PHYSICAL REVIEW A VOLUME 55, NUMBER 4 APRIL 1997

Isospectral deformation of quantum potentials and the Liouville equation

V. M. Eleonsky and V. G. Korolev
Lukin Research Institute of Physical Problems, Zelenograd, Moscow 103460, Russia

(Received 18 October 1996)

A quantum problem on an isospectral deformation of one-dimensional potentials (and of corresponding wave functions) is considered. The isospectral deformation defined in the form of a phase flow is shown to obey a system of coupled Liouville equations. In a simple case of an individual flow the well-known integrable Liouville equation arises; its solution provides known families of isospectral potentials. Operators performing this deformation are studied; their unitary property is proved. An evolution of spectral shift operators is determined using those unitary operators. An asymptotical behavior of both a potential and wave functions under this isospectral deformation is studied. It is shown, in particular, that the deformation of the Rosen-Morse potential and that of the harmonic oscillator's potential have common analytical properties. The approach used in the paper can be extended to the case of a deformation leading to a shift of one selected energy level. In the case of the simplest individual flow we get a generalization of the integrable Liouville equation. [S1050-2947(97)03303-9]

PACS number(s): 03.65.Ge

I. INTRODUCTION

Various approaches have been used to investigate deformation of one-dimensional potential and spectra of the corresponding Schrödinger problem. Among them are the technique based on integral equations of Gelfand-Levitan and Marchenko's type [1–4], the factorization (supersymmetry) method [5,6] (involving, in particular, a generalization of the creation and/or annihilation operators for the harmonic oscillator [7]), and various extensions of the Darboux transformation [8,9].

Nevertheless, *evolutional equations* (PDEs) governing the isospectral deformation and, in particular cases, providing known families of potentials have not yet been derived. This problem is resolved in the present paper.

For this purpose we associate the isospectral deformation with a phase flow [10,8,11] defined by a certain energy functional. As a result, we reveal that the isospectral deformation obeys *a system of coupled Liouville equations*. In a simple case of an individual flow the classical Liouville equation [12] arises; its integration immediately leads to familiar results [5–9]. On the other hand, the Liouville equation itself is found to be an integral of a much more complicated evolutional equation obtained in [12] for a description of isospectral potentials. However, if we do not restrict ourselves to the individual flow, we obtain a system of equations whose solutions may give *new* families of isospectral one-dimensional potentials.

Furthermore, we show that the same isospectral deformation can be described as a *unitary transformation* performed by linear integral operators. Those operators allow us to determine an evolution of spectrum shift operators.

It is also shown in the present paper that for a wide class of isospectral potentials the evolution ends with a local minimum splitting off and moving off to infinity; this local well is asymptotically close to a simple soliton potential containing a unique bound state.

As examples, in this paper we consider the isospectral deformation of the Rosen-Morse (''soliton'') potential and of the harmonic oscillator's potential. Previously, soliton potentials related to the Liouville equation were studied by Andreev [13] using a version of the inverse scattering method. Finally, we demonstrate that the presented approach can be easily extended to a description of potential deformations accompanied by a shift of a selected energy level.

The paper is arranged as follows. The general description of the isospectral deformation using phase flows is presented in Sec. II together with a complete analysis of the individual flow; an interpretation of the obtained results in terms of the double Darboux transformation is discussed. In Sec. III we study unitary operators performing an isospectral deformation; an evolution of spectrum shift operators is investigated. In Sec. IV asymptotical properties of the isospectral deformation of both a potential and of wave functions are studied; the results are illustrated by examples of the Rosen-Morse potential and of the harmonic oscillator. In the last section we derive evolution equations governing a deformation accompanied by a shift of a selected energy level.

II. ISOSPECTRAL FLOW AND EVOLUTIONAL EQUATIONS FOR POTENTIAL AND WAVE FUNCTIONS

A. General case

Consider the Schrödinger problem with the point spectrum

$$-\frac{1}{2}\psi_{xx}+U(x,t)\psi=E\psi,\qquad \lim_{x\to\pm\infty}\psi=0,\qquad x\in\mathcal{R}^1,\tag{2.1}$$

where the potential $U(x,t)$ depends on the parameter t. Following [10,11,8], introduce a phase flow as follows:

$$U_t=\frac{\partial}{\partial x}\sum_k \tau_k\frac{\delta E_k}{\delta U}=\frac{\partial}{\partial x}\sum_k \tau_k\psi_k^2.\tag{2.2}$$

Here $(\psi_k(x,t),E_k(t),k\geqslant 0)$ are eigenelements of the Schrödinger problem (2.1), $\delta E_k/\delta U$ is a variational derivative of the functional

$$E_k=\int_{-\infty}^{+\infty}dx\left\{\frac{1}{2}[(\psi_k)_x]^2+U\psi_k^2\right\}, \tag{2.3}$$

τ_k, $k=0,1,\ldots,$ is either a finite number set or a rapidly decreasing (as $k\to\infty$) number sequence. Finally, it is supposed that $\int_{-\infty}^{+\infty}\psi_k^2=1$.

Let us derive evolutional equations for both the potential $U(x,t)$ and the wave functions $\psi_m(x,t)$, $m=0,1\ldots$ [and simultaneously verify that the deformation obeying the phase flow (2.2) is really isospectral, i.e., $(E_m)_t=0$].

By virtue of Eqs. (2.1) and (2.2), the evolution of the eigenelements of the Schrödinger problem induced by the above phase flow satisfies the following relations:

$$(E_m)_t\psi_m^2+\frac{1}{2}[\psi_m(\psi_m)_{xt}-(\psi_m)_x(\psi_m)_t]_x$$

$$=\sum_k\tau_k\psi_m^2(\psi_k^2)_x,\quad k,m=0,1,\ldots. \tag{2.4}$$

Using the identities (see, for example, [10])

$$\psi_m^2(\psi_k^2)_x=\frac{1}{2}(\psi_m^2\psi_k^2)_x+\frac{1}{2}[\psi_m^2(\psi_k^2)_x-\psi_k^2(\psi_m^2)_x]$$

$$=\frac{1}{2}(\psi_m^2\psi_n^2)_x+(\psi_m\psi_k)[\psi_m(\psi_k)_x-\psi_k(\psi_m)_x], \tag{2.5}$$

$$(E_m-E_k)\psi_k\psi_m=\frac{1}{2}[\psi_m(\psi_k)_x-\psi_k(\psi_m)_x]_x \tag{2.6}$$

transform relations (2.4) to the form

$$2(E_m)_t\psi_m^2=\left\{-[\psi_m(\psi_m)_{xt}-(\psi_m)_x(\psi_m)_t]+\sum_k\tau_k\psi_m^2\psi_k^2+\sum_{k\neq m}\tau_k\frac{[\psi_m(\psi_k)_x-\psi_k(\psi_m)_x]^2}{2(E_m-E_k)}\right\}_x,$$

$$k,m=0,1,\ldots. \tag{2.7}$$

Integrating these expressions in x and taking into account that $\lim_{x\to\pm\infty}\psi_m=0$, we find that $(E_m)_t=0$, whereas the evolution of the eigenfunctions is determined by the following system of equations:

$$\left[\frac{(\psi_m)_x}{\psi_m}\right]_t-\tau_m\psi_m^2=\sum_{k\neq m}\tau_k\left\{\psi_k^2+\frac{[\psi_m(\psi_k)_x-\psi_k(\psi_m)_x]^2}{2(E_m-E_k)\psi_m^2}\right\}. \tag{2.8}$$

The substitution $\psi_m=\exp S_m$ allows us to write Eq. (2.8) in the form of *the system of coupled Liouville equations*:

$$(S_m)_{xt}-\tau_m\exp(2S_m)=\sum_{k\neq m}\tau_k\exp(2S_k)\times\left\{1+\frac{[(S_m)_x-(S_k)_x]^2}{2(E_m-E_k)}\right\}. \tag{2.9}$$

Thus, the solutions of system (2.8) [or (2.9)] together with relation (2.2), when compatible with the Schrödinger problem (2.1), determine isospectral deformations both of the potential $U(x,t)$ and of the wave functions ψ_m.

B. Individual flow

Let all $\tau_m=0$ except for some $\tau_n=1$ corresponding to the eigenelement (ψ_n,E_n). In this case the flow is defined by the expression

$$U_t=(\psi_n^2)_x. \tag{2.10}$$

System (2.8) takes the form

$$\left[\frac{(\psi_n)_x}{\psi_n}\right]_t=\psi_n^2, \tag{2.11}$$

$$\left[\frac{(\psi_m)_x}{\psi_m}\right]_t=\psi_n^2+\frac{J_{mn}^2}{2(E_m-E_n)\psi_m^2},\quad m\neq n, \tag{2.12}$$

where $J_{mn}\equiv\psi_m(\psi_n)_x-\psi_n(\psi_m)_x$. As in the general case (2.8), the substitution $\psi_n=\exp S_n$ transforms system (2.11) and (2.12) to the form

$$(S_n)_{xt}=\exp 2S_n, \tag{2.13}$$

$$(S_m)_{xt}=\exp 2S_n\left\{1+\frac{[(S_m)_x-(S_n)_x]^2}{2(E_m-E_n)}\right\},\quad m\neq n. \tag{2.14}$$

Equation (2.13) for ψ_n is completely integrable: it is the hyperbolic Liouville equation. Its general solution is [12]

$$S_n(x,t)=f_n(x)-g_n(t)-\ln\left\{C_n\int^x dx'e^{2f_n(x')}+C_n^{-1}\int^t dt'e^{-2g_n(t')}\right\}. \tag{2.15}$$

Here $f_n(x)$, $g_n(t)$ are arbitrary functions and C_n is an arbitrary parameter. For the initial problem $\psi_n(x,0)=\psi_n^0$ we find that for $t>0$ expression (2.15) yields

$$\psi_n(x,t)=\frac{\psi_n^0(x)e^{-g_n(t)}}{1+\int_0^t dt'e^{-2g_n(t')}\int_x^{+\infty}dx'[\psi_n^0(x')]^2}, \tag{2.16}$$

where $g_n(t=0)=0$. The condition for the norming integral to be independent of t can be satisfied if we take $g_n(t)=-t/2$. Under such a choice the above result (without relation to the Liouville equation) was obtained by McKean and Trubowitz [8].

To integrate evolutional equations (2.12) for ψ_m, $m\neq n$, we introduce auxiliary functions

$$A_{nm}(x,t)=\int_x^{+\infty}dx'\,\psi_n(x',t)\psi_m(x',t),\quad m\neq n. \tag{2.17}$$

Subtracting Eq. (2.11) from Eq. (2.12), we get

$$\left\{\frac{J_{mn}}{\psi_m\psi_n}\right\}_t=-\frac{J_{mn}^2}{2(E_m-E_n)\psi_m^2}, \tag{2.18}$$

whence, using identity (2.6) for excluding J_{mn}, we get after simple transformations that system (2.11) and (2.12) can be written in the form

$$\left[\frac{(\psi_n)_x}{\psi_n}\right]_t=\psi_n^2, \tag{2.19}$$

$$\left[\frac{(A_{nm})_x}{A_{nm}}\right]_t=\psi_n^2,\quad m\neq n, \tag{2.20}$$

so that

$$\left(\ln\frac{A_{nm}}{\psi_n}\right)_{xt}=0. \tag{2.21}$$

Thus, once a solution of Eq. (2.11) for ψ_n is known, solutions for all the functions A_{nm} can be obtained directly from the relation

$$A_{nm}=A_{nm}^0\frac{\psi_n}{\psi_n^0}\;e^{z(t)}, \tag{2.22}$$

where $A_{nm}^0\equiv A_{nm}(x,0)$, $\psi_n^0\equiv\psi_n(x,0)$, and $z(t)$ is an arbitrary function such that $z(0)=0$.

Under the choice $g_n(t)=-t/2$, $z\equiv 0$ formula (2.16) takes the form

$$\psi_n=\psi_n^0\;\frac{e^{t/2}}{\theta_n},\quad \theta_n\equiv 1+(e^t-1)\int_x^{+\infty}dx'[\psi_n^0(x')]^2. \tag{2.23}$$

Then from Eq. (2.22) we get

$$\int_x^{+\infty}dx'\,\psi_n(x',t)\psi_m(x',t)=\int_x^{+\infty}dx'\,\psi_n^0(x')\psi_m^0(x')\frac{e^{t/2}}{\theta_n(x,t)},\quad m\neq n. \tag{2.24}$$

Integrating this, we arrive at the expression for the evolution of the eigenfunctions $\psi_m(x,t)$, $m\neq n$,

$$\psi_m(x,t)=\psi_m^0-(e^t-1)\frac{\psi_n^0}{\theta_n}\int_x^{+\infty}dx'\,\psi_n^0(x')\psi_m^0(x'),\quad m\neq n, \tag{2.25}$$

presented in [8].

C. Evolution of potential

Now let us find an equation for the evolution of the potential $U(x,t)$ induced by the nth individual flow. Introduce the function $W(x,t)$: $U=W_x$ [by virtue of Eq. (2.2) $W_t=(\psi_n)^2$]. From Eq. (2.11) it follows that $[\ln\psi_n]_{xt}=W_t$, hence the equation for W_t takes the form

$$[\ln(W_t)]_{xt}=2W_t, \tag{2.26}$$

which can also be written in the form of the Liouville equation

$$\Pi_{xt}=\exp\Pi \tag{2.27}$$

for the function $\Pi(x,t)$ defined by the expression $2W_t=\exp\Pi$. The solution of the equation determines the potential:

$$U(x,t)=U^0-\frac{d^2}{dx^2}\ln\left\{1+\int_0^t dt'\,e^{-2g_n(t)}\int_x^{+\infty}dx'\times\int^{x'}dx''U_t^0(x'')\right\}, \tag{2.28}$$

where $U^0\equiv U(x,t=0)$, $U_t^0\equiv U_t(x,t=0)$. [This expression for $g_n(t)=-t/2$ also coincides with the result of [8].] It should be stressed that within the framework of our problem the ''initial'' functions $U^0(x)$ and $U_t^0(x)$ cannot be given independently, since flow (2.10) is defined using the eigenfunction ψ_n of the Schrödinger operator for the potential $U(x)$. However, from a more general viewpoint, potentials U related to solutions of Eq. (2.27) as

$$U=\frac{1}{2}\frac{d}{dx}\int_0^t dt'\exp\Pi(x,t') \tag{2.29}$$

can have unexpected physical importance, not being necessarily isospectral.

Note that integration of the relation $[\ln\psi_n]_{xt}=W_t$ leads to the expression

$$U(x,t)=U^0+\frac{d^2}{dx^2}\ln\frac{\psi_n}{\psi_n^0} \tag{2.30}$$

whence solution (2.28) for U can be obtained at once, if solution (2.16) of the Liouville equation for the function ψ_n is given.

Now return again to the evolutional equation for the potential in the form (2.26). As a consequence of both this and the Schrödinger equations we obtain the following equation:

$$\frac{1}{2}W_{xxt}W_t-\frac{1}{4}(W_{xt})^2=2(W_t)^2(W_t-E_n). \tag{2.31}$$

Its differentiation in t leads to the relation

$$\left[\frac{W_{xt}}{W_t}\right]_{xt}+\left[\frac{W_{xt}}{W_t}\right]\left[\frac{W_{xt}}{W_t}\right]_t=4W_{xt}, \tag{2.32}$$

which, clearly, turns into an identity if W_t satisfies Eq. (2.26). On the other hand, differentiating Eq. (2.31) in x yields

$$\frac{1}{8}W_{xxxt}=W_{xt}(W_x-E_n)+\frac{1}{2}W_{xx}W_t. \tag{2.33}$$

This equation was presented (up to a change of variables) in the book [12] in the context of another approach to the problem on isospectral deformation of potentials. Thus Eq. (2.26), representable also in the form of the Liouville equation, is an integral of Eq. (2.33).

D. Solution of the Liouville equation as a double Darboux transformation

Interpret a sense of the expressions for the evolution of $U(x,t)$ and $\psi_n(x,t)$ in terms of the Darboux transformation. Recall that the ''kernel'' of the Darboux approach is the following simple algebraic observation:

If some function φ is an arbitrary solution (not necessarily eigenfunction) of the Schrödinger equation with the potential V for some value of the parameter E, then the function $1/\varphi$ (defined in the points $\varphi\neq 0$) is a solution of the Schrödinger equation with the potential $\tilde{V}=V-(d^2/dx^2)\ln\varphi$ for the same value E.

Show now that expressions (2.16) and (2.30) for the isospectral deformation of ψ_n and U in the case of individual flow (2.10) can also be interpreted as a result of the above transformation applied twice. Let the pair (ψ_n^0,E_n^0) be an eigenelement of the Schrödinger problem with the potential U_0 (i.e., ψ_n^0 is a normalized function). Then the function $\phi^0=1/\psi_n^0$ is a solution (now unnormalized) of the Schrödinger problem with the new potential $\tilde{U}=U_0-(d^2/dx^2)\ln\psi_n^0$, but for the same E_n^0.

The general solution of the Schrödinger equation with the potential $\tilde{U}$ and the parameter E_n^0 is given by the expression

$$\phi^\alpha=\phi^0\left(1+\alpha\int_x^{+\infty}\frac{dx'}{(\phi_n^0)^2}\right), \tag{2.34}$$

where the parameter α arises (which is an analog of the deformation parameter t).

Applying the above transformation a second time, we find that the function $\psi^\alpha=1/\phi^\alpha$ is a solution of the Schrödinger equation with the potential $U^\alpha=\tilde{U}-(d^2/dx^2)\ln\phi^\alpha$ and with the same value of E_n^0. Combining these two transformations, we get the final expression for the function ψ^α:

$$\psi^\alpha=1/\phi^\alpha=\frac{1}{\phi^0\left(1+\alpha\int_x^{+\infty}\frac{dx'}{(\phi^0)^2}\right)}=\frac{\psi_n^0}{1+\alpha\int_x^{+\infty}dx'(\psi_n^0)^2} \tag{2.35}$$

which corresponds to U^α,

$$\begin{aligned}U^\alpha&=\tilde{U}-\frac{d^2}{dx^2}\ln\phi^\alpha=U^0-\frac{d^2}{dx^2}\ln\psi_n^0-\frac{d^2}{dx^2}\ln\phi^\alpha\\&=U^0+\frac{d^2}{dx^2}\ln\frac{\psi^\alpha}{\psi_n^0}\end{aligned} \tag{2.36}$$

with E_n^0.

Analyzing the asymptotics of the expression for ψ^α, we see that the double Darboux transformation restores the normability of the function. Thus ψ^α is an eigenfunction for the Schrödinger problem with the potential U^α and with the initial eigenvalue E_n^0.

Considering the parameter α as a function of t, one can easily show that the function $\sqrt{d\alpha/dt}\;\psi^\alpha$ is a solution of the Liouville equation. This function would have norm 1 if $\alpha(t)$ obeys the equation $d\alpha/dt=\alpha+1$; such a choice of α leads to the exponential flow considered in [8]. Expression (2.36) obtained by means of the double Darboux transformation coincides with Eq. (2.30) as well.

III. OPERATORS OF ISOSPECTRAL EVOLUTION

A. Unitary transformation of functions ψ_m, $m\neq n$

We now turn our attention to the evolution of the wave functions ψ_m, $m\neq n$ [hereafter we assume $g_n(t)=-t/2$]. The expression

$$\psi_m=\psi_m^0-(e^t-1)\frac{\psi_n^0}{\theta_n}\int_x^{+\infty}dx'\,\psi_n^0(x')\,\psi_m^0(x'),\quad m\neq n \tag{3.1}$$

obtained previously as a solution of the Liouville equation's counterpart (2.12) can also be interpreted as a result of action of *a linear integral operator* $\hat{\mathbf{S}}_n$ (depending on the function ψ_n^0) onto ψ_m^0:

$$\psi_m=\hat{\mathbf{S}}_n\psi_m^0, \tag{3.2}$$

$$\hat{\mathbf{S}}_n\;f(x)\equiv f(x)-(e^t-1)\frac{\psi_n^0}{\theta_n}\int_x^{+\infty}dx'\,\psi_n^0(x')\;\;f(x'). \tag{3.3}$$

This operator is *unitary* in a subspace of the functions ψ_m, $m\neq n$. Indeed, as is shown in the Appendix, the linear operator $\hat{\mathbf{S}}_n^\dagger$ that is Hermitian-conjugate to $\hat{\mathbf{S}}_n$ has the form

$$\hat{\mathbf{S}}_n^\dagger\;f(x)\equiv f(x)-(e^t-1)\psi_n^0\int_{-\infty}^x dx'\,\frac{\psi_n^0(x')}{\theta_n(x',t)}\;\;f(x'). \tag{3.4}$$

On the other hand, it can also be shown (see the Appendix) that the inverse operator $\hat{\mathbf{S}}_n^{-1}$: $\psi_m^0=\hat{\mathbf{S}}_n^{-1}\;\psi_m$ can be determined in such a way that

$$\hat{\mathbf{S}}_n^{-1}=\hat{\mathbf{S}}_n^\dagger. \tag{3.5}$$

Hence, the evolution of the wave functions ψ_m, $m\neq n$, obeying Eqs. (2.12) or (2.20) is determined by the unitary operator $\hat{\mathbf{S}}_n$. Note that the conservation of the eigenvalues E_m, $m\neq n$, is one of the evident consequences of the transformation's unitary property.

It can be checked by direct calculations that the operator $\hat{\mathbf{S}}_n$ transforms the Hamiltonian H^0 of the original quantum system with the potential U^0 into the Hamiltonian H^t of the quantum system with the new potential U:

$$H^t=\hat{\mathbf{S}}_n\,H^0\,\hat{\mathbf{S}}_n^{-1}, \tag{3.6}$$

$$H^0=-\frac{1}{2}\frac{d^2}{dx^2}+U^0(x),\quad H^t=-\frac{1}{2}\frac{d^2}{dx^2}+U(x),$$

$$U=U^0-\frac{d^2}{dx^2}\ \ln\theta_n. \tag{3.7}$$

Let us dwell on the evolution of the wave function ψ_n, which is described by the expression

$$\psi_n(x,t)=\frac{e^{t/2}\psi_n^0}{1+(e^t-1)\int_x^{+\infty}dx'[\psi_n^0(x')]^2}\equiv\hat{\mathcal{N}}(x,t)\psi_n^0(x). \tag{3.8}$$

Thus, contrary to the evolution of ψ_m, $m\neq n$, determined by a *linear* operator, the evolution of the eigenfunction ψ_n (which defines the flow itself) is determined by the *nonlinear* operator $\hat{\mathcal{N}}$. The inverse operator in that case has the following form:

$$\psi_n^0(x)=\frac{e^{t/2}\psi_n(x,t)}{1+(e^t-1)\int_x^{+\infty}dx'[\psi_n(x',t)]^2}$$
$$\equiv\hat{\mathcal{N}}^{-1}(x,t)\psi_n(x,t). \tag{3.9}$$

The condition $\hat{\mathcal{N}}^{-1}\hat{\mathcal{N}}=1$ can be checked easily by direct calculations. One can also show that the eigenfunctions $\psi_n(x,t)$ and $\psi_m(x,t)$, $m\neq n$, being orthogonal at $t=0$, retain the orthogonality at any $t>0$:

$$(\psi_m(x,t),\psi_n(x,t))=(\psi_m^0(x),\psi_n^0(x))=0. \tag{3.10}$$

B. Evolution of shift operators

Suppose that at $t=0$ there exists a spectrum shift operator $\mathbf{L}(0)$ that translates one eigenfunction $\psi_m^0\equiv\psi_m(x,0)$ corresponding to the eigenvalue E_m into another eigenfunction $\psi_{m'}^0$ corresponding to $E_{m'}$:

$$\psi_{m'}^0=\mathbf{L}(0)\psi_m^0. \tag{3.11}$$

Let $m,m'\neq n$. Applying the operator $\hat{\mathbf{S}}_n$ to both sides of Eq. (3.11) and taking into account that $\psi_{m'}(t)=\hat{\mathbf{S}}_n\psi_{m'}^0$ and $\psi_m^0=\hat{\mathbf{S}}_n^{-1}\psi_{m'}(t)$, we get

$$\psi_{m'}(t)=\hat{\mathbf{S}}_n\mathbf{L}(0)\hat{\mathbf{S}}_n^{-1}\psi_m(t). \tag{3.12}$$

Hence, given the shift operator $\mathbf{L}(0)$ at $t=0$ and the unitary operator $\hat{\mathbf{S}}_n$, one can obtain for any t a shift operator $\mathbf{L}_n(t)$,

$$\mathbf{L}_n(t)\equiv\hat{\mathbf{S}}_n\mathbf{L}(0)\hat{\mathbf{S}}_n^{-1}, \tag{3.13}$$

which acts in the subspace of all the eigenfunctions except for ψ_n. Note that the shift operator $\mathbf{L}_n(t)$ is defined for the ''new'' quantum system with the Hamiltonian H^t specified by expression (3.6).

Consider now shift operators for the state ψ_n. Let

$$\psi_{n+1}^0=\mathbf{L}(0)\psi_n^0. \tag{3.14}$$

Applying $\hat{\mathbf{S}}_n$ to both sides, we get

$$\psi_{n+1}(t)=\hat{\mathbf{S}}_n\psi_{n+1}^0=\hat{\mathbf{S}}_n\mathbf{L}(0)\hat{\mathcal{N}}^{-1}\hat{\mathcal{N}}\psi_n^0=\hat{\mathbf{S}}_n\mathbf{L}(0)\hat{\mathcal{N}}^{-1}\psi_n(t). \tag{3.15}$$

Then the operator $\mathbf{L}_n^{\uparrow}(t)$ can be introduced:

$$\psi_{n+1}(t)=\mathbf{L}_n^{\uparrow}(t)\psi_n(t),\quad \mathbf{L}_n^{\uparrow}(t)\equiv\hat{\mathbf{S}}_n\mathbf{L}(0)\hat{\mathcal{N}}^{-1}. \tag{3.16}$$

The operators $\mathbf{L}_{n+1}^{\downarrow}(t)$, $\mathbf{L}_n^{\downarrow}(t)$, and $\mathbf{L}_{n-1}^{\uparrow}(t)$ are introduced analogously:

$$\psi_n(t)=\mathbf{L}_{n+1}^{\downarrow}(t)\psi_{n+1}(t),\quad \mathbf{L}_{n+1}^{\downarrow}(t)\equiv\hat{\mathcal{N}}\,\mathbf{L}^{\dagger}(0)\hat{\mathbf{S}}_n^{-1},$$

$$\psi_{n-1}(t)=\mathbf{L}_n^{\downarrow}(t)\psi_n(t),\quad \mathbf{L}_n^{\downarrow}(t)\equiv\hat{\mathbf{S}}_n\mathbf{L}^{\dagger}(0)\hat{\mathcal{N}}^{-1},$$

$$\psi_n(t)=\mathbf{L}_{n-1}^{\uparrow}(t)\psi_{n-1}(t),\quad \mathbf{L}_{n-1}^{\uparrow}(t)\equiv\hat{\mathcal{N}}\,\mathbf{L}(0)\hat{\mathbf{S}}_n^{-1}.$$

Thus, the evolution of the shift operators can be determined for any pair of eigenelements of the Schrödinger operator.

Note that in this case the evolution of analogs of the number operators, which we define, following [14,15], as $\mathbf{N}(t)\equiv\mathbf{L}_{n-1}^{\uparrow}(t)\mathbf{L}_n^{\downarrow}(t)$ $[\mathbf{N}^0=\mathbf{L}(0)\mathbf{L}^{\dagger}(0)]$ and $\tilde{\mathbf{N}}(t)\equiv\mathbf{L}_{n+1}^{\downarrow}(t)\mathbf{L}_n^{\uparrow}(t)$ $[\tilde{\mathbf{N}}^0=\mathbf{L}^{\dagger}(0)\mathbf{L}(0)]$ is specified by the operators $\hat{\mathcal{N}}$ and $\hat{\mathcal{N}}^{-1}$:

$$\mathbf{N}(t)=\hat{\mathcal{N}}\,\mathbf{N}^0\hat{\mathcal{N}}^{-1},\quad \tilde{\mathbf{N}}(t)=\hat{\mathcal{N}}\,\tilde{\mathbf{N}}^0\hat{\mathcal{N}}^{-1}. \tag{3.17}$$

For the other states $m\neq n$ the operator $\hat{\mathcal{N}}$ in these expressions should be replaced with $\hat{\mathbf{S}}_n$.

IV. ASYMPTOTICAL ANALYSIS AND EXAMPLES

A. General scenario of isospectral deformation as $t\to\infty$

Investigate the evolution both of an arbitrary potential $U(x)$ and of a wave function of the nth corresponding bound state for large values of t. At $t=0$ for an asymptotics of the function ψ_n as $x\to+\infty$ we use the first term of the quasiclassical approximation [16],

$$\psi_n^0\sim\exp(-I(x)),\quad I(x)\equiv\int^x\sqrt{2[U^0(x')-E_n]}\,dx'. \tag{4.1}$$

This approximation is valid if $U_x^0\ll(U^0-E_n)^{3/2}$ at large x. In particular, this is true both for scattering potentials [with $U^0(x\to+\infty)\to0$, $U_x^0(x\to+\infty)\to0$] as well as for infinitely growing potentials (for example, for anharmonic oscillators of any kind).

The asymptotics of the integral appearing in the solution for $\psi_n(t)$, $t>0$ is determined by the expression

$$\int_x^{\infty}dx'[\psi^0(x')]^2\sim\frac{1}{2\sqrt{2[U^0(x)-E_n]}}\exp[-2I(x)]. \tag{4.2}$$

Letting then that $t\to\infty$, we obtain the following approximate expressions for ψ_n and $\Delta U\equiv U(t)-U^0$:

$$\psi_n(x\to\infty,t\to\infty)\sim\frac{\exp[t/2-I(x)]}{1+\dfrac{1}{2\sqrt{2[U^0(x)-E_n]}}\exp[t-2I(x)]},\tag{4.3}$$

$$\Delta U(x\to\infty,t\to\infty)\sim-\frac{2\sqrt{2[U^0(x)-E_n]}\exp[t-2I(x)]}{1+\dfrac{1}{2\sqrt{2[U^0(x)-E_n]}}\exp[t-2I(x)]}.\tag{4.4}$$

In the neighborhood of the curve $x=x_0(t)$ specified implicitly by the equation $2I(x_0)=t$, we set $x=x_0(t)+y$, $|y|\ll|x_0(t)|$. Then we have $I(x)-t/2=I(x)-I(x_0)\sim\sqrt{2[U^0(x_0)-E_n]}y$. As a result we get

$$\psi_n\approx\sqrt{\frac{\sqrt{2[U^0(x_0)-E_n]}}{2}}\cosh^{-1}[\sqrt{2[U^0(x_0)-E_n]}\times[x-x_0(t)]+\delta(x_0)],\tag{4.5}$$

$$\Delta U\approx-2\sqrt{2[U^0(x_0)-E_n]}\cosh^{-2}[\sqrt{2[U^0(x_0)-E_n]}\times[x-x_0(t)]+\delta(x_0)],\tag{4.6}$$

$$x_0(t):\quad \int^{x_0}\sqrt{2[U^0(x')-E_n]}\,dx'=\frac{t}{2},$$

where $\delta(x_0)=\frac{1}{2}\ln 2\sqrt{2[U^0(x_0)-E_n]}$ is an additional shift, whose value may be refined using the following terms of the quasiclassical approximation.

Note that asymptotical expression (4.5) proves to be normalized to 1 (as with the exact expression for ψ_n), which means that the wave function vanishes rapidly outside the neighborhood of the curve $2I(x_0)=t$.

The normalized wave function (4.5) corresponds to a unique eigenstate with energy $E=E_n-U^0(x_0)$ in the ''potential'' ΔU (4.6). Since the total potential U is $U^0+\Delta U$, we see that the asymptotical analysis of the potential deformation does not violate the isospectrality: $E_n(t)\approx E_n^0$.

Hence, in the general case the evolution of the wave function defining the flow is described asymptotically (at large t) as its transformation into the sech profile, moving with changing (generally) structure parameters and velocity; this wave function becomes strongly localized in a potential minimum of the form $-\text{sech}^2\xi$ moving along with it; the ''law'' of this movement is given by the condition $2I(x)=t$.

Now consider how this scenario is realized for potentials of different structure.

B. The Rosen-Morse potential deformation

Let us study the isospectral evolution of the family of reflectionless ''soliton'' potentials (traditionally called the Rosen-Morse potentials [17]):

$$U_N^0(x)=-\frac{N(N+1)}{2}\,\frac{1}{\cosh^2 x},\quad N=1,2,\ldots.\tag{4.7}$$

Recall that the spectrum of the Schrödinger problem with this potential has exactly N bound states with eigenvalues $E_n=-(N-n)^2/2$, $n=0,1,\ldots,N-1$. Each of the corresponding eigenfunctions defines its own phase flow and, consequently, its own isospectral evolution of the original potential. Thus, for fixed N there exist N different scenarios of isospectral deformation of this potential.

The main properties of this deformation can be formulated as follows.

(i) The isospectral deformation of potentials (4.7) describes a state that splits up (as $t\to\infty$) into a single-soliton potential of the form $\text{sech}^2\xi$ that moves off to $+\infty$ and a claster moving in the opposite direction and (at $t=\infty$) coming to rest at some point $x=-x_0<0$ (a usage of the term ''soliton'' will be justified below).

(ii) The soliton potential that moves off contains a unique bound state with energy corresponding to the nth eigenstate (defining the flow) of the original potential.

(iii) The spectrum of the Schrödinger problem with the potential that is left over in the limit $t=\infty$ coincides with that for the original potential with the exception of the nth state, which is removed.

(iv) At any $0<t<\infty$ the spectrum of the Schrödinger problem coincides with the original one. As $t\to\infty$, the difference between the wave functions ψ_m, $m\neq n$, and the wave functions of the limit case $t=\infty$ becomes infinitely small.

Consider a transformation of the wave function ψ_n and perform the same analysis as in the preceding subsection. At $t=0$, asymptotics of ψ_n, as $x\to+\infty$, has the form $\psi^0(x)\sim C(k)\exp(-kx)$, $k=N-n$. As $t\to\infty$, we have

$$\psi_n(x\to\infty,t\to\infty)\approx\frac{C(k)\exp(-kx+t/2)}{1+\dfrac{C^2(k)}{2k}\exp(-2kx+t)}.\tag{4.8}$$

As with the above-described general case, in the vicinity of the line $-kx+t/2=0$ we set $x=-t/2k+y$, $|y|\ll t/2k$. Then we obtain

$$\psi_n(x\to\infty,t\to\infty)\approx\frac{C(k)\exp(-ky)}{1+\dfrac{C^2(k)}{2k}\exp(-2ky)}=\sqrt{\frac{k}{2}}\frac{1}{\cosh k\left[x-\dfrac{1}{2k}t-\delta\right]},$$

$$\delta\equiv\frac{1}{k}\ln\frac{C(k)}{\sqrt{2k}}.\tag{4.9}$$

Thus, as $x\to+\infty$, the wave function of the ''flow-generating'' nth state forms an asymptotical ''bump'' (4.9) traveling to the right with the constant velocity $1/2k$ without any (up to the approximation) changes in shape. This approximate wave function corresponds to the unique eigenstate with energy $E_0=-k^2/2$ in the potential

$$U_{(n)}^{\rightarrow}=-\frac{k^2}{\cosh^2 k\left[x-\frac{1}{2k}t-\delta\right]}, \tag{4.10}$$

which is an asymptotics of the isospectrally deformed potential (4.7) at large x and $t\rightarrow\infty$.

The same formulas can be obtained using the general asymptotical expressions (4.5) and (4.6).

Consider particular cases of the deformation for potentials (4.7) in detail.

(i) Let $N=1$ (the original potential has the only bound state $E_0=-1/2$, $\psi_0^0=[\sqrt{2}\cosh x]^{-1}$). Using Eq. (2.28) we find that in this case the evolution of the potential is reduced to a simple movement of the potential to the right along the x axis with no change in shape and, obviously, with the bound state energy unaltered.

(ii) First nontrivial examples arise in the case $N=2$ ($U^0=-3/\cosh^2 x$). The spectrum consists of two states:

$$E_0=-2,\quad \psi_0^0=\frac{\sqrt{3}}{2}\frac{1}{\cosh^2 x}$$

and

$$E_1=-1/2,\quad \psi_1^0=\sqrt{\frac{3}{2}}\frac{\sinh x}{\cosh^2 x}. \tag{4.11}$$

For both the flows built on these states an evolution can be treated as a splitting up into two potential wells. One of them moves off to infinity, whereas the other comes to rest at some point $-x_0<0$. In the first case [Fig. 1(a)] the well moving to the right is described asymptotically by the expression $U^{\rightarrow}\sim-4\operatorname{sech}^2 2\xi_1$ and carries away a bound state with energy $E=-2$. The left part of the potential forms (at $t=\infty$) the potential $U^{\leftarrow}\sim-\operatorname{sech}^2\xi_2$ containing the other state with energy $E=-1/2$.

In the second case [Fig. 1(b)] these two asymptotical soliton wells interchange: the well $U^{\rightarrow}\sim-\operatorname{sech}^2\xi_1$ moves off to $+\infty$ carrying away the state with $E=-1/2$, whereas the well $U^{\leftarrow}\sim-4\ \operatorname{sech}^2 2\xi_2$ moves to the left and comes to rest (at $x_0=-\frac{1}{4}\ln 3$, to be exact) (Fig. 2).

Note that in this case the "two-soliton state" has a quite simple form:

$$U(x,t)=-3\ \frac{\cosh^4 x-2T\cosh x\ \sinh x-T^2\sinh^4 x}{(\cosh^3 x-T\sinh^3 x)^2},$$

$$T\equiv\tanh(t/2) \tag{4.12}$$

thereby the "flow-generating" wave function corresponding to $E=-1/2$ behaves as follows:

$$\psi_1(x,t)=\sqrt{\frac{3(1-T^2)}{2}}\frac{\cosh x\ \sinh x}{\cosh^3 x-T\ \sinh^3 x}. \tag{4.13}$$

(iii) Finally, consider the case $N=3$ ($U^0=-6/\cosh^2 x$). The spectrum consists of three bound states:

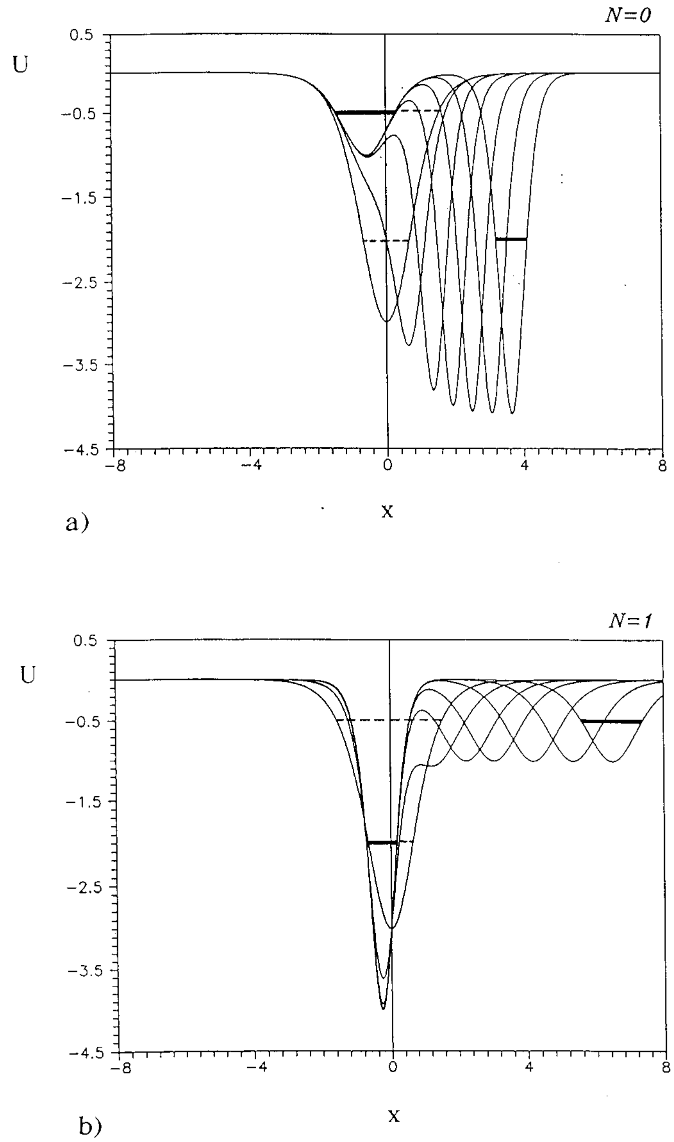

FIG. 1. Evolution of the Rosen-Morse potential containing two bound states: (a) under the flow built on the ground state and (b) under the flow built on the excited state.

$$E_0=-9/2,\quad \psi_0^0=\frac{\sqrt{15}}{4}\frac{1}{\cosh^3 x},$$

$$E_1=-2,\quad \psi_1^0=\frac{\sqrt{15}}{2}\frac{\sinh x}{\cosh^3 x},$$

$$E_2=-1/2,\quad \psi_2^0=\sqrt{3}\left[\frac{1}{\cosh x}-\frac{5}{4\cosh^3 x}\right]$$

(this case includes all three qualitatively different kinds of individual flows, namely, a flow built on a ground state, a flow built on an uppermost excited state, and a flow built on an intermediate state).

Under the flow built on the ground state (Fig. 3) the original potential splits up into two wells: $U^{\rightarrow}\sim-9\operatorname{sech}^2 3\xi_1$ and $U^{\leftarrow}\sim-3\ \operatorname{sech}^2\xi_2$. The first of them (which contains the unique bound state with energy $E=-9/2$) moves off to infinity as $t\rightarrow\infty$, whereas the second well comes to rest: it

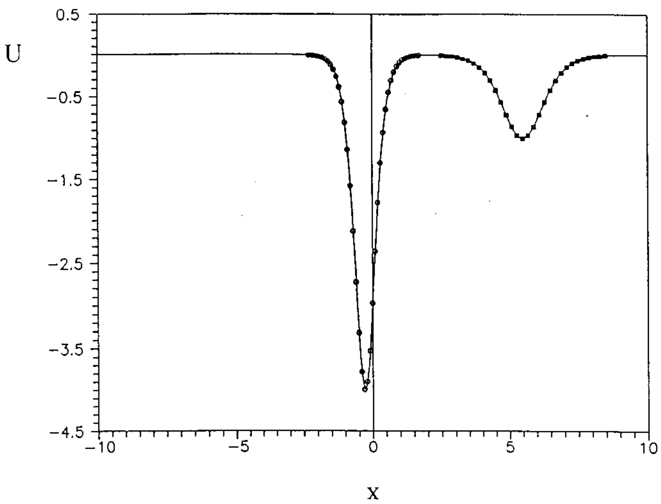

FIG. 2. Splitting up of the Rosen-Morse potential at large t into two sech^2 wells (in the case of the flow built on the excited state). The exact solution is depicted by a solid line; the asymptotical solutions for both the sech^2 wells are marked with rectangles and circles.

contains two bound states corresponding to the other levels ($E=-2$ and $E=-1/2$) and coincides with the original potential of the previous case.

Under the flow built on the first excited state with energy $E=-2$ (Fig. 4) a well that splits off from the original potential has the asymptotical form $U^{\rightarrow}\sim-4\ \mathrm{sech}^2 2\xi$ and carries away to infinity its only bound state with the above energy. The left complicated nonsymmetric well (quite loosely call it ''soliton cluster'') contains two other states with energies $E_0=-9/2$ and $E_2=-1/2$. In this case the creation of a soliton cluster instead of a simple soliton well is ''due'' to a gap in the quadratic spectrum, which arises in the limit $t=+\infty$.

An analogous situation takes place for a deformation of the original potential under the flow built on the upper state (Fig. 5): in this case the potential $U^{\rightarrow}\sim-\mathrm{sech}^2\xi$ moves away, whereas the left nonsymmetric cluster contains two bound states with energies $E=-9/2$ and $E=-2$ (in Fig. 6 a stage of well-distinguishable splitting of the original potential into a cluster and a single-soliton potential is presented; an asymptotics of the latter is depicted).

As is seen in Figs. 3 and 4, eigenfunctions associated with those states that lie above the ''flow-generating'' state transform in such a way that at large t their form becomes asymptotically close to the form of the eigenfunctions for the potential that is left after the soliton bump moves off. In particular, when passing to the limit $t=+\infty$ each of these eigenfunctions loses one zero. Note that the results obtained above are also in agreement with qualitative description of the paper [18].

To prove a soliton character of the potential well moving off to $+\infty$, we have followed numerically an evolution of a disturbed potential (not related to the original Schrödinger problem). It was revealed that a final stage of the evolution is the same for quite arbitrary perturbations; a detailed study of this item is beyond the scope of our paper.

Note also that the existence of the Liouville equation's solutions splitting up into a single soliton and a cluster was previously revealed by Andreev in [13]; he treated those solutions as N solitons. In that paper a preservation of the eigenvalues for the Schrödinger problem with such potentials was also mentioned.

C. Isospectral deformation of harmonic oscillator

For the harmonic oscillator's potential an isospectral deformation defined by the expression (2.28) was presented (without any relation to the Liouville equation) as an illustration in a number of papers [8,7,6,18,19]. Let us show that, in accordance with the general asymptotical results presented above, the transformation of the harmonic oscillator's potential also results in a local potential well's moving off to infinity; this well is also described asymptotically as a sech^2 potential with *varying* structure parameters and velocity.

As in the preceding subsection, in order to verify general asymptotical results, let us prove that the asymptotical procedure performed *ab initio* in this case leads to the same formulas as can be obtained directly from the general approximate expressions (4.5) and (4.6).

Consider a deformation of the harmonic oscillator's potential $U^0(x)=x^2/2$ under a flow defined by the wave function of the nth state. Its asymptotics for $t=0,\ x\rightarrow+\infty$ has the form

$$\psi_n^0\sim x^n\exp(-x^2/2). \tag{4.14}$$

The exact expression for the deformed potential is

$$U(x,t)=\frac{x^2}{2}+\frac{d}{dx}\left(\frac{(e^t-1)W_t^0}{1+(e^t-1)\int_x^{+\infty}dx'\,W_t^0(x')}\right) \tag{4.15}$$

[where, as we remember, the function $W(x,t)$ is defined by the relation $W_x=U$]. For large x we have

$$W_t^0=(\psi_0^0)^2\sim x^{2n}\exp(-x^2), \tag{4.16}$$

$$\int_x^{+\infty}dx'\quad W_t^0(x')\sim\frac{1}{2\sqrt{\pi}}\ \frac{\exp(-x^2)}{x}+\beta(n)x^{2n-1}\times\exp(-x^2),\quad \beta(0)=0. \tag{4.17}$$

As $t\rightarrow\infty$, for $\Delta U=U-U^0$ we get

$$\Delta U\approx2\ \frac{\exp(t-x^2)[-4\sqrt{\pi}x^{3+2n}+x^{4n}\exp(t-x^2)]}{[2\sqrt{\pi}x+x^{2n}\exp(t-x^2)]^2}. \tag{4.18}$$

In the vicinity of the curve $t=x^2$ we set $x=\sqrt{t}+y$, $|y|\ll\sqrt{t}$. Then

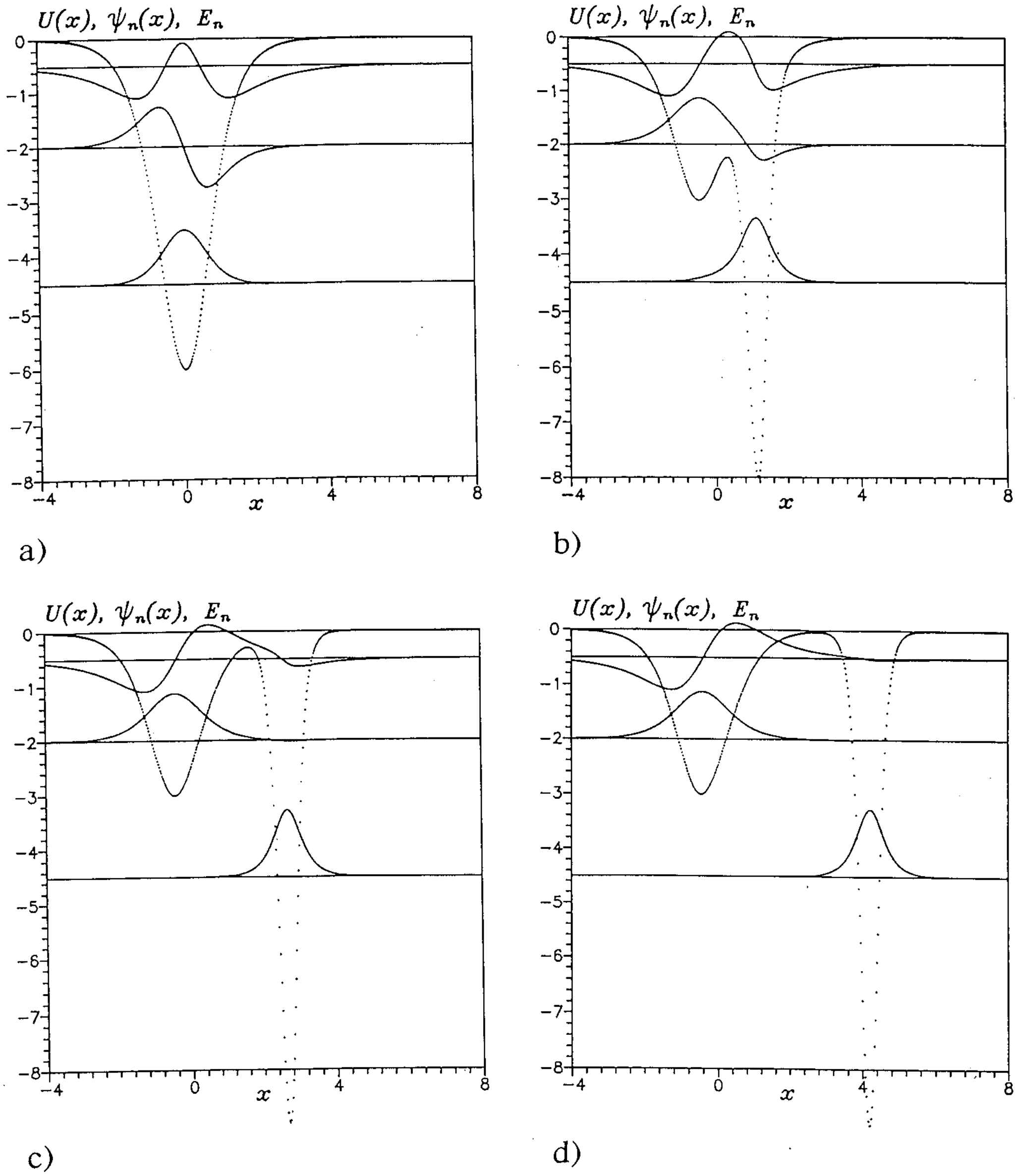

FIG. 3. Evolution of the Rosen-Morse potential containing three bound states, under the flow built on the ground state. For four evolution stages the potential (dotted line), the wave functions (solid lines), and the energy levels are depicted.

$$\Delta U \approx 2\,\frac{\exp(-2\sqrt{t}y)[-4t^{n+1}\sqrt{\pi t}]}{[2\sqrt{\pi t}+t^{n}\exp(-2\sqrt{\pi}y)]^{2}}$$

$$=-\frac{t}{\cosh^{2}\left[\sqrt{t}\left(x-\sqrt{t}+\frac{1}{2\sqrt{t}}\ln\frac{2\sqrt{\pi t}}{t^{n}}\right)\right]}. \quad (4.19)$$

A corresponding expression for the wave function ψ_n is

$$\psi_n \approx \sqrt{\frac{\sqrt{t}}{2}}\,\frac{1}{\cosh\left[\sqrt{t}\left(x-\sqrt{t}+\frac{1}{2\sqrt{t}}\ln\frac{2\sqrt{\pi t}}{t^{n}}\right)\right]}. \quad (4.20)$$

Hence, the local potential well ΔU has the form of a sech^2 potential moving to the right, so that its minimum is located at $x_0 \sim \sqrt{t}$; its depth increases directly with t, whereas its width decreases as $1/\sqrt{t}$. The spectrum of the Schrödinger problem with that potential consists of the only eigenstate with energy $E_0=-t/2$ (note that this potential is reflectionless for any t). Taking into account that the well (4.19) moves up along the ''slope'' of the harmonic oscillator's potential, so that its ''zero'' is located at a height $h \sim (\sqrt{t})^2/2=t/2$, we see that this is in accordance with the isospectrality of the deformation. The numerical investigations have confirmed this result completely (Fig. 7).

Thus, as is shown in Figs. 2, 6, and 7, the general asymptotical analysis of the behavior both of a potential and of a wave function ψ_n at large t allows us to describe with good accuracy both the splitting up of the soliton potentials and the evolution of the harmonic oscillator's potential.

Concluding this section, we would like to remark that an analogous transformation scenario, where one meets with a $\mathrm{sech}^2\xi$ potential well behaving in the same way, was previ-

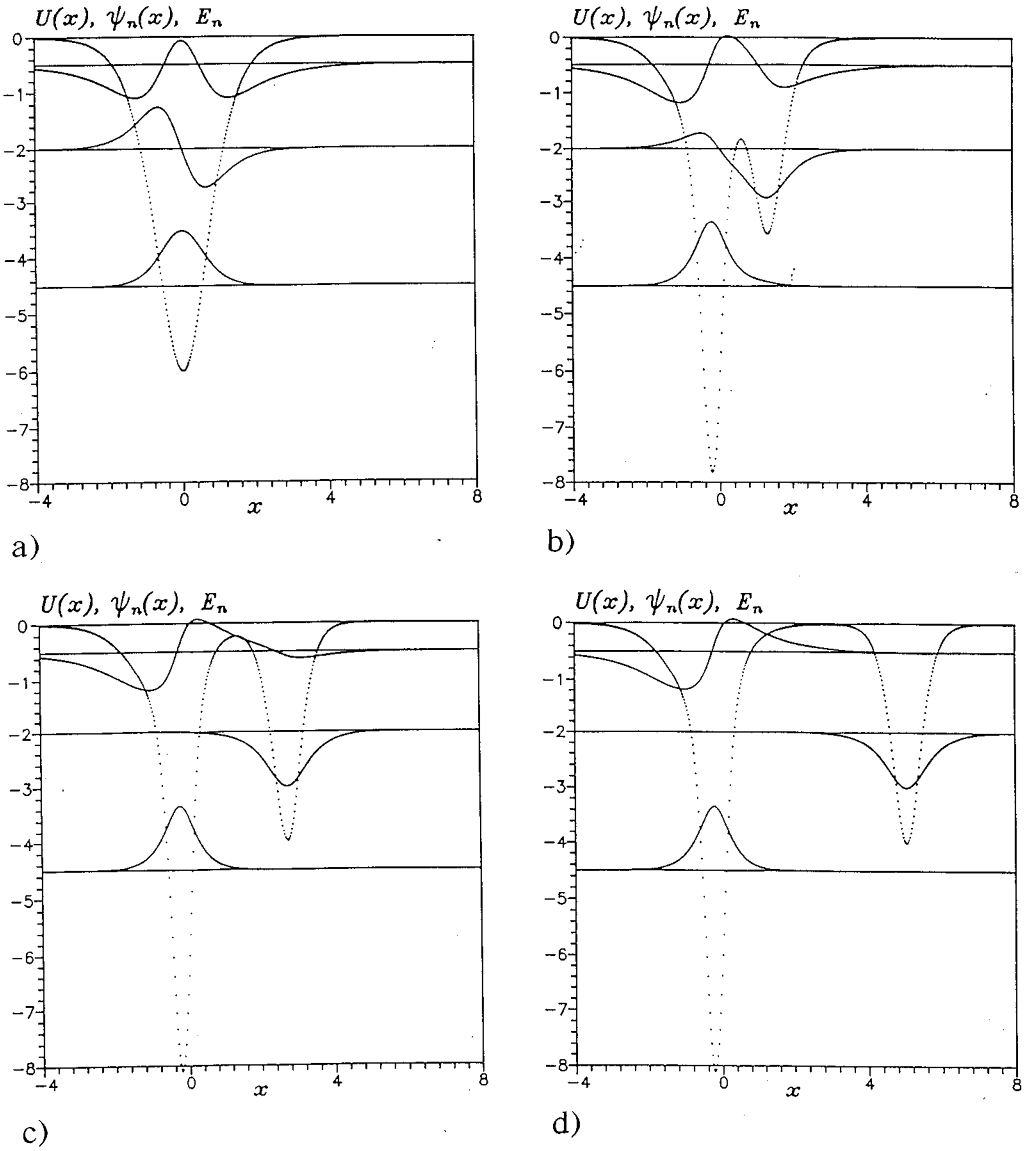

FIG. 4. Same as in Fig. 3, but for the flow built on the first excited state.

ously studied in the problem on anharmonic oscillators admitting shift operators that are polynomials of third degree in the momentum [14].

The potential $U(x)$ was specified there as a solution of the following nonlinear differential equation:

$$\frac{1}{4}V_{xxxx}-\frac{3}{2}(V^2)_{xx}-x^2V_{xx}-3xV_x=0,$$
$$V(x)=U(x)-x^2/2, \tag{4.21}$$

related to the fourth Painlevé equation. Equation (4.21) has two integrals:

$$I_1=x\left\{-\frac{1}{2}V_{xxx}+3(V^2)_x+2x^2V_x\right\}+\frac{1}{2}V_{xx}-3V^2, \tag{4.22}$$

$$I_2=-\frac{1}{4x^2}\left\{-\frac{1}{2}V_{xx}+3V^2+I_1\right\}^2+\frac{(V_x)^2}{4}-V^3-I_1V. \tag{4.23}$$

The creation (or annihilation) of a pair of neighboring energy levels (in the middle of a spectral gap) is accompanied by a pairwise ''arrival'' (or moving off) of symmetric sech^2 wells from left and right infinities. Those additional potential wells are described asymptotically by the expression

$$V^{\leftrightarrow}(x)\approx-\frac{L^2}{\cosh^2[L(x\pm L)]}. \tag{4.24}$$

Obviously, the dependence of the asymptotical potential on the parameter L completely coincides with that dependence in the case of isospectral deformation, if the parameter L is identified with $\sqrt{t}$.

It is worth mentioning that the expression (4.24) is an exact solution of the equation

$$\left\{-\frac{1}{2}V_{xx}+3V^2+I_1\right\}^2-I_1^2=4L^2\left\{\frac{1}{4}(V_x)^2-V^3-I_1V\right\}, \tag{4.25}$$

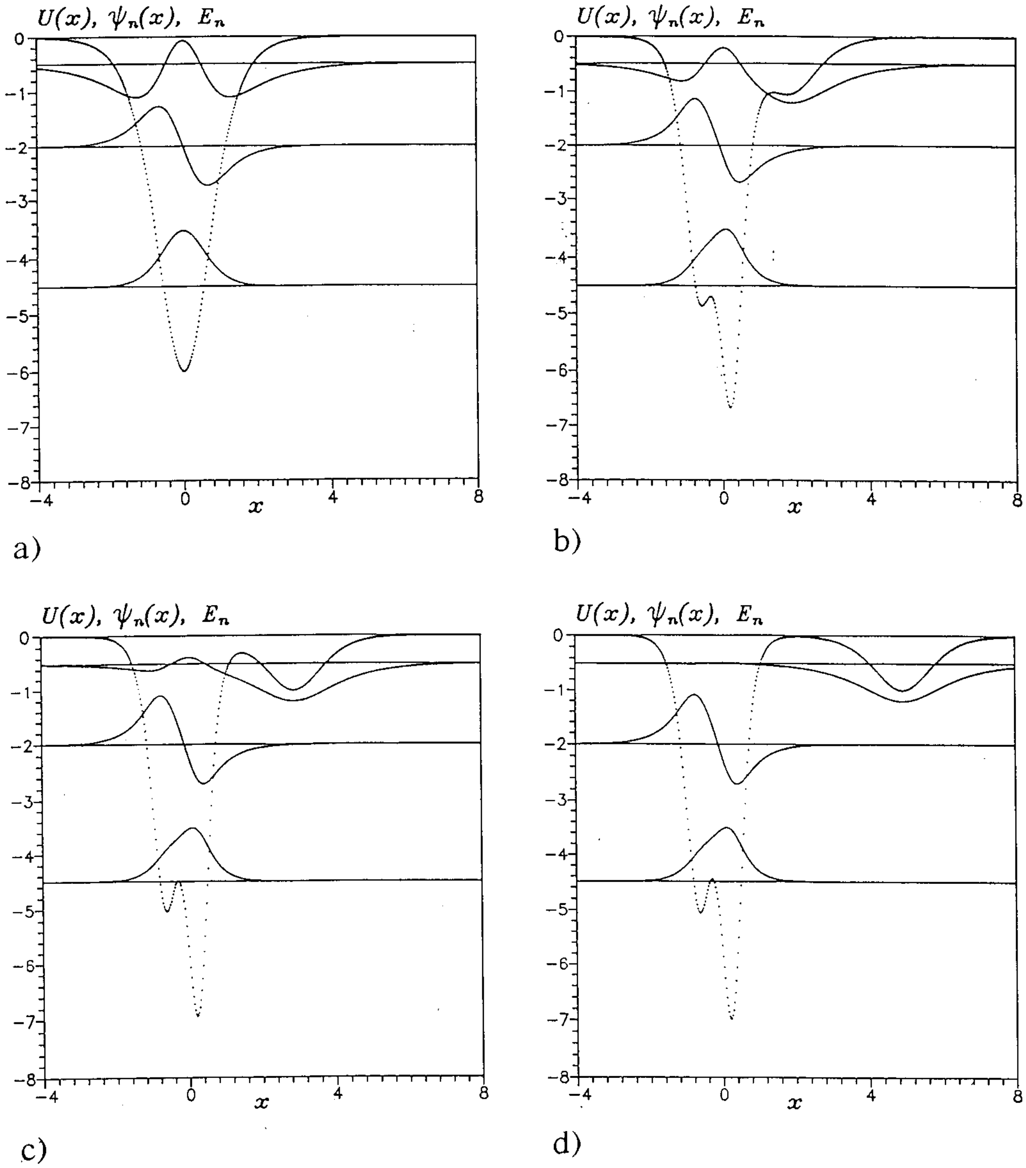

FIG. 5. Same as in Figs. 3 and 4, but for the flow built on the uppermost excited state.

which differs from Eq. (4.23) by the replacement of x^2 with L^2 [a constant term $(-I_1^2-4L^2I_2)$ should also be added to the right side].

V. GENERALIZATIONS

Hence, we have shown that in 1D the description of an isospectral deformation using a phase flow leads to a system of coupled Liouville equations. In the particular case of the ''individual flow'' the well-known integrable Liouville equation arises; its solution specifies a deformation both of the potential and of the eigenfunction ψ_n (which defines the flow). The formulas we have derived in that case are in agreement with the known results obtained by means of both the integral equations formalism and the Darboux transformation.

Using the individual flow as an example, we now show that a direct extension of the approach presented above allows us to derive an evolution equation that is a generalization of the Liouville equation and describe a deformation that leads to a shift of a selected energy level.

Consider a flow defined by the relation

$$U_t=(\psi_n\widetilde{\psi}_n)_x, \tag{5.1}$$

where ψ_n, $\widetilde{\psi}_n$ is a pair of solutions of the Schrödinger equation

$$-\frac{1}{2}y_{xx}+U(x,t)y=E_ny, \tag{5.2}$$

of which only ψ_n is supposed to be normalized (i.e., an eigenfunction): $\int_{-\infty}^{\infty}\psi_n^2(x,t)\,dx<+\infty\ \forall t$. Introduce a Wronskian $w_n(t)$ of the functions $\psi_n,\widetilde{\psi}_n$:

$$w_n(t)=\psi_n(\widetilde{\psi}_n)_x-(\psi_n)_x\widetilde{\psi}_n. \tag{5.3}$$

Using a procedure analogous to that presented in Sec. II, we obtain the following equations:

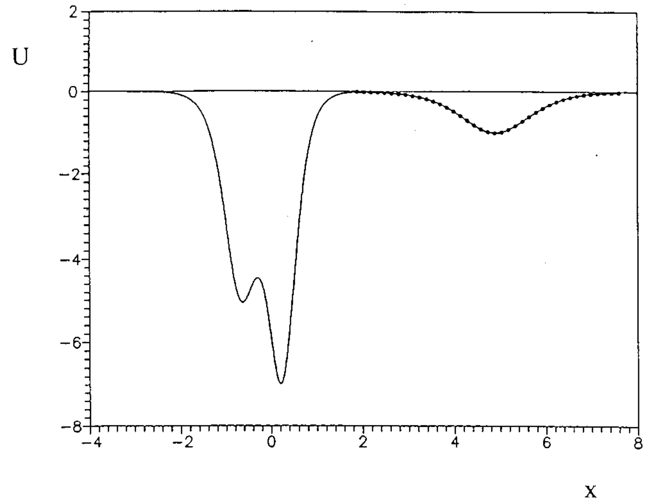

FIG. 6. Splitting up of the Rosen-Morse potential into a soliton well, which moves off, and a complicated cluster, which moves to the left and, finally, comes to rest (for the case presented in Fig. 5). For the well traveling to the right a corresponding asymptotics is depicted by small circles.

$$(E_n)_t = \frac{1}{2}\, w_n(t), \tag{5.4}$$

$$\left[\frac{(\psi_n)_x}{\psi_n}\right]_t = \psi_n \tilde{\psi}_n . \tag{5.5}$$

The first specifies an evolution of the nth eigenvalue. In particular, if we choose $\tilde{\psi}_n$ in such a way that $w_n(t)=1$, then the nth eigenvalue moves with velocity $\frac{1}{2}$ either upward or downward. It can easily be shown that all other eigenvalues are not changed by the flow: $(E_m)_t = 0$.

It follows from Eq. (5.5) that

$$\left[\frac{(\psi_n)_x}{\psi_n}\right]_{xt} = (\psi_n \tilde{\psi}_n)_x = U_t , \tag{5.6}$$

whence, after integration in t, we get, as before,

$$U - U^0 = \frac{d^2}{dx^2} \ln \frac{\psi_n}{\psi_n^0}. \tag{5.7}$$

Expressing $\tilde{\psi}_n$ from relation (5.3) and substituting it in Eq. (5.5), we obtain a generalization of Eq. (2.11):

$$\left[\frac{(\psi_n)_x}{\psi_n}\right]_t = \psi_n^2 \left(1 + w_n(t) \int^x \frac{dx'}{\psi_n^2(x')} \right). \tag{5.8}$$

On the other hand, expressing $\tilde{\psi}_n$ from Eq. (5.5) and substituting it in Eq. (5.3), we get the following equation for the function $\varphi_n \equiv 1/\psi_n$:

$$\left[\frac{(\varphi_n)_{xx}}{\varphi_n}\right]_t = -w_n(t) \tag{5.9}$$

which if of higher degree than Eq. (5.8), but admits an integration in t. As a result we obtain the equation

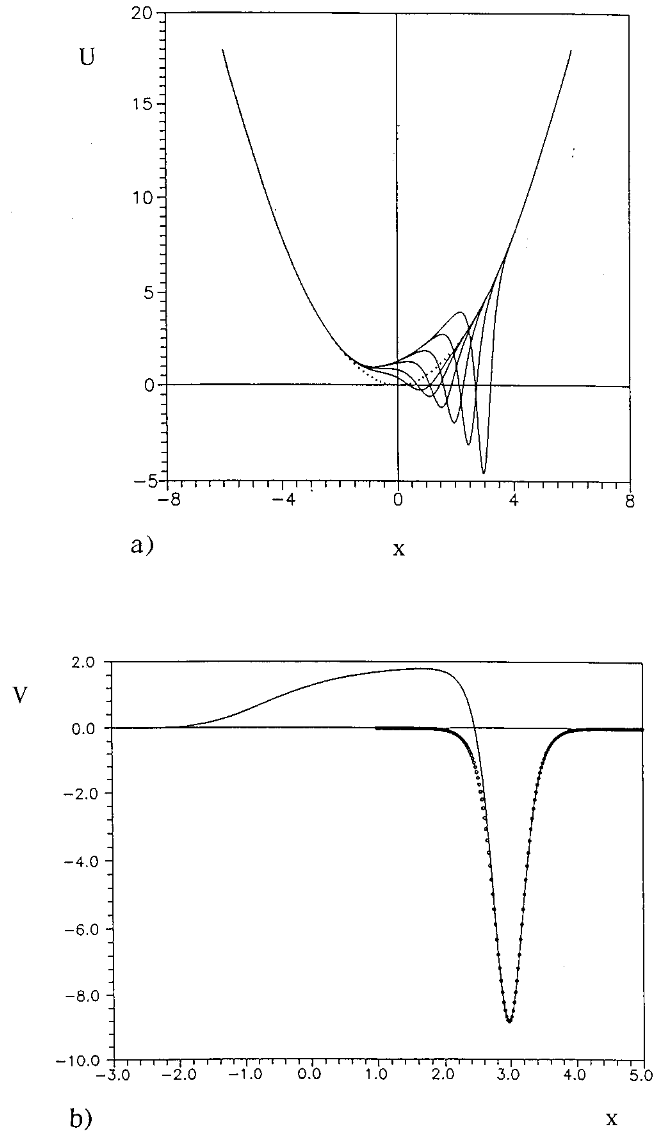

FIG. 7. (a) Evolution of the harmonic oscillator's potential under the individual flow built on the ground state. (b) Function $V(x,t) \equiv U(x,t) - U^0$, $U^0 \equiv x^2/2$, for some $t \gg 1$; the asymptotics of this function is depicted by circles.

$$\frac{(\varphi_n)_{xx}}{\varphi_n} - \frac{(\varphi_n^0)_{xx}}{\varphi_n^0} = -2(E_n - E_n^0). \tag{5.10}$$

At a given function ψ_n^0 (and, consequently, φ_n^0) this is a Schrödinger equation for the function φ_n. Taking into account that $(\psi_n^0)_{xx}/\psi_n^0 = 2(U^0 - E_n^0)$, this equation can be written in the form

$$-\frac{1}{2}(\varphi_n)_{xx} + \left\{ U^0 - \frac{d^2}{dx^2} \ln \psi_n^0 \right\} \varphi_n = E_n \varphi_n . \tag{5.11}$$

Thus, the normal solution $\psi_n = 1/\varphi_n$ for the Schrödinger problem with the potential U (not yet determined) is specified by a suitable non-normalizable solution $\varphi_n[\varphi_n(x \to \pm\infty) \to \pm\infty]$ of the Schrödinger equation with the potential $U^0 - d^2[\ln \psi_n^0]/dx^2$ and the parameter $E_n = E_n^0$

$+\frac{1}{2}\int_0^t w_n(t')dt'$. Having found the eigenfunction ψ_n, we finally obtain the potential $U(x,t)$ using formula (5.7).

In some cases it is convenient to express ψ_n using the solutions of the Schrödinger equation for the original potential U^0, especially when these solutions are known in explicit form. It can be shown that such a formula can easily be obtained from Eq. (5.10):

$$\varphi_n = \frac{\psi_n^0(\chi^0)_x - (\psi_n^0)_x\chi^0}{\psi_n^0}, \quad E_n \neq E_m, \tag{5.12}$$

where χ^0 is a solution of the Schrödinger equation with the ''old'' potential U^0 and the ''new'' parameter E_n. The same formula was previously obtained in [9] using the Darboux transformation.

An equation specifying the evolution of the potential $U(x,t)$ (more exactly, as before, of the function W_t, where $W_x = U$) can be derived from Eq. (5.8) for the function ψ_n, if we take into account expression (5.1) for the flow:

$$\left[\frac{W_{xt}}{W_t}\right]_t = 2W_t + \left(\frac{w_n(t)}{W_t}\right)_t. \tag{5.13}$$

[It can easily be shown that this expression is an integral of Eq. (2.33), where now E is a function of the deformation parameter t.] Clearly, the equation for the function $\rho = \psi_n\tilde{\psi}_n$ has the same form:

$$\left[\frac{\rho_x}{\rho}\right]_t = 2\rho + \left(\frac{w_n(t)}{\rho}\right)_t. \tag{5.14}$$

Introducing again the function Π: $\exp\Pi = 2W_t$, we can rewrite the equation for the potential in the form of *a generalized Liouville equation*:

$$\Pi_{xt} = \exp\Pi + \left(\frac{1}{2}w_n(t)\exp(-\Pi)\right)_t. \tag{5.15}$$

We stress that if we choose $\tilde{\psi}_n \equiv \psi_n$ [i.e., $w_n(t) = 0$], then all the results obtained in the present section for the extended individual flow (5.1) are reduced to the results obtained in Sec. II for the simple individual flow (2.10) [except for expression (5.12), which was derived under the essential condition $E_n - E_n^0 \neq 0$].

Sequentially building the individual flows of type (5.1) on different eigenstates one can obtain families of potentials with a prescribed set of E_n. Note that a structure of an original potential may be quite arbitrary. Moreover, after every use of individual flow (5.1) shifting a selected state, one can insert flow (2.10), which does not change the spectrum. An order of using flows (5.1) seems to be essential (a question on commutativity of such flows was discussed in [8]).

A further extension of a class of exactly (or almost exactly) solvable potentials can be related with flows involving more than one state simultaneously (not sequentially); call them ''multiflows.'' For example, it is of interest to study ''a double flow'' ($\tau_m = 0$, $m \neq n_1, n_2$), which leads, as has been shown in the paper, to the system of two coupled Liouville equations. If it turns out that this system allows one to construct families of explicit solutions, this would also extend a list of completely (or partially) integrable evolutional equations.

APPENDIX

Let us find an operator that is Hermitian-conjugate to $\hat{\mathbf{S}}_n$. For this purpose we write Eq. (3.1) in the form

$$\psi_m = \psi_m^0 - (e^t - 1)\frac{\psi_n^0}{\theta_n}\int_{-\infty}^{+\infty} dx'\, h(x'-x)\,\psi_n^0(x')\,\psi_m^0(x'), \tag{A1}$$

$$h(\xi) = \begin{cases} 1, & \xi > 0 \\ 0, & \xi < 0. \end{cases} \tag{A2}$$

Then we have

$$\int_{-\infty}^{+\infty} dx\ \psi(x)\hat{\mathbf{S}}_n\psi_m^0(x) = \int_{-\infty}^{+\infty} dx\ \psi(x)\psi_m^0(x) - (e^t - 1) \times \int_{-\infty}^{+\infty} dx\ \frac{\psi_n^0(x)}{\theta_n(x,t)}\psi(x) \times \int_{-\infty}^{+\infty} dx'\, h(x-x') \times \psi_n^0(x')\,\psi_m^0(x'). \tag{A3}$$

Transforming the last term

$$\int_{-\infty}^{+\infty} dx\ \frac{\psi_n^0(x)}{\theta_n(x,t)}\psi(x)\int_{-\infty}^{+\infty} dx'\, h(x-x')\,\psi_n^0(x')\,\psi_m^0(x') = \int_{-\infty}^{+\infty} dx'\,\psi_n^0(x')\,\psi_m^0(x') \times \int_{-\infty}^{+\infty} dx\ \frac{\psi_n^0(x)}{\theta_n(x,t)} h(x-x')\,\psi(x) = \int_{-\infty}^{+\infty} dx'\,\psi_n^0(x')\,\psi_m^0(x')\int_{-\infty}^{x'} dx\ \frac{\psi_n^0(x)}{\theta_n(x,t)}\psi(x), \tag{A4}$$

we get

$$\int_{-\infty}^{+\infty} dx\ \psi(x)\hat{\mathbf{S}}_n\psi_m^0(x) = \int_{-\infty}^{+\infty} dx\ \psi_m^0(x)\,\psi(x) - (e^t - 1) \times \int_{-\infty}^{+\infty} dx\ \psi_m^0(x)\,\psi_n^0(x)\int_{-\infty}^{x} dx'\ \frac{\psi_n^0(x')}{\theta_n(x',t)}\psi(x') \equiv \int_{-\infty}^{+\infty} dx\ \psi_m^0(x)\hat{\mathbf{S}}_n^{\dagger}\psi(x). \tag{A5}$$

Thus, the linear operator $\hat{\mathbf{S}}_n^{\dagger}$ is defined as follows:

$$\hat{\mathbf{S}}_n^\dagger \; f(x) \equiv f(x) - (e^t - 1)\psi_n^0 \int_{-\infty}^{x} dx' \frac{\psi_n^0(x')}{\theta_n(x',t)} \; f(x'). \tag{A6}$$

Now let us construct an inverse operator for $\hat{\mathbf{S}}_n$. Introduce a function $Z(x,t)$: $Z_x = -\psi_n^0(x)\psi_m(x,t)$, $Z^0 = Z(x,0)$ and rewrite Eq. (3.1) as follows:

$$Z_x = Z_x^0 + (e^t - 1)\frac{(\psi_n^0)^2}{\theta_n} \; Z^0 \tag{A7}$$

or, using the definition of the symbol θ_n, in the form

$$Z_x^0 - \frac{(\theta_n)_x}{\theta_n} \; Z^0 = Z_x. \tag{A8}$$

Assuming $Z^0 = C(x,t)\theta_n(x,t)$, we find

$$C(x,t) = -\int_x^{+\infty} dx' \frac{Z_{x'}}{\theta_n(x',t)} + \kappa(t). \tag{A9}$$

Substituting this expression in the previous equation, we get

$$Z_x^0 = \kappa(t)(\theta_n)_x - (\theta_n)_x \int_x^{+\infty} dx' \frac{Z_{x'}}{\theta_n(x',t)}. \tag{A10}$$

Instead of the function Z we use now its original definition; letting

$$\kappa(t) \equiv -\int_{-\infty}^{+\infty} dx' \frac{\psi_n^0}{\theta_n}\psi_m, \tag{A11}$$

we obtain

$$\psi_m^0(x) = \psi_m(x,t) - (e^t - 1)\psi_n^0(x) \times \int_{-\infty}^{x} dx' \frac{\psi_n^0}{\theta_n(x',t)}\psi_m(x',t). \tag{A12}$$

Thus, under the above choice of $\kappa(t)$ we find that $\hat{\mathbf{S}}_n^{-1} = \hat{\mathbf{S}}_n^\dagger$.

[1] P.B. Abraham and H.E. Moses, Phys. Rev. A **22**, 1333 (1980); H.E. Moses and R.T. Prosser, J. Math. Phys. **24**, 2146 (1983).

[2] M. Luban and D.L. Pursey, Phys. Rev. D **33**, 431 (1986); D.L. Pursey, *ibid.* **33**, 1048 (1986); **33**, 2267 (1986); **36**, 1103 (1987).

[3] B.M. Levitan, Matem. Sbornik **132**, 73 (1987) (in Russian).

[4] D.L. Pursey and T.A. Weber, Phys. Rev. A **50**, 4472 (1994); T.A. Weber and D.L. Pursey, *ibid.* **52**, 3923 (1995).

[5] M.M. Nieto, Phys. Lett. B **145**, 208 (1984).

[6] C.V. Sukumar, J. Phys. A **18**, L57 (1985); **18**, 2917 (1985); **21**, L455 (1988).

[7] B. Mielnik, J. Math. Phys. **25**, 3387 (1984).

[8] H.P. McKean and E. Trubowitz, Commun. Math. Phys. **82**, 471 (1982).

[9] J. Poschel and E. Trubowitz, *Inverse Spectral Theory* (Academic, New York, 1987).

[10] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, and V.E. Zakharov, *Theory of Solitons: The Inverse Scattering Method* (Consultants Bureau, New York, 1984).

[11] A.C. Newell, *Solitons in Mathematics and Physics* (SIAM, Philadelphia, 1985).

[12] F. Calogero and A. Degasperis, *Spectral Transform and Solitons* (North-Holland, Amsterdam, 1982), Vol. I.

[13] V.A. Andreev, Teor. Mat. Fiz. **29**, 213 (1976).

[14] V.M. Eleonsky, V.G. Korolev, and N.E. Kulagin, Chaos **4**, 583 (1994); V.M. Eleonsky and V.G. Korolev, J. Phys. A **28**, 4973 (1995); Int. J. Bifurc. Chaos **5**, 1727 (1995).

[15] V.M. Eleonsky and V.G. Korolev, J. Phys. A **29**, L241 (1996).

[16] L.D. Landau and I.M. Lifshitz, *Quantum Mechanics (Nonrelativistic theory)* (Pergamon, Oxford, 1984).

[17] N. Rosen and P.M. Morse, Phys. Rev. **42**, 210 (1932).

[18] V.M. Chabanov and B.N. Zakhariev, Phys. Rev. A **49**, R3159 (1994).

[19] D.J. Fernandez, L.M. Nieto, and O. Rosas-Ortiz, J. Phys. A **28**, 2693 (1995).