\star}(2)\phi(3)\phi(4)\\&+\int d1\,d2\langle 1|H_0|2\rangle\delta\psi^{\dagger}(1)\delta\psi(2)+\frac{1}{2}\int d1d2\,d3\,d4\langle 1,2|V|3,4\rangle\\&\times[\delta\psi^{\dagger}(1)\delta\psi^{\dagger}(2)\phi(3)\phi(4)+\phi^{\star}(1)\phi^{\star}(2)\delta\psi(3)\delta\psi(4)+\delta\psi^{\dagger}(1)\phi^{\star}(2)\phi(3)\delta\psi(4)\\&+\delta\psi^{\dagger}(1)\phi^{\star}(2)\delta\psi(3)\phi(4)+\phi^{\star}(1)\delta\psi^{\dagger}(2)\phi(3)\delta\psi(4)+\phi^{\star}(1)\delta\psi^{\dagger}(2)\delta\psi(3)\phi(4)].\end{aligned} \tag{15}$$

No first-order contribution in $\delta\psi$ appears in Eq. (15), a consequence of the fact that the condensate wave function $\phi(l)$ satisfies the nonlinear Schrödinger equation. Hence the effective Hamiltonian (15) is quadratic in the operators $\delta\psi(l)$ and $\delta\psi^{\dagger}(l)$. In order to determine the spectrum of elementary excitations of the condensate, this Hamiltonian is diagonalized via a generalized Bogoliubov transformation to yield

$$\mathcal{H}=\hbar\int dl\,\omega(l)b(l)^{\dagger}b(l), \tag{16}$$

where $\omega(l)$ is the frequency of the elementary mode associated with the complete set of quantum numbers l. The annihilation and creation operators $b(l)$ and $b^{\dagger}(l')$ satisfy the boson commutation relations

$$[b(l),b^{\dagger}(l')]=\delta(l-l'). \tag{17}$$

We recall that in the scalar case these operators are linear combinations of the original fluctuation operators $\delta\psi$ and $\delta\psi^{\dagger}$, obtained via the canonical u-v transformation [19]. They describe ''quasiparticle excitations,'' whose spectrum is directly apparent from the Hamiltonian (16).

A similar transformation can be introduced in the case of a multicomponent field [28] and reads

$$b(l)=\int dl'[u(l,l',t)\delta\psi(l',t)+v(l,l',t)\delta\psi^{\dagger}(l',t)] \tag{18}$$

the integration being performed over the full set of quantum numbers $l'=\{j,\mathbf{r}\}$. The coefficients $u(l,l',t)$ and $v(l,l',t)$ are the matrix elements of the $M\times M$ matrices $\mathbf{U}(t)$ and $\mathbf{V}(t)$, whereby M is the number of field components; $M=2$ in our particular case. Since the generalized Bogoliubov transformation should be canonical, the matrices $\mathbf{U}$ and $\mathbf{V}$ must satisfy the relations [28]

$$\mathbf{U}\mathbf{U}^{\dagger}-\mathbf{V}\mathbf{V}^{\dagger}=\mathbf{I},\quad \mathbf{U}\mathbf{V}^{T}=\mathbf{V}\mathbf{U}^{T}. \tag{19}$$

These properties allow one to express $\delta\psi(l)$ and $\delta\psi^{\dagger}(l)$ in terms of the quasi-particle operators $b(l)$ and $b(l)^{\dagger}$ as

$$\delta\psi(l,t)=\int dl'[u^{\star}(l',l,t)b(l')-v(l',l,t)b^{\dagger}(l')], \tag{20}$$

where $u^{\star}(l',l,t)$ and $v(l',l,t)$ are matrix elements of $\mathbf{U}^{\dagger}(t)$ and $\mathbf{V}^{T}(t)$, respectively. The functions $u(l,l',t)$ and $v(l,l',t)$ can be determined by substituting Eq. (18) into the commutator $[b(l),\mathcal{H}]=\hbar\omega(l)b(l)$.

Since we are considering specifically the case of elementary excitations above a two-component, *homogeneous* condensate, it is advantageous to work in the momentum representation; hence

$$u(l,l',t)\rightarrow u_{ij}(p,p',t)\equiv u_{ij}(p,t), \tag{21}$$

the last step resulting from the fact that the problem is local in momentum space. Assuming that the matrix elements $\mathcal{R}$, V_s, and V_x are real, it is possible to take the classical condensate wave functions ϕ_a and ϕ_b to be real, with $\phi_i=\sqrt{\rho_i}$. Introducing $\widetilde{\delta\psi}(l,t)\equiv\delta\psi(l,t)\exp(i\omega_0 t)$, $\widetilde{u}_{kj}(p,t)$

[2]This is a generalization of the linearization scheme used in a recent study of matter wave phase conjugation off Bose condensates [27].

$=u_{kj}(p,t)\exp(i\omega_0 t)$, and $\tilde{v}_{kj}(p,t)=v_{kj}(p,t)\exp(-i\omega_0 t)$, we obtain the four coupled equations

$$\hbar\omega\tilde{u}_{ja}(p,t)=\left[\frac{p^2}{2m}-\mathcal{R}\left(\frac{\rho_b}{\rho_a}\right)^{1/2}+V_s\rho_a\right]\tilde{u}_{ja}(p,t)+(V_x\sqrt{\rho_a\rho_b}+\mathcal{R})\tilde{u}_{jb}(p,t)-V_s\rho_a\tilde{v}_{ja}(p,t)-V_x\sqrt{\rho_a\rho_b}\tilde{v}_{jb}(p,t), \tag{22}$$

$$\hbar\omega\tilde{u}_{jb}(p,t)=\left[\frac{p^2}{2m}-\mathcal{R}\left(\frac{\rho_a}{\rho_b}\right)^{1/2}+V_s\rho_b\right]\tilde{u}_{jb}(p,t)+(V_x\sqrt{\rho_a\rho_b}+\mathcal{R})\tilde{u}_{ja}(p,t)-V_s\rho_b\tilde{v}_{jb}(p,t)-V_x\sqrt{\rho_a\rho_b}\tilde{v}_{ja}(p,t), \tag{23}$$

$$-\hbar\omega\tilde{v}_{ja}(p,t)=\left[\frac{p^2}{2m}-\mathcal{R}\left(\frac{\rho_b}{\rho_a}\right)^{1/2}+V_s\rho_a\right]\tilde{v}_{ja}(p,t)+(V_x\sqrt{\rho_a\rho_b}+\mathcal{R})\tilde{v}_{jb}(p,t)-V_s\rho_a\tilde{u}_{ja}(p,t)-V_x\sqrt{\rho_a\rho_b}\tilde{u}_{jb}(p,t), \tag{24}$$

$$-\hbar\omega\tilde{v}_{jb}(p,t)=\left[\frac{p^2}{2m}-\mathcal{R}\left(\frac{\rho_a}{\rho_b}\right)^{1/2}+V_s\rho_b\right]\tilde{v}_{jb}(p,t)+(V_x\sqrt{\rho_a\rho_b}+\mathcal{R})\tilde{v}_{ja}(p,t)-V_s\rho_b\tilde{u}_{jb}(p,t)-V_x\sqrt{\rho_a\rho_b}\tilde{u}_{ja}(p,t), \tag{25}$$

where $j=\{a,b\}$.

The spectrum of eigenvalues $\omega(p)$ of this system of equations consists of four branches, but from the time-reversal symmetry of the problem, i.e., symmetry under the transformation $U\leftrightarrow -V$, it follows that if $\omega(p)$ is a solution, so is $-\omega(p)$. Hence we limit our discussion to the physically relevant positive-frequency branches.

Consider first the situation of degenerate condensates $\delta=0$. In this case the two condensates are equally populated $\rho_a=\rho_b=\rho/2$ and the eigenenergies satisfy the dispersion relations

$$\hbar\omega_-(p)=\left[\frac{p^2}{2m}\left(\frac{p^2}{2m}+(V_s+V_x)\rho\right)\right]^{1/2} \tag{26}$$

and

$$\hbar\omega_+(p)=\left[\left(\frac{p^2}{2m}+2\mathcal{R}+(V_s-V_x)\rho\right)\left(\frac{p^2}{2m}+2\mathcal{R}\right)\right]^{1/2}. \tag{27}$$

The first branch of the spectrum $\omega_-(p)$ behaves exactly as in the case of a scalar Schrödinger field [4], except for the trivial substitution $V\rightarrow V_s+V_x$. It is gapless and for repulsive interactions $(V_s,V_x>0)$ it corresponds to real frequencies. For attractive potentials, in contrast, $\omega_+(p)$ can become pure imaginary, illustrating the instability of (free space) condensates for attractive interactions. In the following, we therefore concentrate on the more interesting situation of repulsive interactions.

The second branch $\omega_+(p)$ exhibits a considerably different behavior. It has a gap at $p=0$ and, more importantly, the frequency ω_+ can become imaginary even though the potential is repulsive, provided that the cross-phase modulation matrix element is larger than self-phase modulation. In order to gain some insight into this spectrum, we introduce the new set of states $|\psi_+\rangle=(|\psi_e\rangle+|\psi_g\rangle)/\sqrt{2}$ and $|\psi_-\rangle=(|\psi_e\rangle-|\psi_g\rangle)/\sqrt{2}$. We mentioned earlier that the classical condensate wave function is real and that $\rho_a=\rho_b$, so that $\phi_a=\pm\phi_b$. This implies that the unperturbed system is in either the $|\psi_+\rangle$ or the $|\psi_-\rangle$ state. For concreteness, assume that the latter situation holds, so that $\phi_+=0$. Thus linear perturbations of the condensates give the wave function

$$\Psi_+(\mathbf{r},t)=\delta\psi_+(\mathbf{r},t)e^{-i\omega_0 t},$$
$$\Psi_-(\mathbf{r},t)=[\sqrt{\rho/2}+\delta\psi_-(\mathbf{r},t)]e^{-i\omega_0 t}. \tag{28}$$

In this basis, the nonlinear Schrödinger equation becomes

$$\hbar\omega_0\phi_-=-\mathcal{R}\phi_-+\frac{(V_s+V_x)}{2}\rho\phi_-, \tag{29}$$

$$i\hbar\frac{\partial\delta\psi_-}{\partial t}=\left[\frac{p^2}{2m}+\frac{V_s+V_x}{2}\rho\right]\delta\psi_-+\frac{V_s+V_x}{2}\rho\delta\psi_-^\dagger, \tag{30}$$

$$i\hbar\frac{\partial\delta\psi_+}{\partial t}=\left[\frac{p^2}{2m}+2\mathcal{R}+\frac{V_s-V_x}{2}\rho\right]\delta\psi_++\frac{V_s-V_x}{2}\rho\delta\psi_+^\dagger. \tag{31}$$

In the new basis, the linearized nonlinear Schrödinger equations are decoupled, and one can understand the stability of the system in terms of the two uncoupled scalar fields $\delta\psi_+$ and $\delta\psi_-$. It is straightforward to identify the branch (26) as the spectrum associated with Eq. (30). It corresponds to elementary excitations of atoms in the same state as the condensate state. In contrast, the branch (27) is the spectrum of Eq. (31); it corresponds to the elementary excitations in a state different from the condensate state. This explains the gap in $\omega_+(p)$, which corresponds to the energy required to excite that state.

Note that in contrast to the field $\delta\psi_-$, which is subjected to the mean-field energy proportional to $(V_s+V_x)\rho$ as expected, the $\delta\psi_+$ field is subjected to a mean-field energy proportional to $(V_s-V_x)\rho$. This difference in signs can be interpreted as resulting from the interferences between the two condensates, which are constructive for one of the field superpositions and destructive for the other. These interferences can have a dramatic effect since they can reverse the sign of the nonlinearity effectively acting on $\delta\psi_+$. If that is the case, this field behaves as if subjected to an *attractive* interaction and thereby becomes unstable [29]. Figure 1 illustrates the gain spectra of this cross-phase-induced modulation instability for various strengths of the matrix element $\mathcal{R}$. Note that this spectrum becomes gapless for $\mathcal{R}=0$ as should be expected (the fact that two values of $\mathcal{R}$ lead to the same frequency at $p=0$ is accidental).

As another limiting case, we consider the situation where the coupling between the two components of the Schrödinger

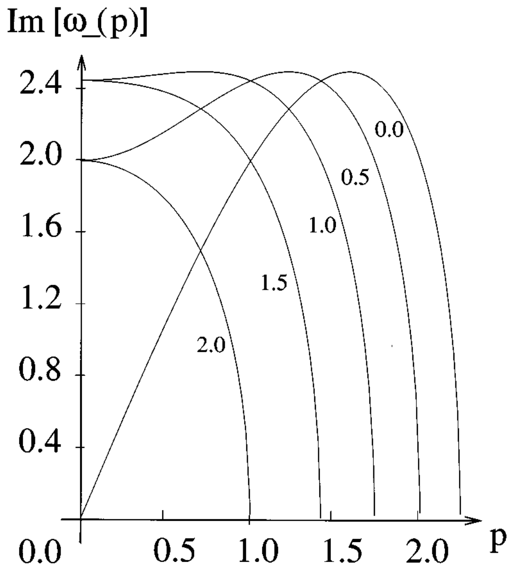

FIG. 1. Gain spectra of cross-phase induced modulation instability for $\mathcal{R}=0,0.5,1.0,1.5,2.0$. Here p is scaled to $\sqrt{m\hbar}$, $V_s/\hbar=2.5$, and $V_x/\hbar=5.0$ in arbitrary units.

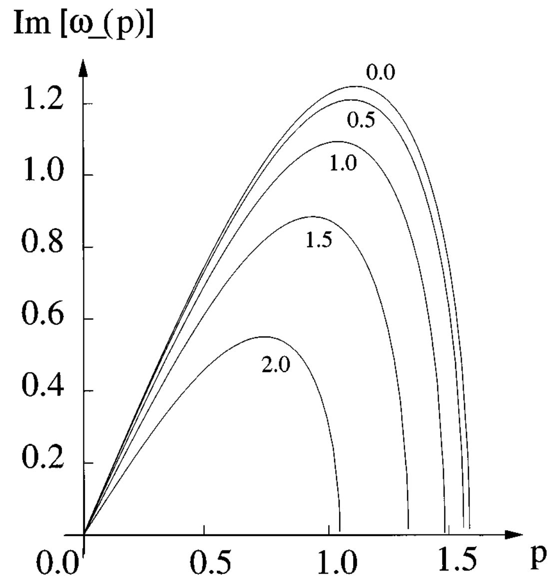

FIG. 2. Gain spectra of cross-phase modulation instability for $\delta=0,0.5,1.0,1.5,2.0$. Here p is scaled to $\sqrt{m\hbar}$, $V_s/\hbar=5.0$, and $V_x/\hbar=10.0$ in arbitrary units.

field results solely from two-body interactions $\mathcal{R}=0$. In this case the two quasiparticle branches have the dispersion relations

$$\hbar\omega_{\pm}(p)=\left\{\frac{p^2}{2m}\left[\left(\frac{p^2}{2m}+V_s\rho\right)\pm[V_s^2(\rho_a-\rho_b)^2+4V_x^2\sqrt{\rho_a\rho_b}]^{1/2}\right]\right\}^{1/2}. \quad (32)$$

Both branches are now gapless and the branch $\omega_+(p)$ is always stable. However, the other branch can become unstable if the cross-modulation matrix element is large enough that

$$V_x^2\rho_a\rho_b>\left(\frac{1}{2}\frac{p^2}{2m}+V_s\rho_a\right)\left(\frac{1}{2}\frac{p^2}{2m}+V_s\rho_b\right). \quad (33)$$

We note that in the case when only one component of the condensate is populated, the system always remains stable. Assuming for concreteness $\rho_a=0$, we find that the elementary excitations in the b state above the condensate satisfy the dispersion relation

$$\hbar\omega_+(p)=\left[\frac{p^2}{2m}\left(\frac{p^2}{2m}+2V_s\rho\right)\right]^{1/2}, \quad (34)$$

while excitations in the a state satisfy

$$\hbar\omega_-(p)=\frac{p^2}{2m}. \quad (35)$$

These relations are in agreement with Ref. [30], where elementary excitations above a multicomponent condensate with only one component populated were considered. In particular, the branch Eq. (34) is phononlike at long wavelengths, while the branch Eq. (35) describes a single ''impurity'' quasiparticle with a free-particle dispersion.

For $\delta=0$, we have as before $\rho_a=\rho_b=\rho/2$, so that condition (33) becomes

$$|V_x\rho|>\left|\frac{p^2}{2m}+V_s\rho\right|, \quad (36)$$

expressing the fact that the mean energy resulting from the cross-coupling between condensates must be able to overcome the energy of the scalar condensate, including its mean energy correction. In the Thomas-Fermi limit, this becomes simply $|V_x|>|V_s|$.

For the wave vectors in the range defined by Eq. (33) fluctuations above the condensate grow exponentially in time or, in other words, a spatial instability develops in the system. Again, the present situation is reminiscent of the cross-phase modulation instability in nonlinear optics. For instance, small perturbations in coupled optical fibers with Kerr nonlinearities are known to lead to the onset of temporal instabilities. This is illustrated in Fig. 2, which shows the dependence of the gain on the detuning δ and is very similar to the gain dependence on the ratio between light intensities in coupled optical fibers (Fig. 7.8 in [22]). One noticeable difference between the two situations is that the predicted instabilities in coupled condensates are spatial, whereas they are temporal in the case of optical fibers.

IV. CONCLUSION

In this paper we have extended the Bogoliubov-Hartree approach to the case of coupled condensates. We determined the spectrum of elementary excitations above an initially homogeneous condensate and examined its main features. We found the appearance of imaginary frequencies, which indicate that the condensate state becomes unstable. The question remains to determine which state the condensate will evolve into under the influence of these instabilities. In particular, it will be important to find the true ground state of the system in those situations where the homogeneous solution is unstable. This is planned to be the subject of a future paper, together with the extension of this work to the case of trapped condensates and of spatially dependent coupling $\mathcal{R}(\mathbf{r})$.

Note added. Recently, an experimental realization of two overlapping condensates in a ^{87}Rb vapor was reported by Myatt *et al.* [31].

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