Analytic contribution of order $\alpha^2(Z\alpha)^5 m$ to the Lamb shift

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The analytic result for the radiative correction of order $\alpha^2(Z\alpha)^5 m$ to the Lamb shift connected with a polarization insertion in one of the two external Coulomb lines is obtained. This correction arises from a gauge-invariant set of diagrams which contain, besides the polarization insertion in the Coulomb leg, all one-loop radiative photon insertions in the electron line with two external Coulomb lines. [S1050-2947(97)09203-7]

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Theoretical work on high-order corrections to hyperfine splitting (HFS) and the Lamb shift in hydrogenlike ions was concentrated recently on calculation of all nonrecoil contributions of order $\alpha^2(Z\alpha)^5$. This work is now successfully completed, and all calculations for the hyperfine splitting [1–6] and for the Lamb shift [7–11,4,12,6] were performed independently by two different groups, and the results of these calculations are in excellent agreement.

Still there exists a certain asymmetry between the results for HFS and Lamb shift with respect to our knowledge of analytical results. For both HFS and Lamb shift, contributions of the order $\alpha^2(Z\alpha)^5$ are induced by six gauge-invariant sets of diagrams in Fig. 1, which correspond to different radiative corrections to the skeleton graph with two Coulomb photons attached to the electron line [1,7]. While corrections to HFS induced by the first three sets of diagrams in Figs. 1(a)–1(c) are known in analytic form [1,2], in the case of the Lamb shift analytic results exist only for the first two sets of diagrams in Fig. 1 [13]. It is the aim of the present paper to obtain the analytic result for the contribution to the Lamb shift induced by the gauge-invariant set of diagrams in Fig. 1(c), containing insertions of one radiative photon in the electron line and simultaneous insertion of a one-loop polarization operator in one of the Coulomb lines. This would imply that contributions of the diagrams in Figs. 1(a)–1(c) are then known analytically for both HFS and the Lamb shift.

As was discussed at length in Ref. [7] the respective contribution to the Lamb shift of S states is given by the expression

$$\Delta E = -\frac{\alpha^2(Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{32}{\pi^2}\int_0^\infty dk I(k) L(k), \quad (1)$$

where

$$I_1(k) = \int_0^1 dv \frac{v^2(1-v^2/3)}{4+(1-v^2)k^2} \quad (2)$$

describes the Coulomb line with one polarization insertion, $L(k)$ corresponds to the sum of radiative corrections to the electron line in Fig. 1, and $k = |\mathbf{k}|$ is the magnitude of the

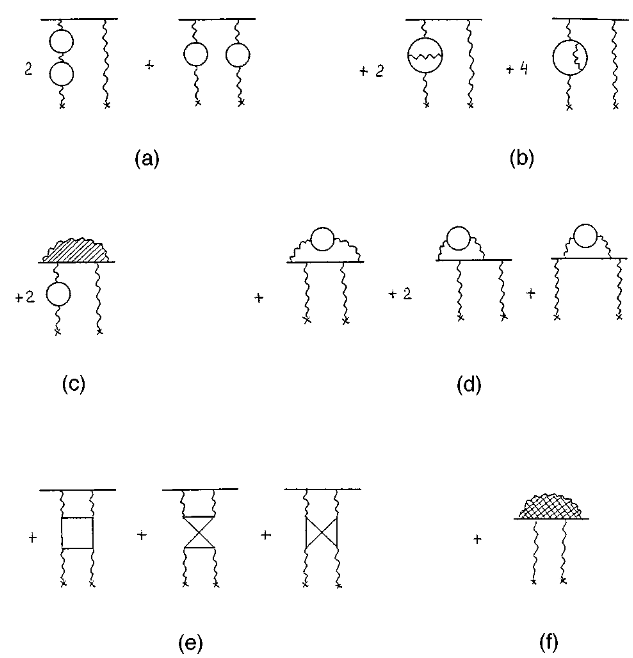

FIG. 1. Six gauge-invariant sets of graphs producing corrections of order $\alpha^2(Z\alpha)^5$.

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spatial momentum of the external Coulomb photons measured in units of electron mass.[1]

Explicit expressions for the electron factor $L(k)$ with on-mass-shell external electron lines in Eq. (1) were obtained earlier in [14,8,9] in the form of apparently different two-dimensional integrals over the Feynman parameters. Of course, one may prove that all these expressions coincide.

For our present goals we need an expression for the electron factor, not in the form of the parametric integral, but as an explicit function of the exchanged momentum. We have started with the parametric expression in [9], and performed all parametric integrations explicitly. The initial expression for the electron factor in [9] contains an auxiliary infrared regularization parameter λ, which was needed to make separate parametric integrals well defined. It is well known that the total expression for the electron factor is infrared finite and should not depend on the infrared photon mass λ. In accordance with this general statement the result of our parametric integration is nonsingular in λ and admits the limit of vanishing λ. Explicitly, after tedious calculations we obtain

$$\begin{aligned}L(k)=&-\frac{1}{4}+\frac{1}{2}\ln k^2+\frac{1}{8}\,\frac{k^2}{1-k^2}\ln k^2-\frac{\sqrt{k^2+4}}{2k}\ln\frac{\sqrt{k^2+4}+k}{\sqrt{k^2+4}-k}\\&+\frac{1}{k\sqrt{k^2+4}}\ln\frac{\sqrt{k^2+4}+k}{\sqrt{k^2+4}-k}\\&-3\left[\frac{1}{k^2}-\frac{\sqrt{k^2+4}}{2k^3}\ln\frac{\sqrt{k^2+4}+k}{\sqrt{k^2+4}-k}\right]\\&+\frac{k}{8}\Phi(k)+\frac{1}{2k}\Phi(k)-\frac{2}{k^2}\left[\frac{1}{k}\Phi(k)+\ln k^2-1\right]\\\equiv&\sum_{i=1}^{i=9}L_i(k),\end{aligned}\tag{3}$$

where

$$\Phi(k)=k\int_0^1\frac{dz}{1-k^2z^2}\ln\frac{1+k^2z(1-z)}{k^2z}.\tag{4}$$

This last integral in the definition of function $\Phi(k)$ may be also calculated in closed form in terms of dilogarithms, but the integral representation is more convenient for further calculations.

The expression for the spin-independent electron factor relevant for the Lamb shift calculation in Eq. (3), has the same general structure as the respective expression for the spin-flip electron factor relevant for the calculation of the contribution to HFS [1]. The same auxiliary function $\Phi(k)$ emerged also in the case of HFS.

It is easy to obtain high- and low-frequency asymptotes of the electron factor in Eq. (3),

$$L(k)_{k\to 0}=\frac{2}{3}\ln k^2-\frac{5}{9},\tag{5}$$

$$L(k)_{k\to\infty}=-\frac{1}{k^2}\left(\frac{1}{3}\ln k^2+\frac{35}{36}\right).$$

For analytic calculation we also need the explicit expression for the polarization operator as the function of the exchanged momentum

$$\begin{aligned}I_1(k)=&\frac{\sqrt{k^2+4}}{k^3}\left[\frac{1}{2}\ln\frac{\sqrt{k^2+4}+k}{\sqrt{k^2+4}-k}-\frac{k}{\sqrt{k^2+4}}\right]-\frac{1}{3}\,\frac{(k^2+4)^{3/2}}{k^5}\\&\times\left[\frac{1}{2}\ln\frac{\sqrt{k^2+4}+k}{\sqrt{k^2+4}-k}-\frac{k}{\sqrt{k^2+4}}-\frac{1}{3}\,\frac{k^3}{(k^2+4)^{3/2}}\right]\\\equiv&I_{11}(k)+I_{12}(k).\end{aligned}\tag{6}$$

Then the contribution to the Lamb shift that is induced by the diagrams in Fig. 1(c) may be written in the form

$$\begin{aligned}\Delta E=&-\frac{\alpha^2(Z\alpha)^5}{\pi n^3}m\left(\frac{m_r}{m}\right)^3\frac{32}{\pi^2}\int_0^\infty dk\sum_{i=1}^{i=9}\sum_{j=1}^{j=2}L_i(k)I_{1j}(k)\\\equiv&\frac{\alpha^2(Z\alpha)^5}{\pi n^3}m\left(\frac{m_r}{m}\right)^3 32\sum_{i=1}^{i=9}\sum_{j=1}^{j=2}\delta\epsilon_{ij}\,.\end{aligned}\tag{7}$$

Calculation of the separate integrals for the terms $\delta\epsilon_{ij}$ is performed directly. We always avoid direct use of the function $\Phi(k)$ with the help of integrating by parts, exploiting the fact that the derivative of $\Phi(k)$ may be easily calculated in terms of elementary functions

$$\Phi'(k)=-\frac{\ln k^2}{1-k^2}-\frac{2}{k\sqrt{k^2+4}}\ln\frac{\sqrt{k^2+4}+k}{\sqrt{k^2+4}-k}-\frac{k}{\sqrt{k^2+4}}.\tag{8}$$

Explicit results for the separate terms on the right-hand side in Eq. (7) are collected in Table I. In the process of calculation two auxiliary integrals

$$\int_0^\infty\frac{k\,dk}{(1-k^2)\sqrt{k^2+4}}\ln k\,\ln\frac{\sqrt{k^2+4}+k}{2}=-\frac{\pi^2}{\sqrt{5}}\ln\frac{1+\sqrt{5}}{2},\tag{9}$$

$$\int_0^\infty\frac{dk}{(1-k^2)}\ln k\,\ln^2\frac{\sqrt{k^2+4}+k}{2}=-\pi^2\left(\ln^2\frac{1+\sqrt{5}}{2}+\frac{\pi^2}{24}\right),$$

were used. These integrals were calculated in [1], and, to the best of our knowledge, do not appear in the mathematical handbooks.

Collecting all terms in Table I we obtain the analytic expression for the contribution to the Lamb shift induced by the gauge-invariant set of diagrams in Fig. 1(c),

$$\begin{aligned}\Delta E=&\left(\frac{8}{3}\ln^2\frac{1+\sqrt{5}}{2}-\frac{872}{63}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}+\frac{628}{63}\ln 2-\frac{2\pi^2}{9}\right.\\&\left.+\frac{67282}{6615}\right)\frac{\alpha^2(Z\alpha)^5}{\pi n^3}m\left(\frac{m_r}{m}\right)^3.\end{aligned}\tag{10}$$

The numerical value of the factor in the parenthesis nicely coincides with the previous numerical results [8,10] within the accuracy of those numerical results.

[1]All integration momenta below are measured in units of electron mass.

TABLE I. Individual contributions to the Lamb shift.

$\delta\epsilon_{11}$	$\frac{1}{64}$
$\delta\epsilon_{12}$	$-\frac{1}{256}$
$\delta\epsilon_{21}$	$-\frac{1}{8}\ln 2-\frac{1}{32}$
$\delta\epsilon_{22}$	$\frac{1}{32}\ln 2+\frac{3}{256}$
$\delta\epsilon_{31}$	$\frac{1}{4}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}-\frac{1}{4}\ln 2-\frac{1}{16}$
$\delta\epsilon_{32}$	$-\frac{5}{12}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}+\frac{11}{24}\ln 2+\frac{35}{288}$
$\delta\epsilon_{41}$	$\frac{7}{48}$
$\delta\epsilon_{42}$	$-\frac{113}{2880}$
$\delta\epsilon_{51}$	$-\frac{1}{96}$
$\delta\epsilon_{52}$	$\frac{13}{5760}$
$\delta\epsilon_{61}$	$-\frac{1}{160}$
$\delta\epsilon_{62}$	$\frac{37}{26880}$
$\delta\epsilon_{71}$	$\frac{1}{8}\ln^2\frac{1+\sqrt{5}}{2}-\frac{1}{4}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}+\frac{1}{4}\ln 2-\frac{\pi^2}{96}+\frac{1}{8}$
$\delta\epsilon_{72}$	$-\frac{1}{24}\ln^2\frac{1+\sqrt{5}}{2}+\frac{2}{9}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}-\frac{17}{72}\ln 2+\frac{\pi^2}{288}-\frac{19}{216}$
$\delta\epsilon_{81}$	$-\frac{5}{12}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}+\frac{5}{12}\ln 2+\frac{5}{36}$
$\delta\epsilon_{82}$	$\frac{5}{12}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}-\frac{73}{160}\ln 2-\frac{7361}{57600}$
$\delta\epsilon_{91}$	$\frac{5}{6}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}-\frac{59}{60}\ln 2-\frac{2677}{14400}$
$\delta\epsilon_{92}$	$-\frac{15}{14}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}+\frac{1013}{840}\ln 2+\frac{219329}{705600}$
$\delta\epsilon_{tot}$	$\frac{1}{12}\ln^2\frac{1+\sqrt{5}}{2}-\frac{109}{252}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}+\frac{157}{504}\ln 2-\frac{\pi^2}{144}+\frac{33641}{105840}$

We would like to mention that exactly the same characteristic combinations of logarithms and square roots of 5 emerged earlier in the contribution of the same diagram to HFS. The result above differs from the respective contribution to HFS only by the values of the factors before these characteristic structures.

We are now able to give the state of the art expression for the total contribution to the Lamb shift of order $\alpha^2(Z\alpha)^5$, combining the analytic results for the diagrams in Figs. 1(a)–1(b) [13], Fig. 1(c) (this work), and the most precise numerical results for the other diagrams [6,10]

$$\Delta E_{tot}=\left(\frac{8}{3}\ln^2\frac{1+\sqrt{5}}{2}-\frac{872}{63}\sqrt{5}\ln\frac{1+\sqrt{5}}{2}+\frac{680}{63}\ln 2-\frac{2\pi^2}{9}-\frac{25\pi}{63}+\frac{24901}{2205}-7.920(1)\right)\frac{\alpha^2(Z\alpha)^5}{\pi n^3}m\left(\frac{m_r}{m}\right)^3$$

$$=-6.861(1)\frac{\alpha^2(Z\alpha)^5}{\pi n^3}m\left(\frac{m_r}{m}\right)^3. \qquad (11)$$

The phenomenological consequences of this result were discussed in detail, e.g., in [6], and we will not reproduce the discussion here. On the theoretical side we would like to mention that the analytic result above nicely confirms qualitative arguments presented in [6] about the natural scale of the order $\alpha^2(Z\alpha)^5$ contributions to the Lamb shift.

We are convinced that at the current state of the art the diagrams in Figs. 1(d)–1(f) do not admit analytic calculation for either the case of HFS or the case of the Lamb shift. In this sense we believe that the semianalytic result in Eq. (11) cannot be improved further, by replacing numerical contributions by analytical results.

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[1] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, Phys. Lett. B **229**, 285 (1989); Yad. Fiz. **50**, 1636 (1989) [Sov. J. Nucl. Phys. **50**, 1015 (1989)].

[2] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, Phys. Lett. B **249**, 519 (1990).

[3] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, Phys. Lett. B **268**, 433 (1991); **316**, 631(E) (1993); **319**, 545(E) (1993); Yad. Fiz. **55**, 466 (1992); **57**, 1343(E) (1994) [Sov. J. Nucl. Phys. **55**, 257 (1992)].

[4] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, Phys. Lett. B **312**, 358 (1993); Yad. Phys. **57**, 1309 (1994) [Phys. Atom. Nuclei **57**, 1240 (1994)]; **57**, 2246 (1994) [**57**, 2158 (1994)].

[5] T. Kinoshita and M. Nio, Phys. Rev. Lett. **72**, 3803(1994); Phys. Rev. D **53**, 4909 (1996).

[6] M. I. Eides and V. A. Shelyuto, Pis'ma Zh. Éksp. Teor. Fiz. **61**, 465 (1995) [JETP Lett. **61**, 478 (1995)]; Phys. Rev. A **52**, 954 (1995).

[7] M. I. Eides, H. Grotch, and D. A. Owen, Phys. Lett. B **294,** 115 (1992).

[8] M. I. Eides and H. Grotch, Phys. Lett. B **301**, 127 (1993).

[9] M. I. Eides and H. Grotch, Phys. Lett. B **308**, 389 (1993).

[10] K. Pachucki, Phys. Rev. A **48**, 2609 (1993).

[11] M. I. Eides, H. Grotch, and P. Pebler, Phys. Lett. B **326**, 197 (1994); Phys. Rev. A **50**, 144 (1994).

[12] K. Pachucki, Phys. Rev. Lett. **72**, 3154 (1994).

[13] S. Laporta, as cited in K. Pachucki, Ref. [10].

[14] G. Bhatt and H. Grotch, Ann. Phys. (N.Y.) **178**, 1 (1987).