Bound states in continuum induced by relativity

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(Received 29 January 1996)

The nonrelativistic reduction of quasipotential equations in QED and Wick-Cutkosky models is considered. It is shown that if we retain an energy dependence of the potential (i.e., make the nonrelativistic reduction only partially), there may arise discrete levels in the continuum for some large critical values of the coupling constant. The fact that the discrete levels disappear in the exact nonrelativistic limit indicates a possible connection with the abnormal solutions of the Bethe-Salpeter equation. Thorough investigation (both analytical and numerical) of the von Neumann–Wigner type potentials is provided. [S1050-2947(96)04612-4]

PACS number(s): 12.20.Ds, 03.65.Ge, 11.10.St

I. INTRODUCTION

Long ago von Neumann and Wigner [1] discovered that a potential decreasing with oscillations at infinity may have discrete levels embedded in the continuum. Further studies revealed that the appearance of such positive eigenvalues in the Schrödinger equation is quite a subtle effect depending on the specific form of the potential [2]. Numerous examples were considered in [3], while in [4] the technique of generating isospectral potentials was applied. Various procedures for constructing bound states embedded in the continuum were compared in [5] using methods based on the Gel'fand-Levitan and Marchenko equations [6]. At present, interest in such potentials has increased due to their applications in the physics of atoms and molecules [7].

Recently, the possibility of an intrinsic connection of such potentials with relativistic physics has been discussed. E.g., in the papers [8,9] of Arbuzov *et al.* a quasipotential equation with a single-photon exchange was employed to study the problem of the narrow e^+e^- resonances discovered in experiments of heavy ion collisions [10]. The unusual spectrum obtained in [8,9] by numerical calculation (see also [11]) caused several responses. In particular, in [12] it was claimed that there can be no analogy between the solutions found in [8,9] and the bound states in continuum (BIC) of the von Neumann–Wigner (VNW) type.

In [13] the Wick-Cutkosky model, and also the ladder approximation for QED and QCD was studied. For positive binding energy ($M>2m$) the Wick's rotation is of no use, so the corresponding Bethe-Salpeter (BS) equation was studied in a pseudo-Euclidean metric. No solutions of the type obtained in [8,9,11] were found. It seems that they were artifacts of the numerical methods applied to the non-Fredholm kernels. Anyway, this question needs further investigation.

The quasipotential method originates from the work of [14]. Later a lot of alternative methods were developed. The main idea of the quasipotential method is elimination of the nonphysical parameter — relative time (or relative energy) from the BS equation. This parameter makes transparent physical interpretation of the BS amplitude difficult. The final product of any quasipotential approach is a Lippman-Schwinger type equation with a potential (quasipotential [14]) depending on the total energy of the system. In general, the potential is a complex, nonlocal function of momenta. Only the on-mass-shell quasipotential becomes local and coincides with the instantaneous kernel of the BS equation.

In [9], and also in [15], it was demonstrated that the nonrelativistic reduction and some additional approximations transform the quasipotential to a local, but energy-dependent expression. The latter is not positive-definite due to overlapping of singularities of the particle propagators and the BS kernel. Below we will see that a sign-changing character of the momentum space quasipotential results in oscillations of the nonrelativistic potential in the coordinate space. These oscillations inspired conjecture that the GSI resonances [10] are solutions of the VNW type potential [9].

II. QUASIPOTENTIAL AND ITS NONRELATIVISTIC FORM

A quasipotential equation is an intermediate stage between the fully covariant BS and nonrelativistic Schrödinger equations. As long as there is an infinite number of ways to eliminate the relative time dependence, there is also an infinite number of corresponding quasipotential equations. But the last step of nonrelativistic reduction leads to the same Schrödinger equation. Hence, below we will concentrate on the simplest example of the quasipotential equation, which is obtained in one-photon exchange approximation for QED with the wave function projected onto the positive frequency states [16,17]

$$2\omega_p[M-2\omega_p]\Psi(\vec{p})=\frac{1}{(2\pi)^3}\int\frac{d\vec{q}}{2\omega_q}V(M;\vec{p},\vec{q})\Psi(\vec{q}). \tag{1}$$

Here $\omega_p\equiv\sqrt{\vec{p}^2+m^2}$, the center-of-mass system is assumed and the fermions are taken with equal masses. The quasipotential V depends on the total energy M and in $O(e^2)$ approximation has the form

$$V(M;\vec{p},\vec{q})=\frac{(2me)^2}{|\vec{p}-\vec{q}|(M-\omega_p-\omega_q-|\vec{p}-\vec{q}|+io)}. \tag{2}$$

It describes the interaction between opposite sign charges. In the case of the same sign charges one must substitute $\alpha=e^2/4\pi\rightarrow-\alpha$. On mass shell $M=2\omega_p=2\omega_q$ the quasipotential reduces to the nonrelativistic Coulomb potential. If we

consider the approximation of small momenta $|\vec{p}|,|\vec{q}| \ll m$, then the quasipotential (2) takes a local form

$$V(M;\vec{p},\vec{q}) = \frac{(2me)^2}{|\vec{p}-\vec{q}|(\mathcal{E}-|\vec{p}-\vec{q}|+io)}. \tag{3}$$

Here $\mathcal{E}=M-2m$ is a binding energy. The same quasipotential is obtained within the Wick-Cutkosky model (massive scalar particles interacting via massless scalar exchange) in the same approximation. The only difference is that one must substitute the coupling $\lambda \equiv g_1 g_2/16\pi^2$ instead of $m^2\alpha/\pi$. For ''ordinary'' bound states (when $M<2m$ and hence $\mathcal{E}<0$) quasipotential (2) as well as potential (3) and the left-hand sides of the corresponding equations never change sign. In that case we have to deal with the Fredholm-type equations which can be studied by standard methods. If for $\mathcal{E}<0$ we neglect $\mathcal{E}$ in the denominator of Eq. (3) we face the usual Coulomb problem. Preserving $\mathcal{E}$ leads to an energy-dependent potential. Of course it is a trace of relativity: dependence over the relative time has transformed into the energy dependence of the potential. It leads to existence of extra solutions, which in our opinion correspond to the abnormal ones of the BS equation.

When $\mathcal{E}>0$, potential (3) is singular [like quasipotential (2)] and taking the final nonrelativistic limit $\mathcal{E}\to 0$ is problematic even for the scattering problem.

So in the approximation considered above we have to deal with Schrödinger equation in the momentum space [9,15]

$$\left(\mathcal{E}-\frac{\vec{p}^2}{m}\right)\Psi_{\mathcal{E}}(\vec{p}) = \frac{\alpha}{2\pi^2}\int d\vec{q}\,\frac{\Psi_{\mathcal{E}}(\vec{q})}{|\vec{p}-\vec{q}|(\mathcal{E}-|\vec{p}-\vec{q}|+io)}. \tag{4}$$

This equation is much easier to study than the initial quasipotential one because the potential here is local and employing the Fourier transformation we can pass to the following differential equation in the coordinate space:

$$\left(\frac{\nabla^2}{m}+\mathcal{E}\right)\Psi_{\mathcal{E}}(\vec{r}) = V(r)\Psi_{\mathcal{E}}(\vec{r}), \tag{5}$$

where for $\mathcal{E}>0$ potential has the form [9,15]

$$V \equiv V_{\mathcal{E}}(r) = \frac{2\alpha}{\pi r}[f(\mathcal{E}r) - \pi e^{i\mathcal{E}r}]. \tag{6}$$

Here

$$f(\mathcal{E}r) \equiv \mathrm{Ci}(\mathcal{E}r)\sin(\mathcal{E}r) - \mathrm{si}(\mathcal{E}r)\cos(\mathcal{E}r) \tag{7}$$

and $\mathrm{Ci}(x)$ and $\mathrm{si}(x)$ are the integral cosine and sine, respectively.

If the binding energy were negative $\mathcal{E}=-E<0$ then

$$V_E(r) = -\frac{2\alpha}{\pi}\frac{f(Er)}{r}. \tag{8}$$

It is worth noting that despite the trigonometric functions, $f(r)$ is quite smooth (see Fig. 1). It has the following asymptotics [18]:

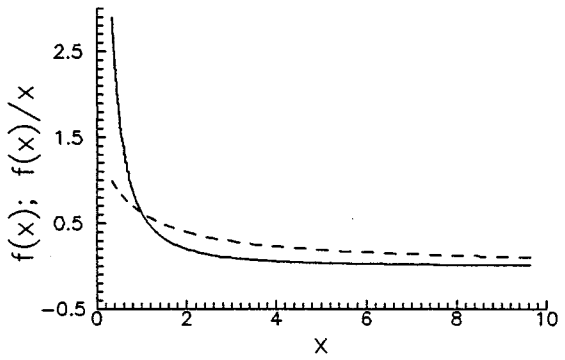

FIG. 1. Functions $f(x)$ (7) (dashed line) and $f(x)/x$ (solid line) which enter potential (6).

$$f(x) \approx \frac{\pi}{2} + x\ln x + (\gamma-1)x - \frac{\pi}{4}x^2 + O(x^3); \quad x \ll 1, \tag{9}$$

$$f(x) \approx \frac{1}{x} + \left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots\right); \quad x \gg 1. \tag{10}$$

Here $\gamma = 0.5772\ldots$ is Euler's constant. Hence for $\mathcal{E}<0$ potential (8) behaves like [17]

$$V_E(r)|_{Er\ll 1} \sim -\frac{\alpha}{r}\left[1 + \frac{2}{\pi}Er\ln(Er) + \frac{2}{\pi}(\gamma-1)Er - \frac{\pi}{4}(Er)^2\right], \tag{11}$$

$$V_E(r)|_{Er\gg 1} \sim -\frac{2\alpha}{Er^2}. \tag{12}$$

According to Eq. (9) when $E=0$ the potential is Coulombic

$$V_{E=0}(r) = -\frac{\alpha}{r}. \tag{13}$$

$V_E(r)$ asymptotically decreases as an inverse square, hence, under some restrictions on α, there will be finite number of eigenvalues [19].

Let us get back to the $\mathcal{E}>0$ case and note that potential (6) in this case is complex, leading to nonconservation of the energy in the scattering process. This absurdity originates from the Feynman io prescription in the denominator of quasipotential (2). This point was criticized by Zastavenko [12]. He proposed the use of the principal value prescription, which is equivalent to neglecting the imaginary part in Eq. (6) or, as it was admitted in [8,9], to neglecting widths of possible resonances. So accepting the principal value prescription, for positive binding energies we have the following potential:

$$V_{\mathcal{E}}(r) = \frac{2\alpha}{\pi r}[f(\mathcal{E}r) - \pi\cos(\mathcal{E}r)]. \tag{14}$$

According to Eq. (10), this potential decreases at infinity with oscillations

$$V_{\mathcal{E}}(r)|_{\mathcal{E}r\gg 1}\sim -\frac{2\alpha}{r}\cos(\mathcal{E}r); \tag{15}$$

while at the origin we have

$$V_{\mathcal{E}}(r)|_{\mathcal{E}r\ll 1}\sim -\frac{\alpha}{r}\left[1-\frac{2}{\pi}\mathcal{E}r\ln(\mathcal{E}r)+\frac{2}{\pi}(1-\gamma)\mathcal{E}r+\cdots\right]. \tag{16}$$

All of the terms in the parentheses in the last equation are positive, i.e., here relativistic corrections to the pure Coulomb potential pull it to the opposite side compared to the case $\mathcal{E}<0$ [see Eq. (11)].

Potential (14) behaves asymptotically quite like the VNW type potential [1–3]. Hence, it is natural to ask whether Eq. (14) allows for discrete levels embedded in the continuum. To answer this question let us recall Atkinson's theorem from the Reed and Simon book [2].

Theorem XI.67 (c). Let the potential have the form

$$V(r)=\sum_{j=1}^{M}\gamma_j\frac{\sin(\sigma_j r)}{r}+Q(r), \tag{17}$$

where the function $Q(r)$ satisfies

$$|Q(r)|\leqslant C(1+r^2)^{-1/2-\tau} \tag{18}$$

for some positive C and τ. Suppose that $k=\sigma_j/2$ for some j. Then there is a solution $u(r)$ of the equation

$$-\phi''+V\phi=k^2\phi,$$

which asymptotically behaves as follows:

$$u(r)=r^{-(\gamma_j/2\sigma_j)}\left[\cos\left(\frac{\sigma_j r}{2}\right)+O(1)\right],\quad \gamma_j/\sigma_j>0,$$

$$u(r)=r^{(\gamma_j/2\sigma_j)}\left[\sin\left(\frac{\sigma_j r}{2}\right]+O(1)\right),\quad \gamma_j/\sigma_j<0. \tag{19}$$

The essence of the theorem is that when $|\gamma_j|>\sigma_j$ the Schrödinger equation may have a square-integrable solution, i.e., there can appear normalizable solutions at the positive binding energies [2].

Comparing Eq. (17) with Eq. (14) we find the difference in the phase of the trigonometric functions, but as long as the phase of asymptotic oscillations can be changed simply by shifting the coordinate r, we foresee no problems in generalizing the theorem for this case, too. Below we will generalize the VNW procedure and construct an explicit example of a potential decreasing with cosine oscillations and having a discrete level in the continuum.

III. EXAMPLE OF THE VNW TYPE POTENTIAL WITH COSINE OSCILLATIONS

For simplicity let us consider an S-wave Schrödinger equation for the radial function

$$\chi''+\frac{2m}{\hbar^2}[\mathcal{E}-U(r)]\chi=0. \tag{20}$$

Denoting

$$k^2\equiv\frac{2m\mathcal{E}}{\hbar^2},\quad V(r)\equiv\frac{2mU(r)}{\hbar^2} \tag{21}$$

we express the potential

$$V(r)=k^2+\frac{\chi''}{\chi}. \tag{22}$$

The VNW procedure consists of decomposing χ [1,3] in the following manner:

$$\chi(r)=\chi_0(r)f(r), \tag{23}$$

where χ_0 is a solution of some (usually free) Schrödinger equation, while $f(r)$ is the so-called modulating function. In the VNW approach the boundary condition at the origin is satisfied by taking $\chi_0(0)=0$. However, to construct a potential with cosinelike (or arbitrary phase) oscillations at infinity we start with

$$\chi_0=\frac{1}{k}\sin(kr+\phi). \tag{24}$$

Then

$$V(r)=\frac{f''}{f}+2k\frac{f'}{f}ctg(kr+\phi). \tag{25}$$

Now we must choose $f(r)$ that will simultaneously ensure the boundary condition at the origin [$\chi(r\to 0)\sim r$], cancel the poles of the $ctg(kr+\phi)$ and produce a potential vanishing at infinity. For that purpose let us define the standard variable [1,3]

$$s(r)=k\int_0^r dz\sin^2(kz+\phi)$$

$$=\frac{kr}{2}-\frac{1}{4}\{\sin[2(kr+\phi)]-\sin(2\phi)\}. \tag{26}$$

So if $\phi\neq 0$ $s(r)\sim kr\sin^2\phi$ for $r\to 0$ and $s(r)\sim\frac{1}{2}kr$ for $r\to\infty$. Consequently, if we take, for instance,

$$f(r)=\frac{s(r)}{A+s^2(r)},\quad A=\text{const}, \tag{27}$$

the above-mentioned requirements will be satisfied, leading to a square-integrable solution $\chi(r)$.

Inserting Eqs. (26) and (27) into Eq. (25) we find

$$V(r)=\frac{A-s^2}{A+s^2}\frac{s''}{s}-2s'^2\frac{3A-s^2}{(A+s^2)^2}+\frac{k^2}{s}\frac{A-s^2}{A+s^2}$$

$$\times\sin[2(kr+\phi)]. \tag{28}$$

Evidently, we will have the desired cosinelike oscillations if $\phi=\pi/4$. Using explicit expressions (below we will simply set $A=1$)

$$\chi_0=\frac{1}{k}\sin\left(kr+\frac{\pi}{4}\right),$$

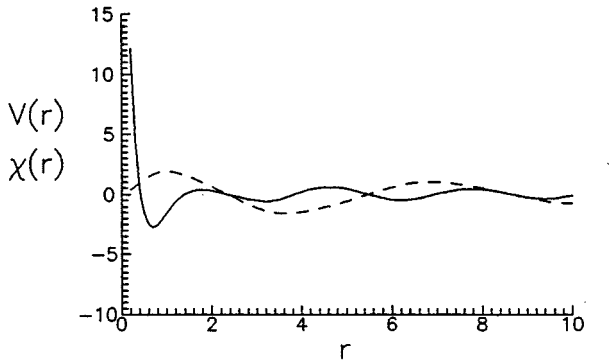

FIG. 2. Potential with cosinelike oscillations (solid line) with its analytically known BIC solution (31) (dashed line). The wave function is not normalized to the unity.

$$s(r)=\frac{kr}{2}-\frac{1}{4}[\cos(2kr)-1],$$

$$s'(r)=\frac{k}{2}[1+\sin(2kr)],$$

$$s''(r)=k^2\cos(2kr),$$

it is straightforward to check that

$$V(r)\to-\frac{4k}{r}\cos(2kr),\quad kr\gg 1, \tag{29}$$

while at the origin

$$V(r)\to\frac{4k}{r},\quad kr\ll 1. \tag{30}$$

Figure 2 shows the potential $V(r)$ and the corresponding wave function

$$\chi(r)=\frac{1}{k}\sin\left(kr+\frac{\pi}{4}\right)\frac{s(r)}{1+s^2(r)}. \tag{31}$$

The latter decreases like r^{-1}. Obviously the asymptotics of Eq. (31) agrees with the Atkinson's theorem. Indeed, in our case [compare to Eq. (19)] $\gamma=-4k$, $\sigma=2k$ and so $\gamma/2\sigma=-1$.

Of course we could construct other examples by different choices of the modulating function $f(r)$ with various behavior at infinity, e.g.,

$$f(r)=\frac{s(r)}{[A+s^2(r)]^n}\to r^{1-2n}. \tag{32}$$

It is easy to establish that the leading asymptotics of the corresponding potential will be

$$V\sim\frac{4k(1-2n)}{r}\cos(2kr).$$

Again this asymptotic behavior agrees with Eq. (19) — $\gamma/2\sigma=1-2n$.

Therefore using the VNW procedure one can construct explicit examples showing that the Atkinson's theorem works in the case of oscillations with an arbitrary phase, too. The only difference is that the correct boundary condition at the origin is provided by the modulating function $f(r)$.

IV. NUMERICAL SOLUTIONS

Let us assume that the Atkinson's theorem is valid for our potential (14) with cosinelike oscillations and rewrite the potential in the form similar to Eq. (17). Obviously we have to put

$$\gamma=-2m\alpha,\quad \sigma=\mathcal{E},\quad k^2=m\mathcal{E}. \tag{33}$$

So according to the theorem a discrete level may emerge only at

$$\mathcal{E}=4m,\quad \alpha>2, \tag{34}$$

i.e., the coupling must be quite large.

It follows that the levels found in [8,11] by numerical methods cannot be of the VNW type because they were found for $\alpha=1/137$. Below we are going to present our method of numerical study of such potentials together with the results obtained. The method we will employ is quite similar to that of searching for ordinary levels described, e.g., in [20]. The latter consists of calculating the large distance behavior of a solution of Schrödinger differential equation starting from relevant boundary conditions at the origin. If this behavior changes discontinuously [i.e., asymptotics of the solution change from $+\exp(\sqrt{|E|}r)$ to $-\exp(\sqrt{|E|}r)$ or vice versa; note that we have set $m=1$] within some small interval of energy, it means that there is a level located somewhere in between. In our case the quantity that will change discontinuously is a phase of the solution. Indeed, when the scattering energy passes the resonance energy, the phase of the scattered wave changes by π [21] and the range ΔE characterizes the width of the resonance. Discrete levels are, in fact, resonances with vanishing widths [3,11] and there must be a jump by π in the phase-energy dependence $\delta(E)$. It is understood that one cannot distinguish between the very small and exactly zero quantities using numerical methods. However, the Atkinson's theorem serves as a guide — when we find a resonance, its energy allows us to determine whether it is a BIC or not without actual calculation of the width. There is also an additional advantage in using this method: as a side product we will find also the ordinary resonance solutions of the equation. As we will see below, this will help us to understand what becomes of BICs when the potential is slightly deformed.

Now let us explain how the mentioned jumps in phase can be detected. Suppose that the resonance energy is somewhere within the interval $E\pm\Delta E$. Then asymptotically we have

$$\Psi_{E-\Delta E}\sim\exp(i\sqrt{E-\Delta E}r+i\delta),$$

$$\Psi_{E+\Delta E}\sim\exp[i\sqrt{E+\Delta E}r+i(\delta+\pi)]|_{\Delta Er\ll 1}\to-\Psi_{E-\Delta E}.$$

It means that if we plot the numerical solutions for $E\pm\Delta E$, then in the interval where the solutions have already reached their asymptotic regime, but ΔEr is still small, we

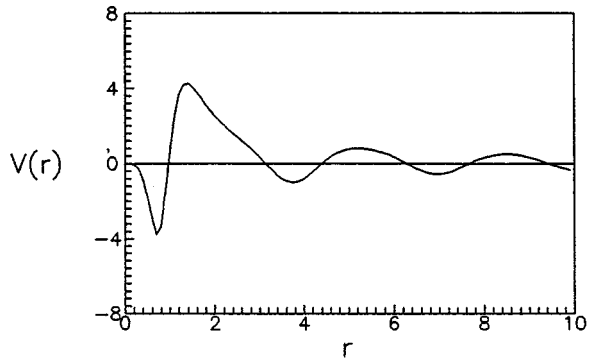

FIG. 3. The von Neumann-Wigner potential [1,3].

must see mirror reflected behavior. For narrow resonances (including BICs) these conditions can be satisfied because ΔE can always be chosen small enough to make $\Delta E r \ll 1$ for large r. For an illustration we will use the original VNW potential [1,3] (see Fig. 3). Solutions for different energies around the analytically known BIC solution are depicted in Fig. 4. Their mirror reflected behavior is transparent. A similar picture will be obtained for arbitrarily small ΔE. While reducing ΔE these solutions coincide with the BIC solution in the increasing interval of r — so the BIC solution can be obtained numerically with a required precision by taking sufficiently small ΔE. Now let us slightly deform the VNW potential by multiplying it by a factor μ and setting μ slightly different from the unity. Presumably even small deformation of the potential must lead to the transformation of the BIC into a resonance. Again, we find the familiar mirror reflected picture but the energy is different than required by the Atkinson's theorem! So even being unable to distinguish between small and exactly zero widths by numerical methods we can conclude that small deformation of the potential has implied transformation of the BIC into a resonance. In fact, near the BIC solution the change of energy of the resonance (when the coupling is varied) turned out larger than the change of the width. So the former can be detected with higher accuracy than the latter. If we increase μ, the resonance energy tends to zero, having a small width, but if we

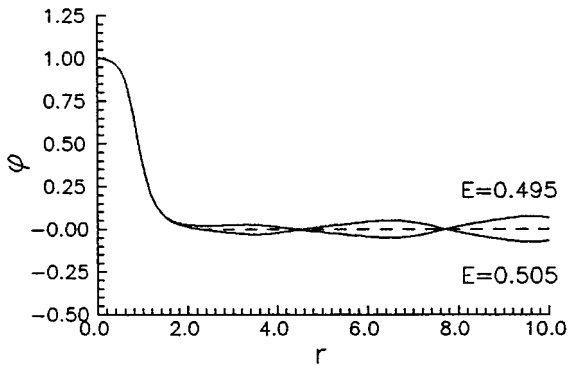

FIG. 4. The analytical BIC solution of the VNW potential (dashed line) and adjacent solutions for two different values of E (solid lines). The mirror reflected character of asymptotics of the solid lines is evident.

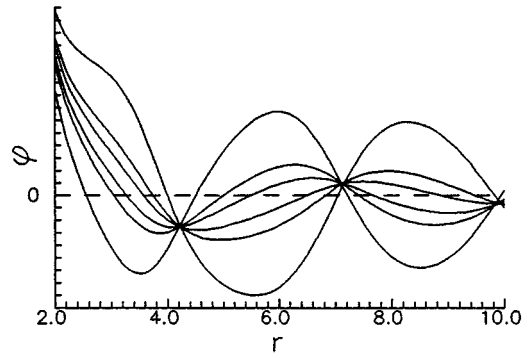

FIG. 5. Several solutions for the VNW potential (which is scaled by a factor $\mu = 0.7$) near the resonance energy within the interval $0.8 \leqslant E \leqslant 0.9$. Unlike Fig. 4 there is no mirror reflected picture, though the phase changes rapidly. Some ΔE solutions have almost mirror reflected asymptotics (change of phase by π), but further reduction of ΔE reveals that these solutions deform into each other continuously.

reduce it, the energy starts to grow. In the latter case resonance becomes broader, and starting from some value of μ (depending, of course, on the accuracy of calculations) we can notice that the familiar mirror reflected picture starts to distort. Such a distorted picture is displayed in Fig. 5. Six numerical solutions with energies in the interval $0.8 \geqslant E \leqslant 0.9$ continuously deform into each other. The first ($E = 0.8$) and the last ($E = 0.9$) ones have opposite phases, but apparently the change of phase proceeds continuously. Further increase of μ results in the further growth of the energy, but simultaneously the width also increases, the resonance becomes broader and, finally, dissolves. At some critical values of μ there again arise BIC solutions and changing μ slightly around these critical values leads to the similar dependence of the resonance energy and width on μ. Dependence of the resonance energy over μ for the VNW potential and for potential (14) is quite similar and, for the latter, is depicted in Fig. 6. Note that for a fixed value of μ there is only one resonance solution.

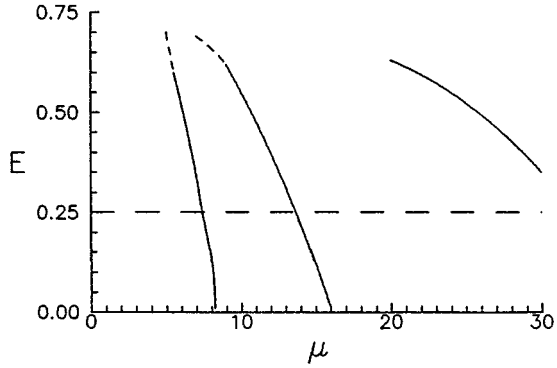

FIG. 6. Dependence of the resonance energy (E) on the coupling μ (35) for potential (14). Dashed parts of the resonance branches correspond to the region where the width of a resonance begins to grow, and finally the resonance dissolves. BICs are located at the interception of the depicted branches with the horizontal dashed line (at $E = 0.25$).

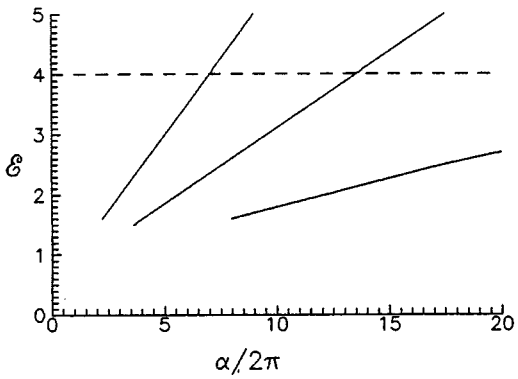

FIG. 7. Dependence of a resonance energy ($\mathcal{E}$) on the coupling α. Clearly, resonances arise one by one when the coupling is increased, so that for a fixed value of the coupling we can have more then one, but always finite number of resonances. BICs at the interceptions with the horizontal dashed line at $\mathcal{E}=\Delta$.

Up to this moment we have discussed the spectrum of the standard Schrödinger equation with the VNW type potential. However, potential (14) is energy dependent. So let us denote

$$x\equiv\mathcal{E}r, \quad \mu\equiv\frac{2\alpha}{\pi\mathcal{E}}, \quad E\equiv\frac{1}{\mathcal{E}} \tag{35}$$

and, for a while, consider μ and E as independent parameters. Then the S-wave version of Eq. (5) with potential (14) can be written in the standard form (we take $m=1$)

$$-\Psi''(x)+\mu V(x)\Psi=E\Psi(x), \tag{36}$$

with

$$V(x)=[f(x)-\pi\cos(x)]/x.$$

Figure 6 shows the resonance branches that we have found by varying the coupling μ. After restoring original parameters, the picture becomes more interesting (see Fig. 7). Again, at some discrete, and rather large [which agree with the estimate (34)] critical values of the coupling constant we have BICs which transform into resonances as soon as we deviate the coupling from the critical value. But now for a fixed value of the coupling there may be several resonances. In fact, as the coupling increases, more and more resonances appear.

V. CONCLUSIONS

The quasipotential method, which is a rather powerful approach in relativistic physics, shows that in many models the simplest one-massless-particle exchange quasipotential for positive binding energies in the small momenta approximation after discarding the imaginary part reduces to the potential that decreases at spatial infinity with cosinelike oscillations [9,15]. As they are damped by the first power of the coordinate this potential imitates the famous von Neumann and Wigner problem [1]. The origin of these oscillations is relativity — pseudoeuclidity of metrics.

It was manifestly shown above that the phase of the potential's oscillations has no decisive importance for the VNW procedure, because the boundary conditions at the origin and infinity can be satisfied by a suitable choice of the modulating function. All relevant results including the Atkinson's theorem remain valid. This theorem gives some indication about the energy of discrete levels embedded in the continuum and the allowed areas of the coupling constant as well. In general, for the pure VNW-type potentials the BICs appear at arbitrary energies, but in our example the discrete level energy has quite definite value and the coupling constant is bounded from below thanks to the above-mentioned theorem. In full agreement with theoretical predictions the numerical method described in Sec. IV confirms existence of the BIC at the indicated point and allows us to find corresponding values of the coupling constant, which happen to be quite large ($\alpha\sim 45;80;\ldots$).

The numerical results show unambiguously that the hypothesis about the relation between the GSI resonances and the VNW-type potentials is not realistic. Such a conclusion was made also in [12] on different grounds. As for Eq. (1), additional investigation is desirable due to the nonlocality of quasipotential (2) and non-Fredholmity of its kernel.

ACKNOWLEDGMENTS

It is a pleasure to thank A. Kvinikhidze and A. Lahiff for reading the manuscript. We would also like to thank A. Pagnamenta and U. Sukhatme and all participants of the Particle Theory Seminar at the University Illinois of Chicago for many helpful discussions.

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