Semiclassical analysis of quasiexact solvability

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Higher-order WKB methods are used to investigate the border between the solvable and insolvable portions of the spectrum of quasiexactly solvable quantum-mechanical potentials. The analysis reveals scaling and factorization properties that are central to quasiexact solvability. These two properties define a new class of semiclassically quasiexactly solvable potentials. [S1050-2947(97)07804-9]

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Quantum-mechanical potentials are said to be *quasiexactly solvable* (QES) if it is possible to find a finite portion of the energy spectrum and associated eigenfunctions exactly and in closed form [1]. QES potentials depend on a parameter J; for positive integer values of J one can find exactly the first J eigenvalues and eigenfunctions, typically of a given parity. QES systems have been classified using an algebraic approach in which the Hamiltonian is expressed in terms of the generators of a Lie algebra [2–4]. This approach generalizes the dynamical symmetry analysis of *exactly solvable* quantum-mechanical systems, whose *entire* spectrum may be found in closed form by algebraic means [5].

In this paper we use higher-order semiclassical methods to examine the boundary between the exactly solvable part of the spectrum and the remaining energy levels in QES systems. We find that the large-J asymptotic behavior of the largest exactly known energy eigenvalue is particularly simple for QES potentials. This study leads to a natural generalization of QES potentials. We discover an infinite tower of potentials; the first is exactly solvable, the second is QES, and the rest share some semiclassical features of QES potentials but are not QES.

The simplest QES potential [6] is

$$V(x)=x^6-(4J-1)x^2. \tag{1}$$

The Schrödinger equation, $-\psi''(x)+[V(x)-E]\psi(x)=0$, has J even-parity solutions of the form

$$\psi(x)=e^{-x^4/4}\sum_{k=0}^{J-1} c_k x^{2k}. \tag{2}$$

The coefficients c_k satisfy the recursion relation

$$4(J-k)c_{k-1}+Ec_k+2(k+1)(2k+1)c_{k+1}=0, \tag{3}$$

where $0\leqslant k\leqslant J-1$ and we define $c_{-1}=c_J=0$.

The simultaneous linear equations (3) have a nontrivial solution for $c_0, c_1, \ldots, c_{J-1}$ if the determinant of the coefficients vanishes. For each integer J this determinant is a polynomial $P_J(E)$ of degree J in the variable E. The roots of $P_J(E)$ are all real and are the J quasiexact energy eigenvalues of the potential in Eq. (1). We have computed the first 40 polynomials; we list the first seven of these below:

$$P_1(E)=E,$$

$$P_2(E)=-8+E^2,$$

$$P_3(E)=-64E+E^3,$$

$$P_4(E)=2880-240E^2+E^4,$$

$$P_5(E)=47104E-640E^3+E^5,$$

$$P_6(E)=-5\,184\,000+331\,456E^2-1400E^4+E^6,$$

$$P_7(E)=-130\,940\,928E+1\,529\,856E^3-2688E^5+E^7.$$

The roots of $P_J(E)$ occur in positive and negative pairs, and successive sets of roots interlace:

$$\text{roots of } P_1(E),\ 0;$$

$$\text{roots of } P_2(E),\ \pm 2\sqrt{2};$$

$$\text{roots of } P_3(E),\ 0,\ \pm 8;$$

$$\text{roots of } P_4(E),\ \pm 3.559\,32,\ \pm 15.077\,51;$$

$$\text{roots of } P_5(E),\ 0,\ \pm 9.211\,35,\ \pm 23.561\,64;$$

$$\text{roots of } P_6(E),\ \pm 4.101\,32,\ \pm 16.707\,78,\ \pm 33.226\,93;$$

$$\text{roots of } P_7(E):\ 0,\ \pm 10.187\,50,\ \pm 25.562\,58,$$

$$\pm 43.940\,52.$$

Let $E_c(J)$ represent the largest root and thus the *critical* energy that marks the upper edge of the quasiexact spectrum. A numerical fit to the large-J asymptotic behavior of $E_c(J)$

for $1 \leqslant J \leqslant 40$ using Richardson extrapolation [8] gives $E_c(J) \sim 3.079\,201\,435\,5 J^{3/2}$ as $J \to \infty$. We recognize the numerical constant:

$$E_c(J) \sim \frac{16}{9}\sqrt{3}\, J^{3/2} \quad (J \to \infty). \tag{4}$$

One may verify this result analytically by finding the minimum of the potential $V(x) = x^6 - (4J-1)x^2$. Minima occur at $x = \pm[(4J-1)/3]^{1/4}$; at these values $V \sim -\frac{16}{9}\sqrt{3} J^{3/2}$ $(J \to \infty)$. Since the quasiexact energy eigenvalues occur in $\pm$ pairs and the zero-point energy is negligible for large J, the asymptotic result in Eq. (4) is confirmed [7]. However, this approach is not useful for all QES systems because in general the spectrum is not symmetric under $E \to -E$.

WKB theory provides a more general derivation of Eq. (4) and gives higher-order corrections. Furthermore, semiclassical analysis reveals features of $V(x)$ that are generic to QES potentials. The leading-order WKB quantization condition is

$$(n + \tfrac{1}{2})\pi \sim \int_{-x_0}^{x_0} dx\ \sqrt{E - x^6 + (4J-1)x^2} \tag{5}$$

for large n, where $\pm x_0$ are turning points [x_0 is the positive zero of $E - V(x)$] and n is the quantum number of the energy level. We seek the Jth *even*-parity energy level $E = E_c(J)$ for which $n = 2J$.

To evaluate the integral on the right side of Eq. (5) for large J, we begin by scaling the variables E_c and x. The scaling $E_c = J^{3/2}\alpha$ and $x = J^{1/4} y$ extracts a factor of J from the integral and reduces Eq. (5) to an exact expression for the numerical constant α:

$$2\pi = \int_{-y_0}^{y_0} dy\ \sqrt{\alpha - y^6 + 4y^2}, \tag{6}$$

where y_0 is the positive root of $\alpha - y^6 + 4y^2 = 0$.

In general, the integral in Eq. (6) is not an elementary function of α. However, for the special value $\alpha = \frac{16}{9}\sqrt{3}$ the polynomial $\alpha - y^6 + 4y^2$ *factors into a product of a linear term times a perfect square*:

$$\frac{16}{9}\sqrt{3} - y^6 + 4y^2 = \left(\frac{4}{\sqrt{3}} - y^2\right)\left(\frac{2}{\sqrt{3}} + y^2\right)^2. \tag{7}$$

Using this factorization, we express the integral in Eq. (6) in terms of beta functions:

$$\int_{-y_0}^{y_0} dy\ \sqrt{\alpha - y^6 + 4y^2} = \frac{8}{3}\int_0^1 dz\ \frac{1+2z}{\sqrt{z}}\sqrt{1-z} = \frac{8}{3}B\left(\frac{3}{2}, \frac{1}{2}\right) + \frac{16}{3}B\left(\frac{3}{2}, \frac{3}{2}\right) = 2\pi. \tag{8}$$

Thus, the WKB quantization condition (6) is satisfied with $\alpha = \frac{16}{9}\sqrt{3}$, which confirms the leading asymptotic form of the critical energy $E_c(J)$ given in Eq. (4).

Two aspects of the above calculation of the WKB integral are crucial. First, the QES potential is such that in the limit $J \to \infty$, the parameter J scales out of the integrand. Second, the factorization in Eq. (7) enables us to evaluate the scaled integral and to obtain the factor of π, which then cancels from the leading-order WKB quantization condition. In fact, merely demanding that $\alpha - y^6 + 4y^2$ factor into the product of a linear term in y^2 times a square of a linear term in y^2 *uniquely specifies* the value of α, which in turn determines the large-J asymptotic behavior of the critical energy $E_c(J)$.

The factorization property extends to general QES x^6 potentials. Consider the scaled factored form $\alpha - V(y) = (a - y^2)(b + cy^2)^2$. (The conditions $a > 0$, b, $c \geqslant 0$ ensure that there is just one pair of turning points and that the WKB integrals give rise to Beta functions of half-integer arguments.) The leading-order WKB condition is

$$2\pi J \sim \int dx\sqrt{E - V(x)} = J\int dy\sqrt{a - y^2}(b + cy^2) = 2\pi J\left(\frac{ab}{4} + \frac{a^2 c}{16}\right),$$

which determines b in terms of a and c: $b = 4/a - ac/4$. This gives the potential

$$V(y) = c^2 y^6 + 2\beta c y^4 + (\beta^2 - 4c)y^2,$$

where $\beta = 4/a - 3ac/4$. This is exactly the large-J scaled form of the *general* nonsingular QES x^6 potential (class VI in Ref. [2]). When $\beta = 0$ and $c = 1$, we obtain the case treated in Eq. (1).

This factorization is a universal feature of quasiexact solvability. In general, at the upper end of the quasiexact spectrum, $E - V(x)$ factors into one term, which fixes the location of the two WKB turning points, multiplied by a second term, which is a perfect square. As another example, consider the QES potential $V(x) = \sinh^2 x - (2J-1)\cosh x$. For this potential there are J even-parity QES states, all of the form $\psi(x) = e^{-\cosh x} P(\cosh x)$, where P is a polynomial. To find the critical energy E_c at the upper boundary of the quasiexact spectrum, we apply the WKB quantization condition

$$\int_{-x_0}^{x_0} dx\ \sqrt{E_c - \sinh^2(x) + (2J-1)\cosh(x)} \sim (2J + 1/2)\pi \quad (J \to \infty). \tag{9}$$

The substitution $x = 2t$ reveals the factorization property of the integrand in Eq. (9). The expression $E_c - V(x)$ factors into one term $4J - 2 - 4\cosh^2 t$ (this term determines the turning points) multiplied by another term in the form of a perfect square, $\sinh^2 t$. This factorization fixes the dependence of E_c on J: $E_c = 1 - 2J$. With this factorization, the right side of Eq. (9) simplifies to $4\int_0^{t_0} dt\ \sinh t\sqrt{4J - 2 - 4\cosh^2 t} \sim 2\pi J$, for large J. Once again, π divides out of the leading-order WKB quantization condition.

The remarkable connection between WKB and quasiexact solvability persists to higher order in WKB theory. The fac-

tor of π cancels from the WKB quantization condition *to all orders* leaving a purely algebraic series. To illustrate this, we do a fifth-order WKB calculation of $E_c(J)$ for the x^6 potential in Eq. (1). We begin by scaling the energy in the Schrödinger equation using $E_c(J)=2[(4J-1)/3]^{3/2}\gamma$. After scaling the independent variable, we obtain the Schrödinger equation

$$\epsilon^2\psi''(u)=(4u^6-3u^2-\gamma)\psi(u), \tag{10}$$

where the small parameter is $\epsilon=3/[2(4J-1)]$. The series representation for the scaled energy γ is

$$\gamma=\sum_{n=0}^{\infty} a_n\epsilon^n, \tag{11}$$

where $a_0=1$ as a result of the factorization in Eq. (7).

The WKB quantization condition to order ϵ^5 is [9]

$$\left(2J+\frac{1}{2}\right)\pi=\frac{1}{2\epsilon}\oint du\ Q^{1/2}-\frac{\epsilon}{96}\oint du\ \frac{Q''}{Q^{3/2}} -\frac{\epsilon^3}{3072}\oint du\ \frac{2Q''''Q-7(Q'')^2}{Q^{7/2}}+\cdots, \tag{12}$$

where $Q(u)=\gamma+3u^2-4u^6$ and where the contours encircle the two turning points (it is crucial that there be only two) and the branch cut joining them.

Next, we expand Eq. (12) in powers of ϵ and perform the resulting contour integrals. Each of these integrals can be evaluated in closed form because of the leading-order factorization $f(u)\equiv-4u^6+3u^2+1=(1-u^2)(2u^2+1)^2$. Indeed, all integrals of the form $\oint du\ u^{2m}[f(u)]^{1/2-n}$, where m and n are non-negative integers, can be expressed as simple multiples of π. For example,

$$\oint du\ [f(u)]^{1/2}=\frac{3\pi}{2},\qquad \oint du\ [f(u)]^{-1/2}=-\frac{2\pi}{\sqrt{3}},$$

$$\oint du\ [f(u)]^{-3/2}=-\frac{10\pi}{9\sqrt{3}},\qquad \oint du\ u^4[f(u)]^{-3/2}=\frac{\pi}{18\sqrt{3}}.$$

We evaluate the appropriate contour integrals and then use Eq. (12) to determine the first five expansion coefficients a_n for the scaled energy γ in Eq. (11). The expansion for the critical energy $E_c(J)$ is

$$E_c(J)\sim 2\left(\frac{4J-1}{3}\right)^{3/2}\left[1-\frac{3\sqrt{3}}{4J-1}+\frac{35}{8(4J-1)^2} +\frac{5\sqrt{3}}{3(4J-1)^3}+\frac{23281}{3456(4J-1)^4} +\frac{88945\sqrt{3}}{5184(4J-1)^5}+\cdots\right]. \tag{13}$$

Numerical results are excellent; when $J=40$ the exact value of $E_c(J)$ is 746.606 715 392 . . . , while the fifth-order WKB result in Eq. (13) gives 746.606 715 384 . . . .

The factorizability property of QES potentials has the consequence that semiclassical analysis can be performed to all orders. All integrals give simple multiples of π and no transcendental functions ever appear. This is in stark contrast with x^{2N} anharmonic oscillator potentials for which WKB analysis leads to elliptic functions [10].

Although factorizability is a property of QES potentials, it extends to a larger class of potentials that are not QES. We call such potentials *semiclassically quasiexactly solvable* (SQES). Consider the potential

$$V(x)=x^{10}-4Jx^4-\delta J^{4/3}x^2. \tag{14}$$

For fixed δ this potential is *not* QES in the conventional group-theoretic sense [2–4]; one cannot find more than one exact algebraically determined eigenstate and corresponding eigenvalue. Nevertheless, for large J we can impose the scaling and factorizability conditions. Generalizing from the factorization of the x^6 potential in Eq. (7), we demand that the quantity $E-V(x)$ factors into a product of a linear term in x^2 multiplied by a perfect square. This requirement fixes the numerical value of $\delta=3(4/5)^{1/3}$ and the asymptotic behavior of the critical energy $E_c(J)\sim\frac{9}{2}(4J/5)^{5/3}$ $(J\to\infty)$, and we have

$$\frac{9}{2}\frac{4J^{5/3}}{5}-x^{10}+4Jx^4+\frac{15}{4}\left(\frac{4J}{5}\right)^{4/3}x^2 =J^{5/3}\left[2\left(\frac{4}{5}\right)^{1/3}-y^2\right]\left[y^4+\left(\frac{4}{5}\right)^{1/3}y^2+\frac{3}{2}\left(\frac{4}{5}\right)^{2/3}\right]^2, \tag{15}$$

where $x=J^{1/6}y$. Because the squared factor has no real zeros, there are *exactly* two real turning points whose locations are determined by the linear factor in y^2.

This factorization enables us to evaluate the large-J leading-order WKB quantization integral exactly:

$$2\pi J\sim\int_{-x_0}^{x_0}dx\ \sqrt{E-V(x)}=\left[\frac{32}{5}B\left(\frac{3}{2},\frac{5}{2}\right)+\frac{16}{5}B\left(\frac{3}{2},\frac{3}{2}\right) +\frac{20}{5}B\left(\frac{3}{2},\frac{1}{2}\right)\right]J.$$

As in the case of the QES potential in Eq. (1), the higher-order WKB quantization integrals can be done exactly.

The potential in Eq. (14) belongs to an infinite hierarchy of SQES x^{4k+2} potentials that exhibit factorization and scaling properties but which are not QES for $k=2,3,4,\ldots$. The kth such potential is [12]

$$V(x)=\frac{16(k+1)^2}{(2k+1)^2}\left[\frac{\Gamma(k+3/2)}{2\sqrt{\pi}\Gamma(k+2)}\right]^{1/k+1} \times J^{2k+1/k+1}\left\{1-\left[{}_2F_1\left[\frac{1}{2},-\frac{1}{2}-k;\frac{1}{2} -k;\left(\frac{\Gamma(k+3/2)}{2\sqrt{\pi}\Gamma(k+2)J}\right)^{1/k+1}x^2\right]\right]^2\right\},$$

where ${}_2F_1$ is a hypergeometric function [11].

To demonstrate the factorization property we substitute $y^2=\{\Gamma(k+3/2)/[2\sqrt{\pi}\Gamma(k+2)J]\}^{1/(k+1)}x^2$ and express $E-V(x)$ as a linear factor in y^2 multiplied by a square of a polynomial of degree k in y^2:

$$\frac{16(k+1)^2}{(2k+1)^2}\left[\frac{\Gamma(k+3/2)}{2\sqrt{\pi}\Gamma(k+2)}\right]^{1/k+1}J^{2k+1/k+1}-V(x)$$
$$=\left[\frac{2J\sqrt{\pi}\Gamma(k+2)}{\Gamma(k+3/2)}\right]^{2k+1/k+1}(1-y^2)$$
$$\times\left[\sum_{n=0}^{k}\frac{\Gamma(n+\frac{1}{2})}{n!\sqrt{\pi}}y^{2k-2n}\right]^2.$$

Note that there are two real turning points at $y=\pm1$.

This factorization fixes the large-J asymptotic behavior of the critical energy:

$$E_c\sim\frac{16(k+1)^2}{(2k+1)^2}\left[\frac{\Gamma(k+3/2)}{2\sqrt{\pi}\Gamma(k+2)}\right]^{1/k+1}J^{2k+1/k+1}. \quad (16)$$

Leading-order WKB verifies this asymptotic behavior:

$$2\pi J\sim\int_{-x_0}^{x_0}dx\ \sqrt{E-V(x)}$$
$$=J\frac{2\sqrt{\pi}(k+1)!}{\Gamma(k+3/2)}\int_0^1du\ \sqrt{\frac{1-u}{\sqrt{u}}}\sum_{n=0}^{k}\frac{\Gamma(n+1/2)}{n!\sqrt{\pi}}u^{k-n}$$
$$=J\frac{2\sqrt{\pi}(k+1)!}{\Gamma(k+3/2)}\sum_{n=0}^{k}\frac{\Gamma(n+1/2)}{n!\sqrt{\pi}}B\left(\frac{3}{2},k-n+\frac{1}{2}\right).$$

Let us examine the hierarchy of SQES potentials for integer [13] values of k using the general formula [11]

$${}_2F_1(1/2,-k-1/2;-k+1/2;z)$$
$$=\frac{\Gamma(1/2-k)}{\Gamma(1/2)}z^{k+1/2}\left(\frac{d}{dz}\right)^k\left[\frac{(1-z)^{k+1/2}}{\sqrt{z}}\right]. \quad (17)$$

For $k=0$ we have $E_c\sim4J$ and ${}_2F_1(\frac{1}{2},-\frac{1}{2};\frac{1}{2};z)=\sqrt{1-z}$, and we obtain the *exactly solvable* harmonic oscillator potential, $V(x)=x^2$. For $k=1$ we have $E_c\sim\frac{16}{9}\sqrt{3}J^{3/2}$ and ${}_2F_1(\frac{1}{2},-\frac{3}{2};-\frac{1}{2};z)=(1+2z)\sqrt{1-z}$, and we obtain the large-J asymptotic approximation to the x^6 QES potential in Eq. (1): $V(x)=x^6-4Jx^2$. For $k=2$ we have $E_c\sim\frac{9}{2}(4J/5)^{5/3}$ and ${}_2F_1(\frac{1}{2},\frac{5}{2};-\frac{3}{2};z)=(1+\frac{4}{3}z+\frac{8}{3}z^2)\sqrt{1-z}$, and we obtain the x^{10} SQES potential in Eq. (15): $V(x)=x^{10}-4Jx^4-\frac{15}{4}(4J/5)^{4/3}x^2$. In the limit $k\to\infty$ we obtain the square-well potential, for which $E_c\sim4J^2$. Thus, at the bottom of the SQES hierarchy is the exactly solvable harmonic oscillator, followed by the QES x^6 potential, which is then followed by the range of new SQES potentials and, finally, in the $k\to\infty$ limit, by the square-well potential.

For QES potentials the critical energy $E_c(J)$ lies at the upper boundary of the quasiexact spectrum. For SQES potentials, the critical energy $E_c(J)$ is still a significant point in the energy spectrum; at this point we observe a sharp change (a first-order phase transition for large J) in the density of states. If we use ordinary WKB analysis to find the $2n$th energy eigenvalue for a x^{4k+2} potential, where n is not correlated with J, we find that as $n\to\infty$

$$E_{2n}\sim\left[4\sqrt{\pi}(k+1)n\Gamma\left(\frac{k+1}{2k+1}\right)\Big/\ \Gamma\left(\frac{1}{4k+2}\right)\right]^{2k+1/k+1}.$$

Observe that the numerical coefficient in this asymptotic relation is a different function of k from that in Eq. (16). For example, for $k=1$ we have $E_c\sim3.0792J^{3/2}$ but $E_{2n}\sim6.4066n^{3/2}$. Also, for $k=2$ we have $E_c\sim3.1024J^{5/3}$ while $E_{2n}\sim7.4235n^{5/3}$. As $k\to\infty$ we have $E_c\sim4J^2$ while $E_{2n}\sim\pi^2n^2$. However, for the special exactly solvable case $k=0$, where the level spacing is constant, they are the same: $E_c\sim4J$ and $E_{2n}\sim4n$.

Finally, we remark that SQES models are a generalization of the Ising limit in quantum mechanics or quantum field theory. The Ising limit is an asymptotic balance between a mass term and a self-interaction term: For the potential $gx^4-m^2x^2$ we take $g\sim m^2$ with g large and get a double well; g scales out, and we obtain the Ising limit. For SQES models there is an asymptotic balance for large J among all the coefficients of the potential.

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Walczak (unpublished); F. Finkel, A. González-López, and M. A. Rodríguez (unpublished).

[7] For the polynomial potentials in this paper $V(x) \sim E_c(J)$ $(J \to \infty)$ at (complex) points where $V'(x) = 0$.

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[12] This is the simplest SQES hierarchy, just as the QES potential in Eq. (1) is the simplest of the x^6 QES potentials.

[13] Noninteger values of k lead to other interesting classes of SQES potentials.