Time-dependent-harmonic plus inverse-harmonic potential in quantum mechanics

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For a time-dependent system characterized by the potential $V(x,t)=a_2(t)x^2+a_1/x^2$, we first demonstrate the construction of classical and quantum invariants and then obtain an exact solution of the Schrödinger equation for this potential. The role of the invariant so constructed is further discussed in the context of studies of the coherent states. [S1050-2947(97)02704-2]

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I. INTRODUCTION

The study of harmonic plus inverse harmonic (HPIH) potential of the type

$$V(x)=b_{20}x^2+b_1/x^2 \tag{1}$$

in quantum mechanics has been a subject of great interest in a variety of contexts [1–5,7] and with different meanings assigned to x. Here b_{20} and b_1 are constants. For example, the reduced Schrödinger equation (SE) for a central harmonic potential involves an effective potential of the type (1) with x replaced by the radial coordinate r and with b_1 attaining some discrete values. Calogero [1] considered a model three-body quantum problem in one spatial dimension in terms of the potential (1) with x replaced by the relative coordinate and thereby reducing the system to a two-dimensional one. The integrability [6] of such a system, which has wide applications in solid state physics and molecular chemistry has also been investigated. The form (1) is also explored [4,5] in the context of coherent state formalism. Several other mathematical aspects of potential (1) have also been studied by various authors [7]. Even for the one-dimensional case several distinct features of this potential (like that of attaining some discrete values by b_1 for the square integrability of the eigenfunction) are highlighted [2]. Khare and Bhaduri [3] have discussed the exact solution of the SE for the Calogero form of potential (1).

On the other hand, the time-dependent (TD) version of the HPIH potential of the form

$$V(x,t)=a_2(t)x^2+a_1/x^2 \tag{2}$$

also appears to be interesting in the context of classical mechanics [8] and coherent state formalism [5]. No doubt, from the point of view of constructing the invariant for a TD harmonic oscillator (since the corresponding Hamiltonian in this case is not the constant of motion) the problem has been of interest for more than two decades or so (see Refs. [2] and [8], and the references therein), the study of the potential (2) has not been carried out to the same extent. Particularly, the role of the available [8] invariant for the system (2) in the quantum context will require investigation. This will consequently enlarge the scope of applications of this system to various physical problems.

In the next section, we discuss the construction of classical and quantum invariants for the system (2). In Sec. III, the exact solution of the TD SE for Eq. (2) is obtained. The role of the quantum invariant obtained in Sec. II is discussed in the context of coherent state formalism in Sec. IV. Finally, the concluding remarks are made in Sec. V.

II. CONSTRUCTION OF CLASSICAL AND QUANTUM INVARIANTS

For the construction of exact invariants for TD classical dynamical systems several methods have been suggested in the literature [9]. We use here the Lie algebraic approach [8] to construct the invariant for the system (2) described by the Hamiltonian

$$H=\tfrac{1}{2}[p^2+\omega^2(t)x^2+k/x^2]. \tag{3}$$

Note that this approach, which has provided [8] the results for a variety of classical systems, is easily extendable to the corresponding quantum case. First, we briefly present the results for the classical case and then obtain the quantum invariant for the system (3) using the extension of this method by Monteoliva *et al.* [11].

A. Classical invariant

Recall that for the system (3), the dynamical algebra closes [8] for the following choice of the phase space functions:

$$\Gamma_1=\tfrac{1}{2}(p^2+k/x^2),\quad \Gamma_2=px,\quad \Gamma_3=\tfrac{1}{2}x^2, \tag{4}$$

and the classical invariant is found [8] to be

$$I=\frac{1}{2}\left[\bar{k}\left(\frac{x}{\rho}\right)^2+k\left(\frac{\rho}{x}\right)^2+(\rho\dot{x}-\dot{\rho}x)^2\right], \tag{5}$$

where $\rho(t)$ is a solution of the auxiliary equation

$$\ddot{\rho}+\omega^2(t)\rho=\bar{k}/\rho^3. \tag{6}$$

Interestingly, the Hamiltonian structure corresponding to Eq. (6) again turns out to be

$$\bar{H}=\tfrac{1}{2}[\overline{p^2}+\omega^2(t)\rho^2+\bar{k}/\rho^2]\quad (\bar{p}=\dot{\rho}), \tag{7}$$

which is analogous to the form (3). This fact is found to suggest [8] a physical meaning to the invariant (5) in the sense that it generates a mapping between H and $\bar{H}$. Also note that the equation of motion corresponding to Eq. (3) is analogous to that corresponding to Eq. (7) [i.e., Eq. (6)]. Further, for the case when $k=0$ (or $\bar{k}=0$) one can as well recover [8] the well-known results for the TD harmonic oscillator.

B. Quantum invariant

Note that in the quantum context not only x, p, ρ, and $\bar{p}$ but also the corresponding phase space functions Γ become operators. We put a cap on these q-number quantities just to differentiate them from the corresponding c-number quantities. Within the framework of the dynamical algebraic method now one expresses the Hamiltonian operator as

$$\hat{H}(t)=\sum_{i=1}^{n} h_i(t)\hat{\Gamma}_i, \tag{8}$$

where the set of operators $\{\hat{\Gamma}_1,\ldots,\hat{\Gamma}_n\}$ generates a dynamical algebra that is closed and thereby implies

$$[\hat{\Gamma}_i,\hat{\Gamma}_j]=\sum_{l=1}^{n} C_{ij}^{l}\hat{\Gamma}_l, \tag{9}$$

where [,] is the commutator and C_{ij}^{l}'s are the structure constants. Since the invariant operator $\hat{I}$ is also a member of this algebra, the same can also be expressed as

$$\hat{I}(t)=\sum_{i=1}^{n} \lambda_i(t)\hat{\Gamma}_i. \tag{10}$$

Further, the invariant condition expressed in this case as ($\hbar=1$)

$$\frac{d\hat{I}}{dt}=\frac{\partial\hat{I}}{\partial t}+\frac{1}{i}[\hat{I}(t),\hat{H}(t)]=0 \tag{11}$$

leads to a system of linear first-order differential equations in λ_i's, which can be solved in the same way as one proceeds in the classical case [8,10]. However, here one has to keep track of the operator character of various quantities at different stages of the solution of these coupled differential equations. For the system (3), the phase space operators now take the form

$$\hat{\Gamma}_1=\frac{1}{2}(\hat{p}^2+k/\hat{x}^2)=\frac{1}{2}\left(-\frac{d^2}{dx^2}+\frac{k}{x^2}\right),$$

$$\hat{\Gamma}_2=\frac{1}{2}(px+xp)=-\frac{i}{2}\left(\frac{d}{dx}x+x\frac{d}{dx}\right),\quad \hat{\Gamma}_3=\frac{1}{2}\hat{x}^2 \tag{4$'$}$$

and the algebra now closes in the same manner as for the classical case with

$$[\hat{\Gamma}_1,\hat{\Gamma}_2]=-2i\hat{\Gamma}_1,\quad [\hat{\Gamma}_1,\hat{\Gamma}_3]=-i\hat{\Gamma}_2,\quad [\hat{\Gamma}_2,\hat{\Gamma}_3]=-2i\hat{\Gamma}_3.$$

Finally, the quantum invariant for Eq. (3) turns out to be

$$\hat{I}(t)=\tfrac{1}{2}[k(\hat{\rho}/\hat{x})^2+\bar{k}(\hat{x}/\hat{\rho})^2+(\hat{\rho}\hat{p}-\hat{\bar{p}}\hat{x})^2], \tag{12}$$

which is of the same form as Eq. (5).

Next, we use the knowledge of the invariant (5) to obtain an exact analytic solution of the TD SE for the potential (2).

III. EXACT SOLUTION OF THE SCHRÖDINGER EQUATION FOR THE POTENTIAL (2)

A search for exactly solvable TD potentials in the SE ($\hbar=\mu=1$),

$$[-\tfrac{1}{2}(\partial^2/\partial x^2)+V(x,t)]\Psi(x,t)=i[\partial\Psi(x,t)/\partial t] \tag{13}$$

is made by Truax [12] and consequently a classification of these potentials is suggested on the basis of their space-time or kinematical algebras. Here we employ the method of Ray [13] (which is also conveniently extended [14] to the study of a class of two-dimensional systems) to obtain an exact solution of TD SE (13) for the potential (2). This method, based essentially on the generalization of the group-transformation method of Burgan *et al.*, [15] can be carried out in two stages. In the first stage, one performs a scale and a phase transformation of the dependent variable and a scale transformation of the independent space and time variables as

$$\Psi(x,t)=B(t)\exp[i\phi(x,t)]\psi(x,t), \tag{14a}$$

$$x'=x/C(t)+A(t),\quad t'=D(t), \tag{14b}$$

where various symbols have been defined in Ref. [13]. This converts the TD SE into a more complicated form. The arbitrary functions appearing in Eq. (14) are then fixed by setting some of the additional terms in the new equation to zero and subsequently by demanding the form of the TD SE to be invariant under the above transformation. This is done by modifying the potential term in Eq. (13) to the form

$$V'=VC^2+\tfrac{1}{2}C\ddot{C}x^2-(\ddot{A}C^3+2\dot{A}\dot{C}C^2)x+\tfrac{1}{2}C^4\dot{A}^2. \tag{15}$$

In the second stage, another phase transformation of the dependent variable converts this new TD SE into a time-independent (TID) SE in one of the standard forms whose exact solutions are normally known in advance. Interestingly, the Hamiltonian corresponding to this final TID SE turns out to be a constant of motion, which, in fact, has a connection with the corresponding classical invariant. Without going into further detail we rather demonstrate here the use of this method for the potential (2).

Use of Eq. (2) and the transformation (14b) reduces Eq. (15) to the form

$$V'=(a_2C+\tfrac{1}{2}\ddot{C})C^3x'^2-[\ddot{A}C^2+2\dot{A}\dot{C}C+2Aa_2C^2+AC\ddot{C}]C^2x'+\frac{a_1}{(x'-A)^2}+F(t), \tag{16a}$$

where

$$F(t)=(a_2C+\tfrac{1}{2}\ddot{C})C^3A^2+A(\ddot{A}C+2\dot{A}\dot{C})C^3+\tfrac{1}{4}C^4\dot{A}^2. \tag{16b}$$

For the arbitrary functions $A(t)$ and $C(t)$ in Eq. (14) one can set

$$\tfrac{1}{2}\ddot{C}+a_2C=k_1/C^3, \quad A=0, \tag{17}$$

where k_1 is a TID constant. Note that the first equation in (17) conforms to Eq. (6). Since a_1 is TID, V' of (16a) now assumes the form of a TID potential, viz.,

$$V'=k_1x'^2+a_1/x'^2,$$

with $F(t)=0$. The SE (13) now becomes separable in x' and t' variables and can be expressed as

$$[-\tfrac{1}{2}(\partial^2/\partial x'^2)+k_1x'^2+a_1/x'^2]\psi=i(\partial\psi/\partial t'). \tag{18}$$

Also note that since $F(t)=0$ in the present case, no further phase-change transformation [cf. Eq. (3.8) of Ref. [13]] on $\psi(x',t')$ is required. For $k_1>0$, Eq. (18) can be realized as a TID SE for TID HPIH potential (1). Further, for $k_1=0$ one can investigate the special case for an inverse harmonic potential [7].

With regard to the connection between the invariant (5) and the Hamiltonian (now rechristened as I') of Eq. (18), it can be verified that the transformation,

$$I=e^{i\phi}I'e^{-i\phi},$$

with $\phi=(\dot{C}x^2/2C)$, gives rise to the same form of I as Eq. (5) except for an imaginary TD term in it. Note that while this latter term will not affect the physical content of the problem, it can, however, be made to disappear by resetting the TD transformations involved in the method.

Corresponding to the Hamiltonian I', the general solution of Eq. (18) can immediately be written as

$$\psi(x',t')=\sum_n c_n e^{-iE_nt'}u_n(x')$$

with t' given, as before, by [13] $t'=D(t)=\int dt/C^2$. Also, the expansion coefficients c_n can be obtained from the condition

$$c_n=\langle u_n(x'),\psi(x',0)\rangle.$$

For the determination of $u_n(x')$ one has to use the following prescription. Note that the ground state solution of the SE corresponding to the TID potential (1) is obtained [2] as

$$u_0(x')=Nx'^m\exp(-\sqrt{b'_{20}}x'^2/2),$$

with the eigenvalue $E_0=\tfrac{1}{2}(2m+1)\sqrt{b_{20}}$, and the normalization constant

$$N=\left[\frac{2^m b'^{1/2(m+1/2)}_{20}}{\sqrt{\pi}(2m-1)!!}\right]^{1/2}.$$

Here, the integer m takes the values $m\geqslant 2$ and $b'_1=2b_1=m(m-1)$, $b'_{20}=2b_{20}$. For the excited states, however, one can start with the ansatz

$$u_n(x')=\mathcal{N}P_n(x')\exp(-\sqrt{b'_{20}}x'^2/2)$$

[where $\mathcal{P}_n(x')$ is a polynomial] and obtain the solution of the TID SE corresponding to the potential (1) in the same way as done [16] for the other singular potentials. Noting that $x'=x/C(t)$, the exact solution of Eq. (13) for the potential (2) can be written as

$$\Psi(x,t)=\frac{\mathcal{N}}{\sqrt{C}}\exp(i\dot{C}x^2/2C)\sum_n C_n \times\exp\left(-iE_n\int dt/C^2\right)u_n(x/C). \tag{19}$$

As a final remark about using the method of Ray [13] it may be mentioned that the time independence of the parameter a_1 in Eq. (2) is an asset: otherwise it would have been difficult to obtain a TID I' in Eq. (18).

IV. ROLE OF THE INVARIANT IN COHERENT STATE STUDIES

About four decades ago. Husimi [17] studied the quantal treatment of the forced TD harmonic oscillator with reference to a variety of phenomena. While some of his results pertain to the exact treatment of the explicit time dependence of the system, many of them are obtained for the adiabatic or perturbative cases. In fact, some of his results are now well taken care of through the coherent state formalism [5] and also by introducing [18] a linear (but complex) invariant in the formalism. Somehow this linear (in momentum or velocity) invariant has not become so popular as the quadratic (in momentum or velocity) one. In a recent study, Gerry and Kiefer [19] have investigated radial coherent states for the isotropic harmonic oscillator within the framework of the unitary irreducible representation associated with the radial spectrum-generating group Sp(2), which is locally isomorphic to su(1,1) and so(2,1). They study the time evolution of a coherent state wave packet, which, however, corresponds to a potential of the type (1). Here, we first briefly review the status of linear invariant and then discuss the role of the quadratic one in the context of coherent state studies.

A. Use of linear invariant

A TID complex linear invariant has been found useful not only in the studies of coherent and squeezed states [18,20,21] but also of coherent correlated states [22]. Man'ko [20] has emphasized that such studies of the TD SE with reference to the coherent states can be better carried out using the knowledge of this simple integral of motion. In this case one looks for the solution of Eq. (13) with $V(x,t)=\tfrac{1}{2}\omega^2(t)x^2$ or with $V(x,t)=\tfrac{1}{2}\omega^2(t)x^2+f(t)x$ in terms of a TD invariant, which is now an operator. For the first case it turns out to be of the form

$$\mathcal{A}(t)=\frac{i}{\sqrt{2}}[\epsilon(t)p-\dot{\epsilon}(t)x], \tag{20}$$

where $\epsilon(t)$ satisfies the equation

$$\ddot{\epsilon}(t)+\omega^2(t)\epsilon(t)=0 \quad [\epsilon(0)=1, \quad \dot{\epsilon}(0)=i] \tag{21}$$

and the operator $\mathcal{A}$ conforms to $[\mathcal{A},\mathcal{A}^\dagger]=1$. It is found that the packet solutions of the SE of Husimi [17], which now involve the function $\epsilon(t)$ in the form

$$\Psi_\alpha(x,t)=\Psi_0(x,t)\exp\left\{-\frac{|\alpha|^2}{2}-\frac{\alpha^2\epsilon^*(t)}{2\epsilon(t)}+\frac{\sqrt{2}\alpha x}{\epsilon(t)}\right\}, \tag{22}$$

where

$$\Psi_0(x,t)=\pi^{-1/4}[\epsilon(t)]^{-1/2}\exp\{i\dot{\epsilon}(t)x^2/2\epsilon(t)\}$$

may be introduced and interpreted as coherent states since they are eigenstates of the operator $\mathcal{A}(t)$ given by Eq. (20). Here α is a complex number, which appears in the definition of the coherent state $|\alpha\rangle$ of Glauber [23].

Note that the importance of the linear invariant (20) has also been recognized [20] in the study of q-deformed Husimi packet solutions of the SE. Interestingly, a special solution of Eq. (21) as

$$\epsilon(t)=|\epsilon(t)|\exp\left(i\int^t d\tau/|\epsilon(t)|^2\right) \tag{23}$$

converts this equation into the form

$$\frac{d^2|\epsilon(t)|}{dt^2}+\omega^2(t)|\epsilon(t)|=1/|\epsilon(t)|^3, \tag{24}$$

which, in fact, is the same as satisfied by the auxiliary function $\rho(t)$ in Eq. (6). Recently, the quantum and quadratic analogue of the invariant (20) of the form

$$I=\mathcal{A}^\dagger(t)\mathcal{A}(t)=\tfrac{1}{2}\{|\epsilon(t)|^2p^2+|\dot{\epsilon}(t)|^2x^2-\epsilon^*\dot{\epsilon}px-\epsilon\dot{\epsilon}^*xp\} \tag{25}$$

has been studied by Man'ko [20] and its role in the context of coherent and squeezed states is investigated. It is not difficult to verify that if one substitutes $|\epsilon(t)|=\rho(t)$ and uses $|\dot{\epsilon}|^2=\dot{\rho}^2+(1/\rho^2)$ from Eq. (23), then the form (25) immediately reduces to a form similar to Eq. (12) and is well known [8] for the TD harmonic oscillator.

B. Use of quadratic invariant

While the roles of linear [cf. Eq. (20)] and possibly that of quadratic [cf. Eq. (25)] invariant for the TD harmonic oscillator are discussed in the context of coherent state studies, the role of the invariant (12) corresponding to the TD HPIH potential (2) has not been discussed in spite of the use [5] of the latter in such studies. What we shall demonstrate here is that for the potential (2) the set of generators (i.e., the phase space functions) (4), which are required to close the dynamical algebra at the classical level and thereby provide a quadratic invariant also conform to the algebra of su(1,1) as discussed by Perelomo [5] at the quantum level. Interestingly, in this way the role of at least quadratic invariant (12) can be highlighted in the studies of coherent states.

Now the question arises whether we can have a linear invariant like Eq. (20) for the system (3), which can, on the one hand, conform to the quadratic form (12), while on the other, provide the packet solutions such as Eq. (22) of Husimi [17]. In this regard, one can also try a complex form, which is linear in momentum resulting in a quadratic form (12).

In the approach of Perelomov [5] the Hamiltonian (3) in the context of the SE (13) is written as

$$H\equiv\Omega K=\Omega_0K_0-\Omega_1K_1-\Omega_2K_2$$
$$=i(\beta K_+-\overline{\beta}K_--i\gamma K_0),$$

where

$$\Omega_{0,1}=\omega_0\left[\left(\frac{\omega(t)}{\omega_0}\right)^2\pm1\right],\quad \Omega_2=0,\quad \omega_0=\omega(0)=1.$$

Here the standard generators K_0,K_1,K_2 of su(1,1) algebra are defined as

$$K_0=\tfrac{1}{2}H,\quad K_+=K_1+iK_2=-\tfrac{1}{2}B_2^\dagger,$$
$$K_-=K_1-iK_2=-\tfrac{1}{2}B_2, \tag{26}$$

with

$$B_2^\dagger=(a^\dagger)^2-k/(2x^2),\quad B_2=a^2-k/(2x^2),$$

and the usual creation and annihilation operators $a^\dagger$and a are given by [5]

$$a^\dagger=\frac{1}{\sqrt{2}}\left(x-\frac{d}{dx}\right),\quad a=\frac{1}{\sqrt{2}}\left(x+\frac{d}{dx}\right).$$

While $B_2^\dagger$, B_2, and H close the su(1,1) algebra, the solution of the SE (13) is written as

$$\Psi(t)=\exp[-i\varphi(t)]\zeta(t),\quad [|\zeta(t)|<1], \tag{27}$$

where $\zeta(t)$ and $\varphi(t)$ satisfy

$$\dot{\zeta}=\beta-\overline{\beta\zeta}^2-i\gamma\zeta,$$
$$i\dot{\varphi}=c(\beta\overline{\zeta}-\overline{\beta\zeta}+i\gamma).$$

Here various symbols have their usual [5] meanings. Using such a prescription for the solution of Eq. (13) coherent state studies are carried out in terms of transition probability and quasienergy spectrum.

Note that in this approach not only the bilinear forms of the creation and annihilation operators (through B_2 and $B_2^\dagger$) close the algebra but the Hamiltonian as a whole is also a party to this closure process. On the other hand, if we define the creation and annihilation operators in terms of Γ_i's of Eq. (4′) as

$$J_+=-i\sqrt{(i/2)}x^2,\quad J_-=i\sqrt{(i/2)}\left(-\frac{d^2}{dx^2}+\frac{k}{k^2}\right),$$

$$J_3=\frac{i}{4}\sqrt{(i/2)}\left(\frac{d}{dx}x+x\frac{d}{dx}\right),$$

then the algebra closes through the commutation relations

$$[J_3,J_\pm]=\pm iJ_+,\quad [J_+,J_-]=2J_3.$$

Thus, the generators (4′) not only give rise to the invariant (12) but also to the same algebra as su(1,1), which is used in the studies of the coherent states. This can clearly bring in the role of the invariant in such studies.

V. CONCLUSIONS

With a view to demonstrating some further applications of the TD HPIH potential (2) in the quantum domain several of its intriguing features are overviewed. In particular, the role of the invariant that exists for this system is highlighted with reference to Eq. (1) (i) obtaining an exact solution of the TD SE and (ii) using it in the studies of the coherent states. It appears that this potential can play as important a role in describing various physical phenomena as the TD harmonic oscillator. From this point of view a detailed study of the coherent and squeezed states is in progress.

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