Fundamental quantum limit in precision phase measurement

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We show through a series of arguments that, given a total average photon number $\langle N\rangle$, the fundamental limit in the precision phase measurement is set by quantum mechanics to be the so-called Heisenberg limit, i.e., $1/\langle N\rangle$. Some specific types of phase measurement are considered in the discussion. However, the proof based on the general principle of complementarity of quantum mechanics applies to any scheme of phase measurement. From the general argument by the complementarity principle, we are able to find a necessary condition for those states that can achieve the Heisenberg limit if they are employed for precision phase measurement. A general guideline is given for the search of the measurement schemes in which the Heisenberg limit is achieved. We demonstrate the procedure by applying it to a few specific examples. [S1050-2947(97)01804-0]

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I. INTRODUCTION

It is well known that the quantum nature of electromagnetic fields leads to limitations on how precisely a physical quantity of an optical field can be measured. It is generally believed that, given an arbitrary state of light, the Heisenberg uncertainty relation sets the lower bound on the sensitivity of the measurement. On the other hand, if we are allowed to prepare the system in some specific states, according to the quantum theory of measurement, a physical quantity can be measured to arbitrary precision, provided that the states are eigenstates of the operator representing the physical quantity in quantum mechanics. For the phase of an optical field, however, the answer is not so straightforward, mainly because of the fact that there does not exist a Hermitian operator for phase in an infinite-dimensional state space for a quantized optical mode [1,2]. Recent theoretical progress [3,4] in identifying a quantum-mechanical operator for phase in a finite-dimensional state space leads to the following limiting state as the eigenstate of a phase operator (which is also defined by a limiting process):

$$|\theta\rangle=\lim_{s\to\infty}(s+1)^{-1/2}\sum_{m=0}^{s} e^{im\theta}|m\rangle, \tag{1}$$

which resembles to the eigenstate of position operator (defined through a limiting process). Indeed, a reasonable probability distribution density of the phase for a given state can be derived by projecting the state onto this phase state (similar to the wave function of the state) and, when the system is prepared in the phase state in Eq. (1), a measurement of the phase will yield a precise value, as we will show below. One caveat, however, is that the average photon number of this phase state is infinite in the limiting process. Thus it is not a physical state, reflecting the difficulty encountered in the search for a physical phase operator. It becomes a common consensus that, with unlimited resources of energy, it is possible to measure a phase shift to arbitrary precision.

In the meantime, we have only a finite amount of energy in a realistic physical world. Thus we will limit our discussion under the finite energy constraint throughout the paper. But can we still measure the phase to an arbitrary precision with this constraint? In some sense, the failure to find an eigenstate of phase with finite photon number also implies a negative answer to the question. Therefore there exists a limit on the sensitivity of the phase measurement in the case of finite energy. The traditional argument for the limit comes from the Heisenberg uncertainty principle for the phase and photon number [1],

$$\Delta\phi\Delta N\geqslant 1, \tag{2}$$

where $\Delta\phi$ and ΔN are the fluctuations for the phase and photon number, respectively. Therefore, shot noise ($\Delta N=\sqrt{\langle\Delta^2N\rangle}\sim\sqrt{\langle N\rangle}$) due to the particle nature of light will place the so-called shot-noise limit or coherent state limit [5] on the sensitivity of the phase measurement,

$$\Delta\phi\gtrsim\frac{1}{\sqrt{\langle N\rangle}}. \tag{3}$$

On the other hand, quantum mechanics does not set any restriction on the fluctuation ΔN of the photon number. Intuitively, one would argue that because of energy constraint, ΔN should be bounded by the mean number of photons, that is, $\langle\Delta^2N\rangle\sim O(\langle N\rangle^2)$. Thus given a total mean number of photons, the limit in precision phase measurement should be the so-called Heisenberg limit

$$\Delta\phi\gtrsim\frac{1}{\langle N\rangle}. \tag{4}$$

Note that the Heisenberg limit should be understood as an approximate limit at a large mean photon number, that is, the phase uncertainty approaches the order of $\langle N\rangle^{-1}$ for large $\langle N\rangle$. We treat it the same way throughout this paper.

Shapiro and co-workers [6] recently proposed the state

$$|\Phi\rangle_{ssw}=A\sum_{m=0}^{M}\frac{1}{m+1}|m\rangle \quad (M\gg 1,\quad A\simeq\sqrt{6/\pi^2}) \tag{5}$$

as the optimum state in precision phase measurement, and they claimed that a $1/\langle N\rangle^2$ performance in sensitivity can be achieved. Although some difficulties [7–9] associated with the phase distribution of this state prevent it from achieving the promised precision, it is interesting to note that, for this state, the photon number fluctuation $\langle \Delta^2 N\rangle$ is on the order of $\exp(\langle N\rangle/A^2)$ for large M. Hence, from Eq. (2) we have the limit $\Delta\phi \gtrsim \exp(-\langle N\rangle/A^2)$, which is much better than the Heisenberg limit of Eq. (4) for large $\langle N\rangle$.

As a matter of fact, the validity of Eq. (2) is not general [2]. For example, for the vacuum state, the left-hand side of Eq. (2) is obviously zero, thus violating the inequality. Therefore, arguments based on the Heisenberg uncertainty relation in Eq. (2) cannot hold in general, and the question remains: What is the limit in precision phase measurement given the available total mean number of photon?

In the literature there have already been quite a number of papers on precision phase measurement [6–8,10–14]. Most of them concentrated on a specific scheme or data analysis strategy for phase measurement, and therefore cannot apply to general cases. Even so, all the analyses up to now [7,8,10–14] have shown that the performance in precision phase measurement does not exceed the Heisenberg limit, except for the questionable scheme [6] with the state in Eq. (5). Thus we have reason to speculate that the fundamental limit set by quantum mechanics in precision phase measurement is the Heisenberg limit.

In this paper, we will prove through a series of arguments that the ultimate limit in precision phase measurement is the Heisenberg limit. We first start our discussion on some specific types of schemes of phase measurement. Then, by applying the complementarity principle to a single-photon interferometer, we provide the most general proof of the fundamental limit, which is independent of measurement schemes. In Sec. III, we will derive a necessary condition for those states that can achieve the Heisenberg limit in precision phase measurement. Based on the results of Secs. II and III, in Sec. IV we will outline a general guideline in the search for the schemes that can achieve the Heisenberg limit. We then apply the guideline in Sec. V to specific examples including some unconventional interferometers that are not based on beam splitters. Section VI is devoted to discussion and summary.

II. FUNDAMENTAL QUANTUM LIMIT IN PRECISION PHASE MEASUREMENT

In this section, we will run through a number of proofs to show that the fundamental quantum limit in precision phase measurement is the Heisenberg limit.

A. Semiclassical argument

Classically, the phase is just the argument of the complex field amplitude used to describe an optical field. Many factors may change the value of phase. In fact, its measurement plays an essential rule in precision measurement, and has been widely used in practical applications as well as in fundamental studies. The traditional method of measuring phase shift is interferometry. This method relies on the optical interference effect for the comparison of phases in two paths. If we fix the phase delay of one path, any detected change in the output intensity of the interferometer will indicate a phase shift experienced in the other path, thus making a measurement of the phase shift. To be more specific, as shown in Fig. 1, a coherent optical field is split by a beam splitter into two fields which later recombine to form interference fringes. If the interferometer is properly balanced, the output intensity has the form of

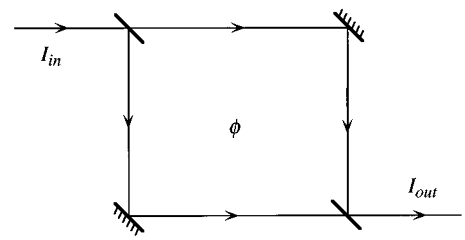

FIG. 1. Mach-Zehnder interferometer for the measurement of the phase difference.

$$I_{\rm out}=I_{\rm in}(1-\cos\phi)/2, \tag{6}$$

where $I_{\rm in}$ is the intensity of the input field, and ϕ is the relative phase shift between the two interfering paths. If we have a well-defined amplitude in the input field, any change $\Delta I_{\rm out}$ in the output intensity must come from the change $\Delta\phi$ in the relative phase. The sensitivity is highest when we set $\phi=\pi/2$:

$$\Delta I_{\rm out}=I_{\rm in}\Delta\phi/2. \tag{7}$$

Classically, there is no limit on how small the change $\Delta I_{\rm out}$ in intensity can be. Therefore, in principle, there is no limit on how small a phase shift $\Delta\phi$ can be measured. In quantum theory, however, the particle nature of light does not allow an infinite division of energy, thus setting a lower limit on $\Delta I_{\rm out}$. We can rewrite Eq. (7) in terms of the photon number as

$$\Delta N_{\rm out}=N_{\rm in}\Delta\phi/2, \tag{8}$$

where $N_{\rm in}$ is the total input photon number and $\Delta N_{\rm out}$ is the change in the output photon number. The minimum $\Delta N_{\rm out}$ that is allowed by quantum theory is simply one corresponding to the change of one quanta. Therefore, the quantum limit for phase measurement is

$$\Delta\phi \geqslant \frac{1}{N} \quad (\text{with } N=N_{\rm in}/2), \tag{9a}$$

which is the Heisenberg limit. $N=N_{\rm in}/2$ is the total number of photons in the arm of the interferometer that experiences the phase shift.

The above semiclassical argument runs equally well if we describe the optical field quantum mechanically. However, it is limited to the specific scheme of interferometry for phase comparison, and to the detection scheme of intensity measurement.

Furthermore, if classical states of light are used as the input to the interferometer, photon statistics is at best the Poisson distribution, i.e., $\Delta N_{\rm out} \geqslant \sqrt{N_{\rm out}}$. But for optimum sensitivity at $\phi=\pi/2$, $N_{\rm out}=N_{\rm in}/2=N$. Hence from Eq. (8) we

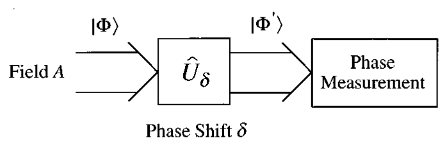

FIG. 2. Quantum description of a phase shift and its measurement.

arrive at the classical limit or the coherent state limit in a conventional interferometer [5],

$$\Delta\phi_c \geq 1/\sqrt{N}. \tag{9b}$$

Thus nonclassical states of light must be employed in order to surpass the coherent state limit to reach the Heisenberg limit.

B. Argument by the change in quantum state produced by the phase shift

Quantum mechanically, a phase shift δ induced by a linear optical element on a single-mode optical field is described by the unitary operator,

$$\hat{\mathcal{U}}_\delta = \exp(i\hat{n}\delta), \tag{10}$$

where $\hat{n}=\hat{a}^\dagger\hat{a}$ is the number operator, with $\hat{a}$ the annihilation operator for the optical mode. If the optical field is in the state $|\Phi\rangle$, the state after the phase shift is then $|\Phi'\rangle=\hat{\mathcal{U}}_\delta|\Phi\rangle$, as shown in Fig. 2.

Next let us write the state $|\Phi\rangle$ in the general form

$$|\Phi\rangle = \sum_m c_m|m\rangle \tag{11}$$

in the basis of photon number Fock state representation. Then the phase-shifted state $|\Phi'\rangle$ can be written as

$$|\Phi'\rangle = \exp(i\hat{n}\delta)\sum_m c_m|m\rangle = \sum_m c_m e^{im\delta}|m\rangle. \tag{12}$$

Since our goal is to detect any change in the state due to the phase shift, we are more interested in the difference $|\Delta\Phi\rangle \equiv |\Phi'\rangle - |\Phi\rangle$. From Eqs. (11) and (12), we can write $|\Delta\Phi\rangle$ as

$$|\Delta\Phi\rangle \equiv |\Phi'\rangle - |\Phi\rangle = \sum_m c_m(e^{im\delta}-1)|m\rangle. \tag{13}$$

If the change in the state is small, so that we are unable to detect it no matter what method we use, it will be impossible to resolve the phase shift δ. The quantity that characterizes the size of the change in the state is the norm of $|\Delta\Phi\rangle$:

$$\||\Delta\Phi\rangle\|^2 \equiv \langle\Delta\Phi|\Delta\Phi\rangle = 4\sum_m |c_m|^2\sin^2(m\delta/2)$$

$$= 4\sum_m P_m \sin^2(m\delta/2), \tag{14}$$

where $P_m \equiv |c_m|^2$ is the photon number distribution for the input field. By using the inequalities $|\sin x| \leq 1$ and $|\sin x| < |x|$, we can easily rewrite Eq. (14) as

$$\||\Delta\Phi\rangle\|^2 \leq 4\sum_m P_m|\sin(m\delta/2)| \leq 4\sum_m P_m m\delta/2 = 2\langle N\rangle\delta, \tag{15}$$

where $\langle N\rangle$ is the total mean number of photon in the input field. From Eq. (15), we see that if $\delta=o(1/\langle N\rangle)$, $\||\Delta\Phi\rangle\|^2$ will be infinitesimally smaller than one when $\langle N\rangle\to\infty$, which means that it is impossible to detect any change in $|\Phi'\rangle$ or the phase shift $\delta\sim o(1/\langle N\rangle)$. We can therefore conclude that the minimum detectable phase shift is at least of the order of $1/\langle N\rangle$ for large $\langle N\rangle$.

However, the above argument only holds for a pure input state. Very often the input field is correlated to other fields, and when we look at the state of the input field alone, it is in a mixed state described by a density operator in the general form of

$$\hat{\rho} = \sum_{m,n} \rho_{mn}|m\rangle\langle n|. \tag{16}$$

For this case, the phase-shifted state can be described by another density operator related to the original density operator by

$$\hat{\rho}' = \hat{\mathcal{U}}_\delta\hat{\rho}\hat{\mathcal{U}}_\delta^\dagger = \sum_{m,n} \rho_{mn}e^{i(m-n)\delta}|m\rangle\langle n|. \tag{17}$$

As before, we are only interested in the change in the density operator,

$$\Delta\hat{\rho} = \hat{\rho}' - \hat{\rho} = \sum_{m,n} \rho_{mn}[e^{i(m-n)\delta}-1]|m\rangle\langle n|, \tag{18}$$

and the quantity that characterizes the size of the change is the sum of the absolute square of all the elements of the matrix $\Delta\hat{\rho}$:

$$\|\Delta\hat{\rho}\|^2 = 4\sum_{m,n} |\rho_{mn}|^2\sin^2(m-n)\delta/2. \tag{19}$$

By using the same trick that leads to Eq. (15), we rewrite Eq. (19) as

$$\|\Delta\hat{\rho}\|^2 < 2\sum_{m,n} |\rho_{mn}|^2|m-n|\delta. \tag{20}$$

It can be easily shown that $|\rho_{mn}|^2 \leq \rho_{mm}\rho_{nn} = P_m P_n$ for any density matrix. Equation (20) then becomes

$$\|\Delta\hat{\rho}\|^2 < 2\sum_{m,n} P_m P_n|m-n|\delta$$

$$< 2\sum_{m,n} P_m P_n(m+n)\delta = 4\langle N\rangle\delta, \tag{21}$$

which is similar to Eq. (15). Thus we have proved for any input field that, in order to have a significant change in the state, the phase shift must be at least of the order of $1/\langle N\rangle$ for large $\langle N\rangle$.

C. Argument by the signal-to-noise ratio in quantum measurement

The argument in Sec. II B relies on the assumption that we are unable to detect an infinitesimal change in the state no matter what method we use. We will prove this assumption in the following by considering the signal-to-noise ratio in a general quantum measurement on the phase-shifted state for the detection of the phase shift. We will base our proof on the pure state in Eq. (11) for the input field.

Consider a general measurement process. Assume that we make a quantum measurement on the output field to find the change due to phase shift δ. Let $\hat{O}$ be the operator corresponding to the measurement. Thus the signal of the measurement is

$$S=\langle\Phi'|\hat{O}|\Phi'\rangle=\langle\Phi|\hat{O}|\Phi\rangle+\langle\Delta\Phi|\hat{O}|\Phi\rangle+\langle\Phi|\hat{O}|\Delta\Phi\rangle +\langle\Delta\Phi|\hat{O}|\Delta\Phi\rangle, \tag{22}$$

where Eq. (13) is used. Since the measurement is for the detection of $|\Delta\Phi\rangle$, it is preferable to have $\langle\Phi|\hat{O}|\Phi\rangle=0$. Otherwise we can always redefine the operator as $\hat{O}-\langle\Phi|\hat{O}|\Phi\rangle$. We rewrite Eq. (22) as

$$\begin{aligned}|S|^2&=|\langle\Delta\Phi|\hat{O}|\Phi\rangle+\langle\Phi'|\hat{O}|\Delta\Phi\rangle|^2\\ &\leqslant 2(|\langle\Delta\Phi|\hat{O}|\Phi'\rangle|^2+|\langle\Delta\Phi|\hat{O}|\Phi\rangle|^2)\\ &\leqslant 2(\|\hat{O}|\Phi'\rangle\|^2+\|\hat{O}|\Phi\rangle\|^2)\||\Delta\Phi\rangle\|^2\end{aligned} \tag{23}$$

where we used the Schwartz inequality in the last inequality. Next, let us find the noise ΔS in the measurement as the variance of $\hat{O}$. Since we are trying to detect the change in the phase, we need to consider the variances for the states before and after the phase shift, that is,

$$(\Delta S)^2=\mathrm{Max}(\langle\Phi|\Delta^2\hat{O}|\Phi\rangle,\langle\Phi'|\Delta^2\hat{O}|\Phi'\rangle). \tag{24}$$

There are two possibilities in the above expression.

(i) If $\langle\Phi|\Delta^2\hat{O}|\Phi\rangle\leqslant\langle\Phi'|\Delta^2\hat{O}|\Phi'\rangle$, we have

$$\begin{aligned}\|\hat{O}|\Phi\rangle\|^2&=\langle\Phi|\hat{O}^2|\Phi\rangle\\ &=\langle\Phi|\Delta^2\hat{O}|\Phi\rangle\\ &\leqslant\langle\Phi'|\Delta^2\hat{O}|\Phi'\rangle\\ &<\langle\Phi'|\hat{O}^2|\Phi'\rangle\\ &=\|\hat{O}|\Phi'\rangle\|^2.\end{aligned} \tag{25}$$

Then the signal-to-noise ratio $R\equiv S/\Delta S$ has the form

$$R^2=\frac{\langle\Phi'|\hat{O}|\Phi'\rangle^2}{\langle\Phi'|\Delta^2\hat{O}|\Phi'\rangle}=\frac{1}{\|\hat{O}|\Phi'\rangle\|^2/S^2-1}. \tag{26}$$

By using the inequalities in Eqs. (23) and (25), we find the upper bound for R^2 as

$$R^2\leqslant\frac{1}{1/4\||\Delta\Phi\rangle\|^2-1}=\frac{4\||\Delta\Phi\rangle\|^2}{1-4\||\Delta\Phi\rangle\|^2}\quad(4\||\Delta\Phi\rangle\|^2<1). \tag{27}$$

(ii) If $\langle\Phi'|\Delta^2\hat{O}|\Phi'\rangle\leqslant\langle\Phi|\Delta^2\hat{O}|\Phi\rangle$ or $\langle\Phi'|\hat{O}^2|\Phi'\rangle\leqslant\langle\Phi|\hat{O}|^2\Phi\rangle+|S|^2$, we have, by using Eq. (23),

$$\|\hat{O}|\Phi'\rangle\|^2\leqslant\|\hat{O}|\Phi\rangle\|^2+2(\|\hat{O}|\Phi'\rangle\|^2+\|\hat{O}|\Phi\rangle\|^2)\||\Delta\Phi\rangle\|^2 \tag{28a}$$

or, equivalently,

$$\|\hat{O}|\Phi'\rangle\|^2\leqslant\|\hat{O}|\Phi\rangle\|^2\,\frac{1+2\||\Delta\Phi\rangle\|^2}{1-2\||\Delta\Phi\rangle\|^2}\quad(2\||\Delta\Phi\rangle\|^2<1). \tag{28b}$$

Combining Eqs. (23), (24), and (28b). we find the upper bound for R^2 in the case

$$\begin{aligned}R^2&=\frac{|S|^2}{\langle\Phi|\hat{O}^2|\Phi\rangle}\\ &\leqslant 2(1+\|\hat{O}|\Phi'\rangle\|^2\|\hat{O}|\Phi\rangle\|^2)\||\Delta\Phi\rangle\|^2\\ &\leqslant\frac{4\||\Delta\Phi\rangle\|^2}{1-2\||\Delta\Phi\rangle\|^2}.\end{aligned} \tag{29}$$

Therefore, the upper bound for the signal-to-noise ratio R in both cases is

$$R^2\leqslant\frac{4\||\Delta\Phi\rangle\|^2}{1-4\||\Delta\Phi\rangle\|^2}\quad(4\||\Delta\Phi\rangle\|^2<1). \tag{30}$$

So the size $\||\Delta\Phi\rangle\|^2$ of the change in the state is a good measure for the sensitivity in precision phase measurement. Furthermore, from Eq. (15), we have

$$R^2\leqslant\frac{8\langle N\rangle\delta}{1-8\langle N\rangle\delta}\quad(8\langle N\rangle\delta<1). \tag{31}$$

Hence, R can be larger than 1 only when $\delta\gtrsim O(1/\langle N\rangle)$ and we have proven that given the total mean number of photon $\langle N\rangle$, the minimum detectable phase shift is the Heisenberg limit.

D. Argument by the complementarity principle of quantum mechanics

All the previous proofs relied on some specific kind of measurement schemes even though they were general. For example, in the proof in Sec. II C, we made the assumption that there exists a Hermitian operator $\hat{O}$ for the corresponding quantum measurement process of the phase. As will be seen below in some cases the noise of the measurement is not simply the variances given in Eq. (24), and the criterion for detecting a phase shift is not the size of the signal-to-noise ratio. Therefore, all of them have some kind of limitations. In the following, we will run through another proof which is based on the complementarity principle of quantum

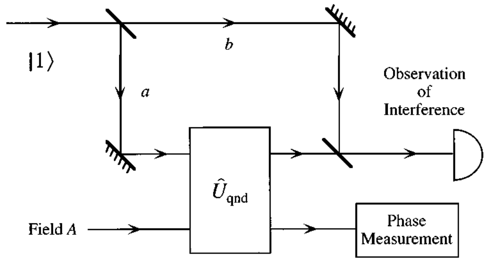

FIG. 3. Single-photon interferometer with a QND device in one of the arms for the which-path information.

mechanics. It is independent of the measurement scheme and thus is the most general proof so far.

The complementarity principle of quantum mechanics [15] concerns the particle and wave duality of light. Although light exhibits both wavelike and particlelike behaviors, it is impossible to observe both of them simultaneously. When we apply the complementarity principle to the phenomena of interference, we find that it is impossible to obtain the complete which-path information for the two possible interfering paths of a photon, and to observe in the meantime the interference effect in a single experiment. In other words, the interference effect will disappear if we know exactly from which one of the two possible interfering paths the photon approaches the detector, whereas the appearance of interference is always a manifestation of the intrinsic indistinguishability of the path of the photon. In more quantitative language, the mutual coherence and indistinguishability of the photon path are related in such a way that the degree of the interference effect (e.g., the visibility of the interference pattern) depends on the precision of our knowledge about which path the photon goes through [16]. The visibility will be zero if we know exactly which path the photon goes through, whereas no knowledge of the which-path information at all will give rise to 100% visibility in the interference pattern. If we have some partial information about which path the photon goes through, the visibility of interference will lie between 0 and 1. Furthermore, if, without disturbing the interference system, there exists a possibility, even in principle, for the distinction of two interfering paths, all interference is wiped out. Notice that it is not necessary to actually carry out an experiment for the distinction in order for the interference to disappear. The mere possibility that it can be performed is sufficient to suppress the interference effect. This supplement of the complementarity principle is the key in our next argument for the fundamental quantum limit in precision phase measurement.

Consider the single-photon interferometer shown in Fig. 3. In one of the interfering paths, we add in a device that makes a quantum nondemolition measurement (QND) of the photon number. Therefore, it is possible to obtain which-path information for the single photon without destroying it (no disturbance to the interference system). It is known [17,18] that the optical Kerr effect can be used to implement a QND measurement of the photon number. In this case, the measured photon imposes a phase shift on another beam called the probe beam. Measurement of the phase shift on the probe beam provides the information about the photon number, and will influence the interference pattern. Thus this interference system provides a platform for a discussion of the precision of the phase measurement in connection with the complementarity principle.

In the QND measurement of the photon number by the optical Kerr interaction, two fields a and A (one is called the signal while the other the probe) are coupled through a Kerr medium, and the state evolution is determined by the unitary operator [18]

$$\hat{\mathcal{U}}_{\mathrm{QND}}=e^{i\kappa\hat{a}^{\dagger}\hat{a}\hat{A}^{\dagger}\hat{A}}, \tag{32}$$

where κ is a parameter that depends on the strength of the interaction and is adjustable. To examine the physical meaning of κ further, let the input state to the QND device be a single-photon state for the signal field a and a general state $|\Phi\rangle$ given in Eq. (11) for the probe field A. Then the output state for the two fields is

$$\hat{\mathcal{U}}_{\mathrm{QND}}|1\rangle_a|\Phi\rangle_A=|1\rangle_a e^{i\kappa\hat{A}^{\dagger}\hat{A}}|\Phi\rangle_A=|1\rangle_a|\Phi'\rangle_A. \tag{33}$$

Thus, according to Eq. (12), the probe field A is subject to a phase shift κ imposed by the input of a single photon in the signal field a. Although this statement comes from the assumption that the probe field A is in a pure state of Eq. (11), it can easily be proved to be correct even for the mixed state described by Eq. (16) for the probe field.

Next, we perform some measurement of the probe field A to estimate the phase shift (Fig. 2). If we can detect the phase shift in field A with precision better than κ by whatever means, we will be able to tell whether a photon is in path a or not. Hence, if we use this device in one arm of the single-photon interferometer, we will know the which-path information, and, according to the complementarity principle, the interference effect will disappear. On the other hand, if we can observe a 100% visibility in the single-photon interferometer, it will be impossible to detect the phase shift κ in field A no matter what kind of method or strategy we use for the extraction of the phase shift. Therefore, the visibility of the interferometer is directly related to our ability to resolve the phase shift κ due to a single photon.

Let us now examine the visibility of the single-photon interferometer with the QND device in path a. For the system of Fig. 3, the visibility of the interferometer has already been derived by Sanders and Milburn [19]. To make the presentation self-contained, in the following we will derive the visibility along the lines of Ref. [19]. Assume that a single-photon state is fed into one of the input ports of the interferometer. To be more general, we assign a mixed state described by the density operator $\hat{\rho}_A$ in the form of Eq. (16) to the probe field A. Thus the input state for the total system is described by the density operator

$$\hat{\rho}_{\mathrm{tot}}=|1\rangle\langle 1|\otimes\hat{\rho}_A. \tag{34a}$$

After the first beam splitter, the state for the system becomes

$$\hat{\rho}'_{\mathrm{tot}}=|\psi\rangle\langle\psi|\otimes\hat{\rho}_A, \tag{34b}$$

with

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|1\rangle_a|0\rangle_b+|0\rangle_a|1\rangle_b).$$

After passing the QND device, the state of the system has the form of

$$\hat{\rho}''_{\rm tot}=\hat{\mathcal{U}}_{\rm QND}\hat{\rho}'_{\rm tot}\mathcal{U}^{\dagger}_{\rm QND}=e^{i\kappa\hat{a}^{\dagger}\hat{a}\hat{A}^{\dagger}\hat{A}}\hat{\rho}'_{\rm tot}e^{-i\kappa\hat{a}^{\dagger}\hat{a}\hat{A}^{\dagger}\hat{A}}$$
$$=\frac{1}{2}\,(|1_a,0_b\rangle\langle 1_a,0_b|e^{i\kappa\hat{A}^{\dagger}\hat{A}}\hat{\rho}_A e^{-i\kappa\hat{A}^{\dagger}\hat{A}}$$
$$+|0_a,1_b\rangle\langle 0_a,1_b|+|0_a,1_b\rangle\langle 1_a,0_b|\hat{\rho}_A e^{-i\kappa\hat{A}^{\dagger}\hat{A}}$$
$$+|1_a,0_b\rangle\langle 0_a,1_b|e^{i\kappa\hat{A}^{\dagger}\hat{A}}\hat{\rho}_A). \tag{34c}$$

From this state, we can calculate the probability of detecting a photon at one of the output ports of the interferometer: $P=\langle\hat{a}^{\dagger}_{\pm}\hat{a}_{\pm}\rangle$ with $\hat{a}_{\pm}=(\hat{a}\pm e^{i\phi}\hat{b})/\sqrt{2}$. It has the form of

$$P=\tfrac{1}{2}[1\pm v\ \cos(\phi-\epsilon)], \tag{35}$$

with the visibility

$$v=|{\rm Tr}(e^{i\kappa\hat{A}^{\dagger}\hat{A}}\hat{\rho}_A)|, \tag{36a}$$

and ϵ the phase of ${\rm Tr}(e^{i\kappa\hat{A}^{\dagger}\hat{A}}\hat{\rho}_A)$. Therefore the visibility of the interference pattern is

$$v=\left|\sum_m P_m e^{im\kappa}\right|, \tag{36b}$$

where Eq. (16) is used for $\hat{\rho}_A$. For the purpose of comparison with the unit visibility, let us calculate the quantity $1-v$ as follows:

$$1-v=1-\left|\sum_m P_m e^{im\kappa}\right|$$
$$\leqslant\left|1-\sum_m P_m e^{im\kappa}\right|$$
$$=2\left|\sum_m P_m e^{im\kappa/2}\sin m\kappa/2\right|$$
$$\leqslant 2\sum_m P_m|\sin m\kappa/2|. \tag{37}$$

By using the inequality $\sin x<x$ in expression (37), we end up with the following inequality:

$$1-v<\langle N\rangle\kappa \quad \text{or} \quad \langle N\rangle>\frac{1-v}{\kappa}, \tag{38a}$$

with $\langle N\rangle$ the average photon number in field A. This inequality sets a lower limit on the total mean number of photons required in field A in order to resolve the phase shift of κ in the phase measurement of field A. The argument runs as follows: When it is possible, by whatever means, to resolve the phase shift κ in field A, we can tell whether the photon entering the interferometer passes through path a or b. Since we know the which-path information, according to complementarity principle the interference effect in the interferometer will disappear or, equivalently, $v\sim 0$. Thus from Eq. (38a) we find that the total mean photon number in field A must satisfy $\langle N\rangle\gtrsim 1/\kappa$, which provides a lower bound on the photon number required in field A in order to resolve a phase shift κ. On the other hand, Eq. (38a) can also be written as

$$\kappa>\frac{1-v}{\langle N\rangle}, \tag{38b}$$

which sets a lower limit on the minimum detectable phase shift, given the total mean number of photons available in the field A. If a phase shift κ can be resolved by whatever means, as the previous argument shows, this will result in the disappearance of the interference pattern, or $v\sim 0$. From Eq. (38b), we have $\kappa\gtrsim 1/\langle N\rangle$. Thus the minimum detectable phase shift in field A is of the order of $1/\langle N\rangle$ or the Heisenberg limit.

The above argument is for the case when there is only a single mode in field A. With a multimode field probing the phase shift, the problem is equivalent to the multiple-phase measurement schemes that have attracted much attention lately [6–8,14]. The multiple measurement schemes divide the available energy into multiple parts, each sensing the same phase shift. Optimization of the measurement strategy can be performed based on quantum information theory for the estimation of the phase shift. A modification of the unitary operator for QND measurement can be made to include the multimode coupling. The modified unitary operator has the form

$$\hat{U}_{\rm QND}=\exp\left[i\kappa\hat{a}^{\dagger}\hat{a}\sum_j \hat{A}^{\dagger}_j\hat{A}_j\right], \tag{39}$$

where a multimode field A with modes characterized by the annihilation operators $\{\hat{A}_j\}$ is coupled to a single mode of $\hat{a}$ that is one arm of a single-photon interferometer. It can be easily checked that a single photon in field a will induce a phase shift κ in all the modes $\{\hat{A}_j\}$. Joint measurements on all the modes can be performed to estimate the phase shift. By following the same line of argument as in the single-mode case above, we can easily show that the precision in the joint phase measurement cannot be better that $\langle N_{\rm tot}\rangle^{-1}$, with $\langle N_{\rm tot}\rangle=\Sigma_j\langle\hat{A}^{\dagger}_j\hat{A}_j\rangle$ the total mean photon number in all the modes. Therefore, we have generalized the proof for the fundamental limit to the multimode case. Furthermore, we did not specify here how the energy is distributed among different modes. Thus the argument applies to the case of an uneven distribution of energy as well as to the case of equal partition of energy as in most recent investigations of multiple-phase measurement schemes [6–8,14].

Although the unitary operator in Eq. (39) is not practical in reality because the optical Kerr effect will produce different coupling constants κ for different modes due to dispersion, and will involve coupling between modes, Eq. (39) is used here purely for the sake of argument. It is allowed in quantum mechanics: in principle, with a proper arrangement of modes and a precise control of the coupling, it is possible to eliminate the cross terms and to have equal coupling strengths for all modes. In essence, the existence of Eq. (39) does not violate any law of quantum mechanics.

Notice that although the loss of interference ($v\sim 0$) relies on the ability to resolve the phase shift κ, it does not require actually performing the measurement of the phase shift. The passing of the probe field A is sufficient to wipe out the single-photon interference effect, provided that the probe field A is in such a state that in principle there exists a measurement scheme on the field A, by which we are able to resolve the phase shift κ.

Before we go to Sec. III, let us find the explicit form of visibility for some known states. It is straightforward to calculate the visibility for various states from Eq. (36b):

(i) For a coherent state $|\alpha\rangle$, $v=e^{-|\alpha|^2(1-\cos\kappa)}\approx e^{-\langle N\rangle\kappa^2/2}$ for $\kappa\ll1$ [19]. $v\sim0$ when $\langle N\rangle\gg1/\kappa^2$, which is consistent with the shot-noise limit of $1/\sqrt{\langle N\rangle}$ in phase measurement sensitivity for coherent state interferometry.

(ii) For the thermal state described by the density matrix $\hat{\rho}_{\rm th}=\Sigma_n P_n|n\rangle\langle n|$, with $P_n=\langle N\rangle^n/(\langle N\rangle+1)^{n+1}$,

$$v=\frac{1}{[1+4\langle N\rangle(\langle N\rangle+1)\sin^2\kappa/2]^{1/2}}$$
$$\simeq\frac{1}{[1+\langle N\rangle^2\kappa^2]^{1/2}}\quad\text{for }\langle N\rangle\gg1\quad\text{and}\quad\kappa\ll1.$$

Notice that v is significantly different from unity only when $\kappa\gtrsim1/\langle N\rangle$, which is consistent with the Heisenberg limit.

(iii) For the phase state in Eq. (1),

$$v=\lim_{s\to\infty}\frac{1}{s+1}\left|\frac{\sin(s+1)\kappa/2}{\sin\kappa/2}\right|=0\quad\text{for any}\quad\kappa\neq0,$$

which reflects the fact that it is possible to make a precise measurement of phase in this state no matter how small the phase shift is. With a finite s, on the other hand, we have $v=|\text{sinc}[(s+1)\kappa/2]/\text{sinc}(\kappa/2)|$, and v is different from 1 only when $\kappa\gtrsim2/s=1/\langle N\rangle_\theta$.

(iv) For a number state $|M\rangle$, $v=1$, and it is impossible to resolve a phase shift no matter how large $\langle N\rangle=M$ is. This reflects the random phase property in the photon number state.

(v) For the phase state of Eq. (5), $v\simeq1-6\kappa/\pi$ when $\langle N\rangle\gg1$ and $\kappa\ll1$. Therefore, for $\kappa\ll1$, $v\simeq1$, which means that the state in Eq. (5) is not suitable for the probe field A for sensing a small phase shift.

Superficially, we notice from example (ii) above that, for the thermal state, the disappearance of the interference pattern ($v\sim0$) is not necessarily related to the existence of a scheme of measurement on the state to resolve the phase shift κ, for from example (ii) we have $v\sim0$ when $\langle N\rangle\kappa\gg1$, but the phase-shifted thermal state $\rho'=\hat{U}\hat{\rho}\hat{U}^\dagger=\rho$ does not contain any information about the phase shift. This fact seems to contradict the complementary principle, which states that interference should always occur whenever there does not exist in principle a method to find the which-path information. On the other hand, we know that mixed states are a result of our lack of interest or ability to know other correlated fields (e.g., the reservoir fields for the thermal state). Once we enlarge the state space to bring in these correlated fields to make a pure state, the whole system will carry information about the phase shift. The question is then: Does there always exist a phase measurement scheme that can resolve the phase shift κ whenever this modified state is utilized in field A for sensing the phase shift, and which causes $v=0$ in the single-photon interferometer? We will address this question in Sec. IV when we discuss the general scheme of phase measurement.

III. A NECESSARY CONDITION FOR THE HEISENBERG LIMIT

It is known that squeezed state interferometry can achieve the Heisenberg limit [10]. Recently, some other schemes [11–13] were discovered that have the same sensitivity. However, it is not common for a phase measurement scheme to achieve the fundamental limit. For example, coherent state interferometry only reaches $1/\sqrt{\langle N\rangle}$ sensitivity. As we proved in Sec. II A, if classical sources are used in the conventional interferometer shown in Fig. 1, the sensitivity is always limited by $1/\sqrt{\langle N\rangle}$, or the coherent state limit. To achieve the Heisenberg limit, nonclassical sources must be used. What are the general requirements for the optical fields which can achieve the Heisenberg limit when they are employed in a phase measurement scheme?

Let us now consider those states which have relatively small photon number fluctuations, so that

$$\langle\Delta^2N\rangle\ll\langle N\rangle^2\quad\text{for large }\langle N\rangle.\tag{40a}$$

We will use these states in the probe field A in the single-photon interferometer with a QND measurement device having a coupling constant

$$\kappa\sim\frac{1}{\langle N\rangle},\tag{40b}$$

which is also the phase shift in field A induced by a single photon in field a. Assume further that photon distribution P_m for these states is smooth, so that $P_m\sim0$ for those m with $|m-\langle N\rangle|>\sqrt{\langle\Delta^2N\rangle}$. Then the contribution to the sum in the visibility formula in Eq. (36b) only comes from those terms with $|m-\langle N\rangle|\lesssim\sqrt{\langle\Delta^2N\rangle}$, and we can approximate Eq. (36b) as

$$v\approx\left|\sum_{|m-\langle N\rangle|\lesssim\sqrt{\langle\Delta^2N\rangle}}P_me^{im\kappa}\right|.\tag{41}$$

For $|m-\langle N\rangle|\lesssim\sqrt{\langle\Delta^2N\rangle}$, because $\kappa\sqrt{\langle\Delta^2N\rangle}\ll1$ as derived from Eq. (40), we can approximate $e^{im\kappa}$ with $e^{i\langle N\rangle\kappa}$, and Eq. (41) becomes

$$v\approx\left|\sum_{|m-\langle N\rangle|\lesssim\sqrt{\langle\Delta^2N\rangle}}P_me^{i\langle N\rangle\kappa}\right|\approx\left|e^{i\langle N\rangle\kappa}\sum_m P_m\right|=1.\tag{42}$$

Therefore, with states satisfying Eq. (40a) in field A and a phase shift of size $\kappa\sim1/\langle N\rangle$, we can observe an interference pattern with 100% visibility, indicating that it is impossible to resolve the phase shift of size κ no matter what we do on field A. Thus in order to obtain the sensitivity set by the Heisenberg limit in the phase measurement, we must utilize states satisfying

$$\langle\Delta^2N\rangle\gtrsim\langle N\rangle^2\tag{43}$$

for sensing the phase shift. Notice that the condition in Eq. (43) is only a necessary condition. It can be easily checked that the phase measurement schemes that have been discovered so far to achieve Heisenberg limit utilize states satisfying the condition in Eq. (43).

IV. GENERAL CONSIDERATION IN THE SEARCH FOR SCHEMES REACHING THE FUNDAMENTAL LIMIT

From the necessary condition derived in Sec. III, we find that, in a search for phase measurement schemes that have a sensitivity reaching the fundamental limit, we must first look for those states that satisfy the necessary conditions in Eq. (43). Then we need to construct a scheme which employs these states in field A for sensing the phase shift. So far, there have been a number of schemes that reach the fundamental limit in phase measurement [10–13]. Among them, some are conventional interferometers with different detection methods [10,12], while other utilize unconventional interferometers which do not use beam splitters as their wave dividers [11,13]. In the following, we will not limit our discussion to a particular type of interferometer.

In order to detect the phase shift, we will make measurement on field A. However, direct photodetection does not reveal any information about the phase of the field. Therefore, we first need to transform the state of the field A into some other state for which photodetection is sensitive to the phase (e.g., homodyne). Let the state of field A be $|\Phi\rangle$ or $|\Phi'\rangle$ with or without the phase shift. Consider a unitary operator $\hat{U}$ which operates on the state $|\Phi\rangle$ or $|\Phi'\rangle$, and results in the state

$$|\Psi\rangle=\hat{U}|\Phi\rangle \quad \text{or} \quad |\Psi'\rangle=\hat{U}|\Phi'\rangle, \tag{44}$$

which will be phase sensitive; that is, detection on $|\Psi'\rangle$ will result in a significantly different outcome from $|\Psi\rangle$. Our goal now is to detect the difference between $|\Psi\rangle$ and $|\Psi'\rangle$. This can be easily achieved if we select those states for $|\Psi\rangle$ such that detection on it will yield null result, whereas detection on $|\Psi'\rangle$ gives a nonzero result. One such state that can serve as $|\Psi\rangle$ is simply the vacuum state. Thus any detection of a photon in state $|\Psi'\rangle$ will be an indication of a phase shift.

Furthermore, phase is a relative quantity. We often need a reference in order to find the change in phase. Therefore, we will bring in another field called B as the reference (e.g., the field in the other arm of the conventional optical interferometers with beam splitters or the local oscillator in homodyne detection). After we find the state (or mixed state) $|\Phi\rangle$ satisfying the condition in Eq. (43), it is useful to enlarge the state space to include the field B, so that fields A and B are correlated. Therefore, the state for the total system has the general form

$$|\Phi\rangle_{\rm tot}=\sum_{m,n} c_{mn}|m\rangle_A|n\rangle_B \tag{45}$$

in the Fock state basis. The unitary operator $\hat{U}$ acts on the enlarged state space of two modes. Normally, the output will also consists of two modes. Detection can then be performed on two modes, and comparison is made between the two modes for the extraction of the phase shift.

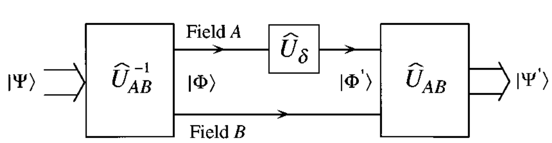

FIG. 4. General scheme of the phase measurement.

If the state $|\Psi\rangle$ is easily available, as in the case of vacuum state, we can generate the special state $|\Phi\rangle$ as the phase-sensing state by the inverse process of $\hat{U}$. Then we form a general type of interferometer, as shown in Fig. 4. Notice that $\hat{U}$ is a general type of unitary operator that satisfies our requirement for producing a unique state $|\Psi\rangle$ from $|\Phi\rangle$. Thus we have generalized our discussion to a broad class of unconventional interferometers.

Before examining specific examples, let us answer the question raised at the end of Sec. II in connection with the complementarity principle. We will look for a scheme for phase measurement. Consider first the case with a pure state in the general form of Eq. (11) for the field A. For any state $|\Phi\rangle$ with a nonzero norm in a Hilbert space, it is possible to find a unitary transformation $\hat{U}_\Phi$ so that

$$\hat{U}_\Phi|\Phi\rangle=|0\rangle, \tag{46}$$

where $|0\rangle$ is the vacuum state and contains no photon. So it can serve as the state $|\Psi\rangle$ with a special feature for distinction. Thus the interferometer has the form shown in Fig. 5. Notice that only single-mode field is used in the interferometer.

Obviously, from Eqs. (11) and (46) we have

$$c_m=\langle m|\hat{U}_\Phi^\dagger|0\rangle=\langle 0|\hat{U}_\Phi|m\rangle^*. \tag{47}$$

With a phase shift on the state $|\Phi\rangle$, we find that the output state becomes

$$|\Psi'\rangle=\hat{U}_\Phi|\Phi'\rangle=\hat{U}_\Phi e^{i\hat{n}\delta}\hat{U}_\Phi^\dagger|0\rangle, \tag{48}$$

where Eq. (46) is used. With no phase shift, the output field is simply in $|0\rangle$ and has no photon, but, with a nonzero phase shift, the output state is no longer the vacuum state and will contain photons. Thus detection of any photon in the output field is an indication of a nonzero phase shift. A better measure for this will be the probability $\overline{P}$ of detecting any photon in the output. Obviously, we have $\overline{P}=1-P_0$, with P_0 being the probability of no photon. With the output state in Eq. (48), we find

$$P_0=|\langle 0|\Psi'\rangle|^2=|\langle 0|\hat{U}_\Phi e^{i\hat{n}\delta}\hat{U}_\Phi^\dagger|0\rangle|^2=\left|\sum_m |c_m|^2 e^{im\delta}\right|^2, \tag{49a}$$

where we used the closure relation $\Sigma_m|m\rangle\langle m|=1$, and Eq. (47) in the last equality. Therefore, for the probability of detecting any photon we have

$$\overline{P}=1-P_0=1-\left|\sum_m P_m e^{im\delta}\right|^2=1-v^2, \tag{49b}$$

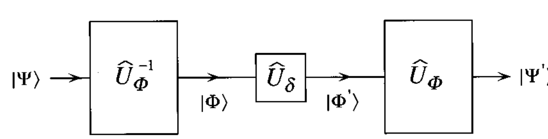

FIG. 5. General scheme of the phase measurement with a single-mode field.

where we used Eq. (36b) with $\kappa=\delta$ for the visibility v of the single-photon interferometer. Therefore, if we use this scheme for detecting a phase shift in the single-photon interferometer discussed in Sec. II D, we find that, whenever $v=0$, $P=1$, indicating that we are able to detect the phase shift of δ. Thus we have shown that whenever the interference disappears ($v=0$), we will have at least in principle a method of knowing whether the single photon passes the path a or not with 100% probability. A good example for the pure state is the phase state in Eq. (1) with finite s. As a matter of fact, such a scheme achieves the Heisenberg limit. From Eq. (49b), we see that the quantity v as expressed in Eq. (36) is a good measure in the search of optimum phase measurement schemes.

Next let us consider a more general case with a mixed state of

$$\hat{\rho}_A=\sum_{m,n}\rho_{mn}|m\rangle\langle n|. \tag{50a}$$

As we discussed at the end of Sec. II, let us assume that, when we enlarge the state space, we are able to obtain a pure state of the form

$$|\Phi\rangle_{AB}=\sum_{m,\lambda}c_m(\lambda)|m\rangle_A|\lambda\rangle_B, \tag{50b}$$

which, after tracing over fields B, will reproduce the mixed state in Eq. (50a). The states $\{|\lambda\rangle_B\}$ characterize the other states in fields B that are correlated with field A. It is always possible to make the states $\{|\lambda\rangle_B\}$ a set of orthonormal states with $\langle\lambda'|\lambda''\rangle=\delta_{\lambda'\lambda''}$. Therefore, after tracing over field B and comparing with Eq. (50a), we have

$$\rho_{mn}=\sum_{\lambda}c_m(\lambda)c_n^*(\lambda).$$

Consider now the vacuum state $|0\rangle_A|0\rangle_B$ for all the relevant modes in fields A and B. As before, it is possible to find a unitary operator $\hat{U}_{AB}$ so that $\hat{U}_{AB}|\Phi\rangle_{AB}=|0\rangle_A|0\rangle_B$. We can then run through the same argument as the case of a pure state for field A. The only thing different here is that the criterion for finding the phase shift is the detection of any photon in any mode of fields A and B. Therefore, we have proved that if we can write the state of the system in the form of a pure state after enlarging the state space, it is always possible to find a measurement scheme to resolve the phase shift due to a single photon whenever the visibility of the single-photon interferometer is zero.

V. SOME EXAMPLES OF PHASE MEASUREMENT SCHEMES WITH THE HEISENBERG LIMIT

Among the states that are available in laboratories, only thermal and squeezed states have a variance of photon number on the order of the square of the mean, thus satisfying the condition in Eq. (43). More specifically, the variance is $\langle\Delta^2N\rangle=2\langle N\rangle(1+\langle N\rangle)$ for the squeezed states of the total mean photon number of $\langle N\rangle$ and $\langle\Delta^2N\rangle=\langle N\rangle(1+\langle N\rangle)$ for the thermal states.

Let us first consider squeezed states. There are a number of ways to utilize squeezed states to form interferometers. As a matter of fact, the first interferometer that beats the coherent state limit for sensitivity in precision phase measurement employs a squeezed vacuum state in the unused input port of a conventional interferometer [20]. It has been proved further that by utilizing coherent squeezed states, one can achieve the sensitivity set by the Heisenberg limit [10]. On the other hand, a squeezed vacuum state is known to be phase sensitive; thus we can form an interferometer directly with squeezed vacuum states without the need for a nonzero coherent component.

It is known [21] that two single-mode fields in squeezed vacuum states with the same squeezing parameter, when combined with a beam splitter, can produce the so-called twin-photon state with a zero-photon-number difference in the two output fields. The zero-photon-number difference is a special feature that can be used to distinguish $|\Psi'\rangle$ from $|\Psi\rangle$. Therefore, the system for the evolution from $|\Phi\rangle$ to $|\Psi\rangle$ will be a normal lossless beam splitter. $|\Phi\rangle$ will be the squeezed state for both fields A and B, and $|\Psi\rangle$ will be the twin photon state. The actual interferometer is the same one shown in Fig. 4 where the unitary operator $\hat{U}$ is simply that for a lossless beam splitter, and the two input ports $\hat{a}_0$ and $\hat{b}_0$ are in a twin-photon state described by

$$|\Psi\rangle=\hat{S}(\mu,\nu)|\text{vac}\rangle \tag{51}$$

with

$$\hat{S}(\mu,\nu)=\exp[i\kappa(\hat{a}_0\hat{b}_0-\text{H.c.})]\quad(\mu=\cosh\kappa,\nu=\sinh\kappa).$$

It is easy to find that $\langle(\hat{a}_0^\dagger\hat{a}_0-\hat{b}_0^\dagger\hat{b}_0)^m\rangle=0$ for any nonzero integer m. The input-output relation for the interferometer with a phase shift of δ is given as

$$\hat{a}=\hat{a}_0\cos\frac{\delta}{2}+\hat{b}_0\sin\frac{\delta}{2},\quad \hat{b}=-\hat{a}_0\sin\frac{\delta}{2}+\hat{b}_0\cos\frac{\delta}{2}, \tag{52}$$

and the photon number difference at the output $\Delta N\equiv\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}$ has the form

$$\Delta N=(\hat{a}_0^\dagger\hat{a}_0-\hat{b}_0^\dagger\hat{b}_0)\cos\delta+(\hat{a}_0^\dagger\hat{b}_0+\hat{b}_0^\dagger\hat{a}_0)\sin\delta. \tag{53}$$

It is easily found that $\langle\Delta N\rangle_{\text{out}}=0$ and $\langle\Delta^2N\rangle_{\text{out}}=4\mu^2\nu^2\sin^2\delta\simeq4\langle N\rangle^2\sin^2\delta(\langle N\rangle\gg1)$, where $\langle N\rangle=|\nu|^2$ is the total mean number of photons in one arm of the interferometer. In this case, since $\langle\Delta N\rangle_{\text{out}}=0$ for any value of δ, we cannot use it as the signal for the detection of the phase shift δ. However, $\langle\Delta^2N\rangle_{\text{out}}\neq0$ for nonzero δ, so it can be used as the signal. But the noise is not simply $\sqrt{\langle(\Delta^2N)^2\rangle}$. In fact, when $\delta=0$, we have $\langle(\Delta N)^m\rangle=0$ for any nonzero integer m, so that any detection of nonzero ΔN is an indication of $\delta\neq0$. However ΔN is quantized and only takes $0,\pm1,\pm2,\ldots$. Thus the noise of ΔN is simply 1. Therefore, $\text{SNR}=\langle\Delta^2N\rangle\simeq4\langle N\rangle^2\delta^2$ for $\langle N\rangle\gg1$ and $\delta\ll1$, and the minimum detectable phase shift is $\delta_{\min}\sim1/2\langle N\rangle$, which is the Heisenberg limit. To further demonstrate this, let us examine the probability P of detecting $\Delta N\neq0$. Obviously, $P=1-P_0$, with P_0 the probability of detecting $\Delta N=0$. We find P_0 by first calculating the characteristic function for ΔN,

$$C(r)\equiv\langle e^{ir\Delta N}\rangle\simeq1/(1+4\langle N\rangle^2\delta^2\sin^2r)^{1/2}\quad(N_{\text{tot}}\gg1,\delta\ll1), \tag{54}$$

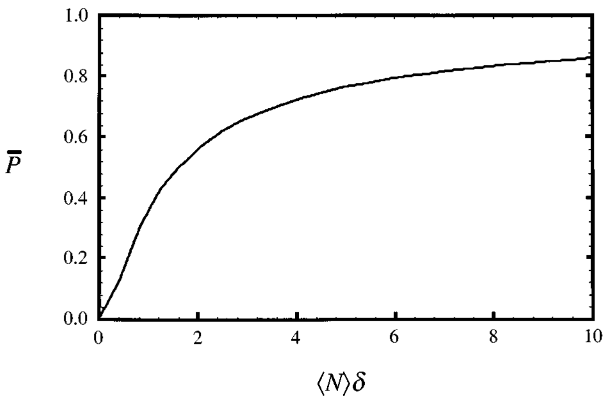

FIG. 6. Probability $\overline{P}$ of detecting any nonzero ΔN as a function of $\langle N\rangle\delta$ for a twin-photon interferometer.

and then making a finite Fourier transformation

$$P_m=\frac{2}{\pi}\int_0^{\pi/2}dr\,\frac{\cos 2mr}{(1+4\langle N\rangle^2\delta^2\sin^2 r)^{1/2}} \tag{55}$$

Thus $P_0=2K(-4\langle N\rangle^2\delta^2)/\pi$, with $K(x)$ the complete elliptic function of the first kind. Figure 6 shows P as a function of $\langle N\rangle\delta$. It is obvious that $P\simeq 0$ when $\langle N\rangle\delta\ll 1$, and starts to rise when $\langle N\rangle\delta\sim 1$, which is an indication of a nonzero phase shift δ. Thus we find in this figure that the minimum detectable phase shift is of the order of $1/\langle N\rangle$, and the interferometer is operated at the Heisenberg limit. This scheme is very similar to the twin Fock state interferometry proposed by Holland and Burnett [12] except that we used the twin-photon state in Eq. (51) as the input to the interferometer. The same scheme was also treated by Sanders and Milburn [22] along the line of phase estimation for interferometry.

Another way to utilize the squeezed state is to follow the general scheme discussed at the end of Sec. IV, and to form a single-mode interferometer. It is known that a squeezed vacuum state can be generated from vacuum by applying the so-called squeezing operator

$$\hat{S}(\xi)=\exp\left[\frac{i\xi}{2}(\hat{a}^2-\text{H.c.})\right]. \tag{56}$$

Then, from Eq. (46), we have $\hat{U}_\Phi=\hat{S}^{-1}(\xi)$. Thus the single-mode squeezed vacuum state interferometer has the same form shown in Fig. 5. Without the phase shift, the output state of the system is simply $|\Psi\rangle_{\text{out}}=\hat{S}^{-1}\hat{S}|\text{vac}\rangle=|\text{vac}\rangle$. With a phase shift δ on the state $|\Phi\rangle=\hat{S}|\text{vac}\rangle$, the output state becomes

$$|\Psi'\rangle_{\text{out}}=\hat{S}^{-1}e^{i\hat{a}^\dagger\hat{a}\delta}\hat{S}|\text{vac}\rangle, \tag{57}$$

and the average photon at the output of the interferometer is then

$$\langle\Psi'|\hat{a}^\dagger\hat{a}|\Psi'\rangle_{\text{out}}=4\langle N\rangle(1+\langle N\rangle)\sin^2\delta, \tag{58}$$

where $\langle N\rangle=\sinh^2\xi$ is the average photon number in the state $|\Phi\rangle$. As before, the noise in the output is simply one photon. Thus we have the signal-to-noise ratio (R_{SNR}) as (SNR, R_{SNR})

$$R_{\text{SNR}}=4\langle N\rangle(1+\langle N\rangle)\sin^2\delta\approx 4\langle N\rangle^2\delta^2\quad(\langle N\rangle\gg 1,\delta\ll 1). \tag{59}$$

Therefore, the signal-to-noise ratio (SNR) is significant only when $\delta\sim 1/\langle N\rangle$. Since the output of the interferometer is vacuum when there is no phase shift, and detection of a single photon will indicate a nonzero phase shift, a better quantity to characterize the sensitivity of the interferometer is the probability P of finding any photon in the output. Obviously $P=1-P_0$, with P_0 being the probability of finding no photon in the output. P_0 can be calculated as

$$\begin{aligned}P_0&=\langle\Psi'|{:}e^{-\hat{a}^\dagger\hat{a}}{:}|\Psi'\rangle_{\text{out}}\\&=\frac{1}{\sqrt{1+4\langle N\rangle(1+\langle N\rangle)\sin^2\delta}}\\&\approx\frac{1}{\sqrt{1+4\langle N\rangle^2\delta^2}},\quad\text{for }\langle N\rangle\gg 1,\quad\delta\ll 1.\end{aligned} \tag{60}$$

This expression can also be derived from Eq. (49b) through the quantity v for the squeezed state [19]. Thus P is significantly different from zero when $\delta\gtrsim 1/\langle N\rangle$, and the interferometer is operated at the Heisenberg limit. This scheme was also discussed by Yurke, McCall, and Klaude [11], who used the signal-to-noise ratio for the sensitivity.

As for the thermal state, it is impossible to implement an interferometer based on such a state alone due to its lack of phase coherence. However, as we discussed above, we can always enlarge the state space to include the field B to form a pure state $|\Phi\rangle_{AB}$, so that we may find coherence between fields A and B. It is known [23] that a nondegenerate parametric down-conversion process produces two correlated fields each having thermal statistics, or, in other words, $\langle\Delta^2N\rangle_{A,B}=\langle N\rangle(\langle N\rangle+1)$. Such a state is actually the twin-photon state or two-mode squeezed state described by Eq. (51), from which we find that it can be generated from vacuum by applying the operator $\hat{S}(\mu,\nu)$. Now let us reverse the process by assigning the two down-converted fields as fields A and B, and set $|\Phi\rangle_{AB}=\hat{S}(\mu,\nu)|\text{vac}\rangle$ and $|\Psi\rangle=|\text{vac}\rangle_{AB}$. Then the unitary operator required in Eq. (44) is $\hat{U}_{AB}=\hat{S}^{-1}(\mu,\nu)$ which can also be realized in a parametric down-conversion process. The interferometer for the whole system then has the same form as the two-mode interferometer shown in Fig. 4. In this case, $|\Psi\rangle_{AB}=\hat{S}^{-1}(\mu,\nu)|\Phi\rangle_{AB}=|\text{vac}\rangle_{AB}$ when the phase shift is zero. But when we have a nonzero phase shift δ, the output state becomes

$$|\Psi'\rangle_{AB}=\hat{S}^{-1}(\mu,\nu)e^{i\delta\hat{A}^\dagger\hat{A}}\hat{S}(\mu,\nu)|\text{vac}\rangle_{AB}. \tag{61}$$

Obviously, the state $|\Psi'\rangle_{AB}$ has a nonzero average photon number. By measuring the photon number in one of the outputs of the interferometer, we can detect the phase shift δ.

To find the average photon number in the output of the interferometer, we find it easier to consider the evolution of the operators than that of the states. Referring to Fig. 4, the fields A and B are related to the input vacuum fields a_0 and b_0 as

$$\hat{A}=\mu\hat{a}_0+\nu\hat{b}_0^\dagger,\quad\hat{B}=\mu\hat{b}_0+\nu\hat{a}_0^\dagger. \tag{62a}$$

Assume that field A experiences a phase shift of δ. Then the output fields a and b of the interferometer are related to fields A and B as

$$\hat{a}=\mu\hat{A}e^{i\delta}-\nu\hat{B}^{\dagger},\quad \hat{b}=\mu\hat{B}-\nu\hat{A}^{\dagger}e^{-i\delta}, \tag{62b}$$

Hence

$$\hat{a}=\left(1+2ie^{i\delta/2}\mu^2\sin\frac{\delta}{2}\right)\hat{a}_0+2ie^{i\delta/2}\mu\nu\sin\frac{\delta}{2}\hat{b}_0^{\dagger}$$
$$\equiv G\hat{a}_0+g\hat{b}_0^{\dagger}, \tag{63}$$

with $G\equiv 1+2ie^{i\delta/2}\mu^2\sin\delta/2$ and $g\equiv 2ie^{i\delta/2}\mu\nu\sin\delta/2$. The photon number operator at output field a is then

$$\hat{n}_a=|G|^2\hat{a}_0^{\dagger}\hat{a}_0+|g|^2\hat{b}_0\hat{b}_0^{\dagger}+Gg^*\hat{a}_0\hat{b}_0+G^*g\hat{a}_0^{\dagger}\hat{b}_0^{\dagger}. \tag{64}$$

It is easily found that $\langle\hat{n}_a\rangle=|g|^2\simeq 4\langle N\rangle^2\sin^2\delta/2$ for $(\langle N\rangle\gg 1)$, with $\langle N\rangle\equiv|\nu|^2$ being the average photon number in field A. Similar to the squeezed state interferometers discussed above, the noise Δn_a is one photon so that $R_{\rm SNR}=4\langle N\rangle^2\sin^2\delta/2$. Therefore, $R_{\rm SNR}\sim 1$ only when $\delta\sim 1/\langle N\rangle$. Furthermore, similar to the squeezed state interferometers, we find the probability of detecting any photon in the output field a as

$$\overline{P}=1-P_0=1-\langle :e^{-\hat{n}_a}:\rangle$$
$$=\frac{4\langle N\rangle(\langle N\rangle+1)\sin^2\delta/2}{1+4\langle N\rangle(\langle N\rangle+1)\sin^2\delta/2}$$
$$\simeq\frac{\langle N\rangle^2\delta^2}{1+\langle N\rangle^2\delta^2}\quad\text{for }\langle N\rangle\gg 1,\quad \delta\ll 1. \tag{65}$$

Therefore, the probability of detecting a phase shift of size δ is significantly different from zero only if $\delta\gtrsim 1/\langle N\rangle$. Thus the minimum detectable phase shift is $1/\langle N\rangle$, or the Heisenberg limit. Such a scheme was also discussed in Ref. [11] along the line of the SU (1,1) interferometer. Notice that the criterion here for detecting a phase shift is simply the detection of any photon in field a alone without considering field b. This is quite different from the criterion discussed in Sec. III for the general case of mixed states. The general criterion is the detection of any photon in any relevant modes which will include both modes a and b. The reason for the difference is that the output fields a and b are actually in a twin-photon state as Eq. (63) indicates, and the photon numbers of modes a and b are perfectly correlated in such a way that the two modes have exactly the same photon number all the time. Thus detection of a photon in mode a alone is equivalent to the detection of a photon in any of the two fields.

As another example, let us consider the state

$$|\Phi\rangle_M=\frac{1}{\sqrt{2}}(|M\rangle_A|0\rangle_B+|M\rangle_B|0\rangle_A). \tag{66}$$

Obviously, $\langle\Delta^2N\rangle_A=M^2/4=\langle N\rangle_A^2$, thus it also satisfies the necessary condition in Eq. (43). However, it is not so easy to find an evolution process to produce a distinctive state $|\Psi\rangle$ as discussed in Sec. IV. Fortunately, Ref. [13] provides some clues on how to construct an effective beam splitter of the M photon as one entity. Consider the evolution operator

$$\hat{U}_M=\exp\left(\frac{\pi}{4M!}[(\hat{A}\hat{B}^{\dagger})^M-\text{H.c.}]\right), \tag{67}$$

which comes from the Hamiltonian given by

$$\hat{H}_I=i\hbar\xi[(\hat{A}\hat{B}^{\dagger})^M-\text{H.c.}]. \tag{68}$$

It is easy to check that

$$\hat{U}_M|0\rangle_A|M\rangle_B=\frac{1}{\sqrt{2}}(|0\rangle_A|M\rangle_B-|M\rangle_A|0\rangle_B),$$
$$\hat{U}_M|M\rangle_A|0\rangle_B=\frac{1}{\sqrt{2}}(|0\rangle_A|M\rangle_B+|M\rangle_A|0\rangle_B), \tag{69a}$$

so that

$$\hat{U}_M|\Phi\rangle_M=|0\rangle_B|M\rangle_B\equiv|\Psi\rangle_M. \tag{69b}$$

With a phase shift of δ in field A, the state $|\Phi'\rangle$ becomes

$$|\Phi'\rangle=e^{i\delta\hat{A}^{\dagger}\hat{A}}|\Phi\rangle=\frac{1}{\sqrt{2}}(|0\rangle_A|M\rangle_B+e^{iM\delta}|M\rangle_A|0\rangle_B) \tag{70a}$$

and the output state then has the form of

$$|\Psi'\rangle_M=\hat{U}_M|\Phi'\rangle=e^{iM\delta/2}\left(\cos\frac{M\delta}{2}|0\rangle_A|M\rangle_B\right.$$
$$\left.+i\sin\frac{M\delta}{2}|M\rangle_A|0\rangle_B\right). \tag{70b}$$

Thus if we measure the photon number at the output port A, any detection of the photon will indicate a phase shift. The probability of detecting any photon in the output port A is $\overline{P}=\sin^2 M\delta/2$, which is significantly different from zero only when $\delta\sim 1/(M/2)=1/\langle N\rangle_A$. Therefore, such a scheme reaches the Heisenberg limit.

The criterion here for the detection of a phase shift is once again different from the general one in Sec. III, because of the form of the unitary operator in Eq. (67). It does not annihilate all the photons to produce a vacuum state, as required for the general scheme, but rather preserves the total photon number. Because of the unusual form of the unitary operator (which depends on the total input photon number M), this scheme is not likely to be practical as compared to the other schemes discussed earlier with single-mode or two-mode squeezed states. It is used here as another example of unconventional interferometers which can achieve the Heisenberg limit.

VI. SUMMARY AND DISCUSSION

In this paper we proved, through a number of arguments, that, with a finite number $\langle N\rangle$ of photons for probing a phase shift, the minimum detectable phase is of the order of $1/\langle N\rangle$, or the Heisenberg limit. In particular, the argument based on the complementarity principle is independent of the schemes of phase measurement, and thus provides true proof of the fundamental quantum limit on the sensitivity of phase measurement. Furthermore, since we do not involve a phase operator in the argument, we manage to avoid some of the difficulties associated with the phase operator. We have also derived, through the argument of the complementarity principle, a necessary condition for those states that, when utilized for sensing a phase shift, can achieve the sensitivity of the Heisenberg limit. With the states satisfying the necessary condition, we have outlined a general guideline for the search for a specific measurement scheme of phase with the sensitivity at the Heisenberg limit.

There is still one question unanswered in this paper. This question is actually the one raised at the end of Sec. II: Does there always exist, even in principle, a measurement scheme that is able to resolve the phase shift induced by a single photon in the single-photon interferometer with a QND measurement device whenever the visibility of the interferometer is zero? Although we answered this question at the end of Sec. III, we made an assumption that any mixed state can be written as a pure state of light if the state space is enlarged. However, we find very often that a mixed state is the result of correlation not only with other optical fields but also with other degrees of freedom of a larger system that do not correspond to the modes of optical fields. For example, spontaneous emission produces an optical field in a thermal state which, when the system is enlarged to include atoms, can be thought of as an optical field correlated with atomic states. For a more general case like this one, our analysis fails. On the other hand, if one believes the principle of complementarity, the answer to the question should be ‘‘yes.’’ Otherwise, the interference effect would occur and the visibility would not be zero.

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