Dynamics of the four-level Λ system in a two-mode cavity

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In this paper we investigate the properties of a four-level atomic system interacting with two modes of the electromagnetic field in a cavity. By linearization of the Hamiltonian we show that the corresponding mathematical model is exactly solvable. To obtain simpler effective Hamiltonians the method of multiple scales is applied. It is shown that under some special values of the detunings and coupling constants, exactly predictable collapses and revivals in the atomic populations appear. Also, due to the specific interactions of the atom with the cavity modes and the ensuing measurement of energy or angular momentum and energy, two-mode Schrödinger-cat states arise in the cavity. Finally, the stabilization and trapping properties of the system are demonstrated in a lossy cavity within the Wigner-Weisskopf approximation. The existence of stationary states is shown and it is also shown that interference effects can cause cancellation of one line in both transmitted light and spontaneous emission spectra. [S1050-2947(97)08002-5]

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I. INTRODUCTION

There exists a constant interest in simple quantum electrodynamical systems, consisting of an atom—approximated by taking into account just two, three, or four levels—and one or only a few cavity electromagnetic modes. This is because in such a system one observes very interesting and fundamental effects such as collapses and revivals in atomic inversion [1], squeezing, vacuum Rabi splitting [2], antibunching of photons, and the production of Schrödinger-cat-type states [3]. These effects demonstrate the quantum ''grainlike'' nature of light in a much more explicit way than spontaneous emission or resonance fluorescence.

Simple quantum electrodynamical systems are also very interesting from a formal point of view since they are often exactly integrable if one neglects losses. In an important recent paper [4], it has been shown that the exact solvability of the Jaynes-Cummings model is connected to the fact that the Jaynes-Cummings Hamiltonian can be expressed in terms of the generators of the Lie algebra su(2). This discovery produces the possibility to exactly solve few-level and few-mode generalizations of the Jaynes-Cummings model.

The purpose of this communication is twofold. On a formal level, we show quite a complicated system which we can prove is exactly integrable, in the sense that the time-evolution operator can be written in an explicit form. This is done via the appropriate linearization of the Hamiltonian. In a sense, our model is a maximal one which possesses this property. However, since the structure of the exact time-evolution operator is very complicated and unmanageable, we use the method of multiple scales (MMS) and apply it on the level of the Schrödinger equation for the time-evolution operator to extract effective Hamiltonians which are useful from a physical point of view. Our system is a four-level generalization of the three-level nonresonant ''Λ system'' considered, e.g., in [5,6]. There has been some discrepancy between the authors of [5] and [6] regarding the structure of the effective Hamiltonian which describes the coupling between two lower levels. We believe that the method of multiple scales, since it is universal and efficient, brings resolution to this discrepancy. It also provides several other interesting effective Hamiltonians not considered in [5] and [6].

On a physical level, we find a two-mode system which can exhibit regular and predictable collapses and revivals—this is a special type of the four-level Λ system with two degenerate upper levels. Unlike in the similar one-mode case, however, the dynamics of inversion, though described by a closed-form expression, can also be only quasiperiodic and thus quite erratic. Furthermore, we propose a Stern-Gerlach type of experiment to produce a Schrödinger-cat-type state in the cavity. Again, the best candidate for such an experiment is the degenerate (in the two upper levels) system if the coupling constants are such that the Stark shifts are identical. Also, the trapping properties of the system are analyzed within the Wigner-Weisskopf approximation.

The rest of the paper is organized as follows. In Sec. II, we present the model and attempt to solve it exactly. In Sec. III, we apply the method of multiple scales to extract a family of effective Hamiltonians for a generalized Λ system. Section IV is devoted to analyzing the production of Schrödinger-cat states of the cavity modes and the two-mode regular collapse-and-revival pattern. In Sec. V, we analyze the stabilization properties and spectra of transmitted and spontaneously emitted radiation under the Wigner-Weisskopf approximation and Sec. VI contains some final remarks.

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II. THE MODEL AND ITS SOLVABILITY

Let us consider a model system consisting of a four-level atom interacting with two modes of the electromagnetic field. We will assume a very specific coupling between the modes which is a direct generalization of the Λ system considered, e.g., in [5]. More precisely, the system is defined by the following Hamiltonian:

$$H=\sum_{i=1}^{4} E_i\sigma_{ii}+\sum_{i=1,2} \omega_i a_i^\dagger a_i + \sum_{j=3,4} (g_{1j}a_1\sigma_{j1}+g_{2j}a_2\sigma_{j2}+\text{H.c.}). \quad (1)$$

Thus, the first (lowest) level is coupled to the third and fourth only via the first mode of the cavity field, while the second level is coupled to the upper ones only via the second mode. So far, we have not assumed any relationship between the atomic energies and the mode frequencies; we will try to solve the problem exactly. By ''exact solution'' we mean an explicit expression for the time-evolution operator. With this aim in mind, we start by linearizing the Hamiltonian. This is simplified by the observation that for some given m photons in the first mode and n photons in the second, the following states of the system can be populated: $|4,m,n\rangle$, $|3,m,n\rangle$, $|1,m+1,n\rangle$, and $|2,m,n+1\rangle$. This suggests that the problem can be reduced to the four-level system in an external time-independent (within the rotating-wave approximation) field if we find appropriate constants of motion. Let us further observe that, besides the Hamiltonian itself, the system indeed possesses at least two further constants of motion (the excitation number operators) which turn out to be very useful:

$$N_1=a_1^\dagger a_1+\sigma_{22}+\sigma_{33}+\sigma_{44} \quad (2)$$

and

$$N_2=a_2^\dagger a_2+\sigma_{11}+\sigma_{33}+\sigma_{44}. \quad (3)$$

The operators N_1 and N_2 commute not only with the Hamiltonian (and with one another) but also with the free part of the Hamiltonian and with all products of the type $a_i\sigma_{ji}$, $i=1,2$; $j=3,4$, as well as $a_i^\dagger a_j\sigma_{ij}$, $i,j=1,2$. Let us define the new operators as

$$S_{31}=\frac{1}{\sqrt{N_1}}\,a_1\sigma_{31}, \quad S_{41}=\frac{1}{\sqrt{N_1}}\,a_1\sigma_{41},$$

$$S_{32}=\frac{1}{\sqrt{N_2}}\,a_2\sigma_{32}, \quad S_{42}=\frac{1}{\sqrt{N_2}}\,a_2\sigma_{42},$$

$$S_{ii}=\sigma_{ii}, \quad i=1,2,3,4, \quad S_{34}=\sigma_{34}. \quad (4)$$

These operators, together with their Hermitian conjugates and the operator $S_{12}=S_{13}S_{32}=S_{14}S_{42}$ (and its conjugate) span the Lie algebra su(4) (cf. [4]) with the usual commutator as the Lie product. Actually, of even more importance is the fact that they form an associative matrix algebra, isomorphic to that spanned by the operators σ_{ij}. Let us for instance calculate the product $S_{12}S_{24}$:

$$\begin{aligned}S_{12}S_{24}&=S_{13}S_{32}S_{24}\\&=\frac{1}{\sqrt{N_1}}\,a_1^\dagger\sigma_{13}\,\frac{1}{\sqrt{N_2}}\,a_2\sigma_{32}\,\frac{1}{\sqrt{N_2}}\,a_2^\dagger\sigma_{24}\\&=\frac{1}{\sqrt{N_1}}\,a_1^\dagger\sigma_{13}\,\frac{1}{N_2}\,(a_2^\dagger a_2+1)\sigma_{34}.\end{aligned}$$

But

$$\begin{aligned}&\frac{1}{N_2}\,(a_2^\dagger a_2+1)\sigma_{34}-\frac{1}{N_2}\,N_2\sigma_{34}\\&\quad=\frac{1}{N_2}\,(a_2^\dagger a_2+1-N_2)\sigma_{34}\\&\quad=\frac{1}{N_2}\,(a_2^\dagger a_2+1-a_2^\dagger a_2-\sigma_{11}-\sigma_{33}-\sigma_{44})\sigma_{34}=0,\end{aligned}$$

hence $S_{12}S_{24}=(1/\sqrt{N_1})a_1^\dagger\sigma_{14}=S_{14}$.

One can now write the Hamiltonian in terms of the operators S_{ij}:

$$H=\sum_{i=1}^{4} E_iS_{ii}+\sum_{i=1,2} \omega_i a_i^\dagger a_i+\sum_{i=1,2}\sum_{j=3,4} (g_{ij}S_{ji}+\text{H.c.}). \quad (5)$$

By applying the unitary transformation

$$W=\exp[-i(\omega_1N_1+\omega_2N_2)t], \quad (6)$$

we get the following ''interaction'' Hamiltonian which only contains the operators S_{ij} and N_i without the explicit presence of the free field terms:

$$\begin{aligned}H'&=(E_1-\omega_2)S_{11}+(E_2-\omega_1)S_{22}+(E_3-\omega_1-\omega_2)S_{33}\\&\quad+(E_4-\omega_1-\omega_2)S_{44}+\sum_{i=1,2}\sum_{j=3,4}\sqrt{N_i}(g_{ij}S_{ji}+g_{ij}^*S_{ij}).\end{aligned} \quad (7)$$

If we choose the zero of energy in such a way that $E_4=\omega_1+\omega_2$, we obtain

$$H'=-\sum_{i=1}^{3} \Delta_iS_{ii}+\sum_{i=1,2}\sum_{j=3,4}\sqrt{N_i}(g_{ij}^*S_{ij}+g_{ij}S_{ji}), \quad (8)$$

where $\Delta_1=(E_4-E_1)-\omega_1$, $\Delta_2=(E_4-E_2)-\omega_2$, $\Delta_3=(E_4-E_3)$.

The matrix representation of the Hamiltonian is given by

$$\begin{pmatrix} -\Delta_1 & 0 & \gamma_{13}^* & \gamma_{14}^* \\ 0 & -\Delta_2 & \gamma_{23}^* & \gamma_{24}^* \\ \gamma_{13} & \gamma_{23} & -\Delta_3 & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 0 \end{pmatrix}, \quad (9)$$

where $\gamma_{ij}=g_{ij}\sqrt{N_i}$.

Thus, the problem of the exact solution for our model has been reduced to the problem of the diagonalization of a quadratic matrix of the fourth degree. This can always be performed, but the necessary intermediate step—the computation of the eigenvalues and eigenvectors—introduces a terrible algebraic mess which obscures both the formal structure of the Hamiltonian and the underlying physics. Therefore, despite the fact that the system admits an exact solution, in the next section we will apply an efficient perturbation scheme—the method of multiple scales—to extract the physical information contained in the Hamiltonian H', provided that a natural small parameter exists in it. To conclude this section, however, let us (with aesthetical reasons in mind) obtain an exact solution for a simpler case. Namely, let us suppose that we may restrict ourselves to the three-level model (with energies E_1, E_2, E_3) such that $\Delta_1 = \Delta_2 = \Delta$ and g_{ij} are real. This is the model considered by Gerry and Eberly [5], but our Δ is still arbitrary and we can consider both strongly resonant and dispersive interactions. In this case, it is more convenient to choose the zero of energy such that $E_1 = \omega_2$. Then the Hamiltonian in matrix form is given by

$$\begin{pmatrix} 0 & 0 & \gamma_{13} \\ 0 & 0 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & \Delta \end{pmatrix}. \tag{10}$$

Let e_1, e_2 denote two nonzero eigenvalues (which are of course still operators containing the excitation number operators) of the above matrix with inverse sign, $e_1 = (1/2)[\Delta - \sqrt{\Delta^2 + 4(\gamma_{13}^2 + \gamma_{23}^2)}]$, $e_2 = (1/2)[\Delta + \sqrt{\Delta^2 + 4(\gamma_{13}^2 + \gamma_{23}^2)}]$. Also let the square root term be denoted by f. It is then possible to perform the exponentiation of the matrix $(-itH')$ to obtain

$$\begin{aligned} U' = \exp(-iH't) = {} & \left[\frac{\gamma_{13}^2 [e_2 \exp(-ie_1 t) - e_1 \exp(-ie_2 t)]}{f(\gamma_{13}^2 + \gamma_{23}^2)} + \frac{\gamma_{23}^2}{\gamma_{13}^2 + \gamma_{23}^2} \right] S_{11} \\ & + \left[\frac{\gamma_{23}^2 [e_2 \exp(-ie_1 t) - e_1 \exp(-ie_2 t)]}{f(\gamma_{13}^2 + \gamma_{23}^2)} + \frac{\gamma_{13}^2}{\gamma_{13}^2 + \gamma_{23}^2} \right] S_{22} + \frac{e_2 \exp(-ie_2 t) - e_1 \exp(-ie_1 t)}{f} S_{33} \\ & + \left[\frac{\gamma_{13}\gamma_{23} [e_2 \exp(-ie_1 t) - e_1 \exp(-ie_2 t) - f]}{f(\gamma_{13}^2 + \gamma_{23}^2)} \right] (S_{12} + S_{21}) + \frac{\exp(-i\Delta t)}{f} \\ & \times [\exp(ie_1 t) - \exp(ie_2 t)](\gamma_{13} S_{13} + \gamma_{23} S_{23}) + \frac{1}{f} [\exp(-ie_1 t) + \exp(-ie_2 t)](\gamma_{13} S_{31} + \gamma_{23} S_{32}). \end{aligned} \tag{11}$$

The total time-evolution operator results if we multiply the above expression by W from the left (we have to set $\sigma_{44} = 0$ in N_1 and N_2 contained in W).

As we can see, although U' has a structure fairly similar to the time-evolution operator for the usual Jaynes-Cummings model [4], it is actually too complicated to be useful in practice. It is a well-known fact in nonlinear mechanics that in many cases a perturbative treatment should be applied even if the exact solution is available.

III. APPLICATION OF THE METHOD OF MULTIPLE SCALES

As mentioned in the Introduction, the method of multiple scales (MMS) [7,8] belongs (together with the Krylov-Bogoliubov-Mitropolskii method [9,10]) among the most successful techniques of dealing with a nonlinear system. In particular, it is widely used in nonlinear mechanics. In the context of quantum-dynamical problems it has been applied in [11] to describe the spontaneous emission from both one atom and from a collection of many atoms, and in [12] to study the dynamics of atoms and molecules in laser fields, in [13] it has been used to investigate the dynamics of both the internal and external degrees of freedom of an atom excited by a standing wave, in [14] to obtain the dynamics of systems having both classical and quantum degrees of freedom, and finally in [15] to approximately solve the Heisenberg equations of motion and the Schrödinger equation for the wave function for the case of an anharmonic oscillator. After some preparatory work with MMS applied to the Heisenberg equations of motion for the Jaynes-Cummings model, we have decided to apply it rather on the level of the Schrödinger equation for the time-evolution operator. The reason for this strategy is the following. On the one hand, the time-evolution operator provides, together with some initially given state, the most general information available about quantum systems. On the other hand, due to the linearity, the whole procedure possesses a very transparent structure. We believe that in many cases the method considered here is much superior with respect to the usual Dyson's expansion since it does not contain terms proportional to powers of time and it preserves the unitarity of the evolution up to the required order. We will see that MMS provides corrections to the unperturbed Hamiltonian, therefore giving some effective Hamiltonians.

From now on, in the following subsections, we will assume that there is at least one small parameter contained in the Hamiltonian in Eq. (1). Our general procedure will be as follows. We will define a dimensionless time τ and write a Hamiltonian which will generate the dynamics in the time τ. This Hamiltonian will also contain a dimensionless small parameter in an explicit form and will thus separate into two parts, a free part and a perturbation. Then we will expand the time-evolution operator (still in time τ) into a power series in terms of the small parameter and will use the solvability conditions to eliminate the possible secular terms. The whole procedure is described in detail in Sec. III A.

In this paper it will be assumed that the absolute values of at least two of Δ_i, $i=1,2,3$, are much larger than all $|g_{ij}|\sqrt{N_i}$, where N_i denotes mean excitation numbers. We will investigate the case of strongly resonant interactions elsewhere. Despite this assumption, several internal resonances are possible in the system and hence we have to consider several cases; in some of these we will obtain interesting effective Hamiltonians to be compared with those in [5,6]. We will also assume that the photon number distribution functions remain well localized near their mean values.

We will first outline the general procedure assuming that all three Δ_i are of the same order and that they are much larger than all $g_{ij}\sqrt{N_i}$. The effective coupling constants are also assumed to be of the same order. Let $\mu=\min(|\Delta_1|,|\Delta_2|,|\Delta_3|)$ and $\nu=\max(|g_{ij}|\sqrt{N_i})$. We define a small dimensionless parameter ϵ as

$$\epsilon=\frac{\nu}{\mu}.$$

Let $\delta_i=\Delta_i/\mu$ and $m_{ij}=g_{ij}\sqrt{N_i}/\nu$. We introduce a dimensionless time $\tau=\mu t$. The dynamics in τ are governed by the Hamiltonian

$$h=-\sum_{i=1}^{3}\delta_i S_{ii}+\epsilon\sum_{i=1,2}\sum_{j=3,4}m_{ij}S_{ij}=h_0+\epsilon h_1. \qquad (12)$$

There are two natural time scales of the evolution: one connected with the part of the Hamiltonian containing detunings, the other connected with the coupling constants. Therefore, it is quite natural to look for an approximate (asymptotic) solution as a function of at least two different times, which are ''faster'' and ''slower.'' Actually, it is more convenient and systematic to work from the very beginning with many dimensionless time variables and to keep some of them according to the needs of a particular problem. Let us therefore introduce some new time variables:

$$T_0=\tau, \quad T_1=\epsilon\tau, \quad T_2=\epsilon^2\tau. \qquad (13)$$

An approximate time-evolution operator is assumed to depend on all these time scales separately.

Thus we have

$$\frac{d}{d\tau}=\frac{\partial}{\partial T_0}+\epsilon\frac{\partial}{\partial T_1}+\epsilon^2\frac{\partial}{\partial T_2}+\cdots . \qquad (14)$$

Before proceeding further, let us introduce some ''blanket'' notation. Namely, in all the formulas below, $F(T_0)$, $F_1(T_0)$, $F'(T_0),\ldots$ denote operators which contain exponential functions of T_0 such that the factors multiplying T_0 in the exponents are $O(1)$. Such operators can be safely integrated over T_0 without any danger of obtaining small (or zero) denominators. Let u denote the time-evolution operator (with $\exp[-i(\omega_1 N_1+\omega_2 N_2)t]$ already factored out) as a function of time τ. To obtain an asymptotic expression for u we expand it as

$$u=u_0+\epsilon u_1+\epsilon^2 u_2+\cdots . \qquad (15)$$

Taking into account Eqs. (12) and (14) we get

$$i\frac{\partial u_0}{\partial T_0}=h_0 u_0, \qquad (16)$$

$$i\frac{\partial u_1}{\partial T_0}+\frac{\partial u_0}{\partial T_1}=h_0 u_1+h_1 u_0, \qquad (17)$$

$$i\frac{\partial u_2}{\partial T_0}+\frac{\partial u_1}{\partial T_1}+\frac{\partial u_0}{\partial T_2}=h_0 u_2+h_1 u_1+h_2 u_0 \qquad (18)$$

[in Eq. (12) we have, of course, that $h_2=0$].

Even in the case of purely dispersive interactions we can have internal resonances, the appearance of which completely changes the physical situation. In order to analyze them, we must consider several subcases.

A. Case 1: $\delta_i-\delta_j=O(1)$

We therefore have to consider the case in which the differences between the δ's are of the same order as the δ's themselves. Equation (16) can easily be solved:

$$\begin{aligned}u_0&=\exp[-ih_0T_0]v(T_1,T_2)\\&=\exp[i(\delta_1 S_{11}+i\delta_2 S_{22}+i\delta_3 S_{33})T_0]v(T_1,T_2).\end{aligned} \qquad (19)$$

To solve Eq. (17) we write

$$u_1=\exp[-ih_0T_0]\overline{u_1}(T_0,T_1,T_2), \qquad (20)$$

and Eq. (17) takes the form

$$i\frac{\partial\overline{u_1}}{\partial T_0}+i\frac{\partial v}{\partial T_1}=e^{ih_0T_0}h_1e^{-ih_0T_0}v(T_1,T_2). \qquad (21)$$

The Hamiltonian h_1 in the ''interaction representation'' with respect to h_0 is

$$\begin{aligned}e^{ih_0T_0}h_1e^{-ih_0T_0}=&(m_{13}^{*}S_{13}e^{-i\delta_{13}T_0}+m_{13}S_{31}e^{i\delta_{13}T_0})\\&+(m_{14}^{*}S_{14}e^{-i\delta_1T_0}+m_4S_{41}e^{i\delta_1T_0})\\&+(m_{23}^{*}S_{23}e^{-i\delta_{23}T_0}+m_{23}S_{32}e^{i\delta_{23}T_0})\\&+(m_{24}^{*}S_{24}e^{-i\delta_2T_0}+m_{24}S_{42}e^{i\delta_2T_0}),\end{aligned} \qquad (22)$$

where $\delta_{ij}=\delta_i-\delta_j$.

We see that, in the first order, all the terms on the right-hand side of Eq. (21) are of the type of $F(T_0)$ and hence can be integrated without the appearance of secular terms. Therefore we put

$$\frac{\partial v}{\partial T_1}=0,$$

and integrate Eq. (21); let $x(T_1,T_2)$ denote a solution to the homogeneous equation $i(\partial\overline{u_1}/\partial T_0)=0$.

To solve Eq. (18) we again write

$$u_2=e^{-ih_0T_0}\overline{u_2}(T_0,T_1,T_2). \tag{23}$$

Then Eq. (18) becomes

$$i\,\frac{\partial\overline{u_2}}{\partial T_0}+i\,\frac{\partial\overline{u_1}}{\partial T_1}+i\,\frac{\partial v}{\partial T_2}=e^{ih_0T_0}h_1e^{-ih_0T_0}\overline{u_1}. \tag{24}$$

Here $\partial\overline{u_1}/\partial T_1=\partial x/\partial T_1$ since v does not depend on T_1. Taking into account Eq. (22) we find the right-hand side of Eq. (24):

$$\begin{aligned}e^{ih_0T_0}h_1e^{-ih_0T_0}\overline{u_1}=\Bigg[&\frac{|m_{13}|^2}{\delta_{13}}\,(S_{33}-S_{11})+\frac{|m_{14}|^2}{\delta_1}\,(S_{44}-S_{11})\\&+\frac{|m_{23}|^2}{\delta_{23}}\,(S_{33}-S_{22})\\&+\frac{|m_{24}|^2}{\delta_2}\,(S_{44}-S_{22})\Bigg]v+F(T_0).\end{aligned} \tag{25}$$

Now, the first four terms on the right-hand side of Eq. (25) do not depend on T_0. Thus, if we try to integrate over T_0, this will immediately lead to terms proportional to T_0. These terms must be avoided since they make all the dynamics nonunitary and thus unphysical, which implies the breaking of the canonical commutation relations, etc. We can avoid their appearance very easily by equating $i(\partial v/\partial T_2)$ to the first four terms in Eq. (25). The solution to the equation produced in this way is obvious:

$$\begin{aligned}v(T_2)=\exp\Bigg[-i\Bigg(&\frac{|m_{13}|^2}{\delta_{13}}\,(S_{33}-S_{11})+\frac{|m_{14}|^2}{\delta_1}\,(S_{44}-S_{11})\\&+\frac{|m_{23}|^2}{\delta_{23}}\,(S_{33}-S_{22})+\frac{|m_{24}|^2}{\delta_2}\,(S_{44}-S_{22})\Bigg)T_2\Bigg].\end{aligned} \tag{26}$$

If we combine this result with Eq. (19), and then go back to time t and define the effective Hamiltonian as $H_{\mathrm{eff}}=iU_0^\dagger(dU_0/dt)$, $U_0=Wu_0$, we obtain

$$\begin{aligned}H_{\mathrm{eff}}=\omega_1N_1+\omega_1N_2-\sum_{i=1}^{3}\,\Delta_iS_{ii}+\left(\frac{|g_{14}|^2N_1}{\Delta_1}+\frac{|g_{24}|^2N_2}{\Delta_2}\right)S_{44}\\+\left(\frac{|g_{13}|^2N_1}{\Delta_{13}}+\frac{|g_{23}|^2N_2}{\Delta_{23}}\right)S_{33}\\-\left(\frac{|g_{13}|^2N_1}{\Delta_{13}}+\frac{|g_{14}|^2N_1}{\Delta_1}\right)S_{11}\\-\left(\frac{|g_{23}|^2N_2}{\Delta_{23}}+\frac{|g_{24}|^2N_2}{\Delta_2}\right)S_{22}.\end{aligned} \tag{27}$$

If we set $\Delta_3=0$, $g_{i4}=0$, we may compare the above effective Hamiltonian with that of Alexanian and Bose [6]. They are completely different; our effective Hamiltonian in Eq. (27) is diagonal, which is a characteristic feature of purely dispersive interactions. We will see later that an effective Hamiltonian of a form which is almost identical to that in [6] will arise if an internal resonance is present in the system.

B. Case 2: $\delta_1-\delta_2=O(\epsilon)$

Let us now assume that, although all the Δ's are much larger than the effective coupling constants, one of their differences, $\Delta_1-\Delta_2$, is a small quantity, comparable with these effective couplings. In this case we write $\Delta_2=\Delta_1-(\Delta_1-\Delta_2)$ and define μ as $\min(|\Delta_1|,|\Delta_3|)$, and ν as $\max(|\Delta_{12}|,g_{ij}\sqrt{N_i})$. The dimensionless small parameter is defined as before, $\epsilon=\nu/\mu$ and $\tau=\mu t$. The Hamiltonian which generates the dynamics in time τ is now given by

$$\begin{aligned}h=&-\delta_1(S_{11}+S_{22})-\delta_3S_{33}\\&+\epsilon\left(\delta_{12}S_{22}+\sum_{i=1}^{2}\sum_{j=3}^{4}\,(m_{ij}^*S_{ij}+m_{ij}S_{ji})\right)\\=&\,h_0+\epsilon h_1,\end{aligned} \tag{28}$$

where lower case δ and m_{ij}'s are defined as before and $\delta_{12}=\Delta_{12}/\nu$. However, this is not enough to substantially change the effective Hamiltonian as given by Eq. (27). Indeed, proceeding as before, we obtain almost exactly the same effective Hamiltonian, with the only difference being that all the Δ_2's are replaced by Δ_1's and all the Δ_{23}'s by Δ_{13}'s in all those (and only those) terms which contain $|g_{ij}|^2$. Substantial changes are introduced to the *first-order* correction only; the operator x is no longer T_1 independent: $x(T_1,T_2)=\exp[-i\,\delta_{12}S_{22}T_1]y(T_2)$, where $y(T_2)$ can be expressed in terms of v.

C. Case 3: $\delta_1-\delta_2=O(\epsilon^2)$

Now we assume that we have almost exact internal resonance: $E_1+\omega_1$ is almost equal to $E_2+\omega_2$, so their difference is small even when compared with $|g_{ij}|\sqrt{N_i}$. While μ and ν are defined as before, we have $\delta_{12}=\Delta_{12}\mu/(\nu^2)$. Again, $\epsilon=\nu/\mu$.

The Hamiltonian (for the evolution in time τ) is now given by

$$\begin{aligned}h=&-\delta_1(S_{11}+S_{22})-\delta_3S_{33}\\&+\epsilon\sum_{i,j}\,(m_{ij}^*S_{ij}+m_{ij}S_{ji})+\epsilon^2\delta_{12}S_{22}\\=&\,h_0+\epsilon h_1+\epsilon^2h_2.\end{aligned} \tag{29}$$

In the Hamiltonian above, the index i takes the values 1,2 while the index j takes the values 3,4. The zeroth-order approximation is given by

$$u_0=\exp[i(\delta_1(S_{11}+S_{22})+\delta_3S_{33})T_0]v(T_1,T_2). \tag{30}$$

To obtain the first-order approximation we compute the Hamiltonian h_1 in the ''interaction representation'' with respect to h_0:

$$e^{ih_0T_0}h_1e^{-ih_0T_0}=(m_{13}^*S_{13}e^{-i\delta_{13}T_0}+m_{13}S_{31}e^{i\delta_{13}T_0})+(m_4^*S_{14}e^{-i\delta_1T_0}+m_{14}S_{41}e^{i\delta_1T_0})+(m_{23}^*S_{23}e^{-i\delta_{13}T_0}+m_{23}S_{32}e^{i\delta_{13}T_0})+(m_{24}^*S_{24}e^{-i\delta_1T_0}+m_{24}S_{42}e^{i\delta_1T_0}). \tag{31}$$

Because the right-hand side of Eq. (31) is of the type $F(T_0)$, we have $\partial v/\partial T_1=0$ and hence $\partial\overline{u_1}/\partial T_1=\partial x/\partial T_1$, where $\overline{u_1}$, x, and $\overline{u_2}$ are defined as for Case 1.

We have to solve to the second order an expression similar to Eq. (25):

$$\begin{aligned} i\frac{\partial\overline{u_2}}{\partial T_0}+i\frac{\partial x}{\partial T_1}+i\frac{\partial v}{\partial T_2} &= e^{ih_0T_0}h_1e^{-ih_0T_0}\overline{u_1}+e^{ih_0T_0}h_2e^{-ih_0T_0}v \\ &= \left[\frac{|m_{13}|^2}{\delta_{13}}(S_{33}-S_{11})+\frac{|m_{14}|^2}{\delta_1}(S_{44}-S_{11})\right. \\ &\quad \left.+\frac{|m_{23}|^2}{\delta_{13}}(S_{33}-S_{22})+\frac{|m_{24}|^2}{\delta_1}(S_{44}-S_{22})\right]v \\ &\quad -\left[\left(\frac{m_{13}^*m_{23}}{\delta_{13}}+\frac{m_{14}^*m_{24}}{\delta_1}\right)S_{12}+\text{H.c.}\right]v \\ &\quad +\delta_{12}S_{22}v+F_1(T_0). \end{aligned} \tag{32}$$

Note the presence of nondiagonal terms in the above expression. We may put $i(\partial x/\partial T_1)=0$ and $i(\partial\overline{u_2}/\partial T_0)=F_1(T_0)$. Then the solution for v is obvious and we immediately find the effective Hamiltonian:

$$\begin{aligned} H_{\text{eff}} &= \omega_1N_1+\omega_2N_2-\sum_{i=1}^{3}\Delta_i+\frac{|g_{13}|^2N_1}{\Delta_{13}}(S_{33}-S_{11}) \\ &\quad +\frac{|g_{14}|^2N_1}{\Delta_1}(S_{44}-S_{11})+\frac{|g_{23}|^2N_2}{\Delta_{13}}(S_{33}-S_{22}) \\ &\quad -\left[\left(\frac{g_{13}^*g_{23}\sqrt{N_1}\sqrt{N_2}}{\Delta_{13}}+\frac{g_{14}^*g_{24}\sqrt{N_1}\sqrt{N_2}}{\Delta_1}\right)S_{12}+\text{H.c.}\right]. \end{aligned} \tag{33}$$

In this Hamiltonian one can recognize that obtained in [6] if we (a) restrict ourselves to the three-level system and (b) let Δ_1 be equal to Δ_2 (this corresponds to $\Delta_1=\Delta_3$ in the notation of [6]) in the interaction terms of the Hamiltonian obtained by Alexanian and Bose. This is not an essential difference, since we have assumed that $\Delta_1-\Delta_2$ is very small.

Our formal results achieved so far may be summarized as follows: they support those of [6] in that our effective Hamiltonian contains intensity-dependent Stark shifts of all levels. On the other hand, they support the assumption of Gerry and Eberly that to study an effective interaction between the states $|1\rangle$ and $|2\rangle$ it is enough to restrict oneself to near resonant cases, i.e., two detunings, Δ_1 and Δ_2, which are equal or almost equal. If this is not the case, one can use a *diagonal* effective Hamiltonian and investigate the effective coupling between the two lower levels only as a small correction. Needless to say, having linearized both the total Hamiltonian and all effective Hamiltonians, we could very easily obtain an explicit expression for all effective time-evolution operators.

D. Case 4: $\delta_1-\delta_2=O(\epsilon^2)$, $\delta_3=O(\epsilon^2)$

Let us now consider the case of a very small energy gap between the levels $|3\rangle$ and $|4\rangle$. We define $\mu=\min(|\Delta_1|,|\Delta_2|)$, $\nu=\max(|g_{ij}|\sqrt{(N_i)}$, $\delta_{12}=\Delta_{12}\mu/(\nu^2)$, $\delta_3=\Delta_3\mu/(\nu^2)$, $\epsilon=\nu/\mu$. This case is quite interesting because we obtain an effective coupling not only between the two lower levels, but also between the two upper ones. In fact, calculations of exactly the same type as before lead to the effective Hamiltonian

$$\begin{aligned} H_{\text{eff}} &= -\sum_i\Delta_i+\frac{1}{\Delta_1}\{N_1[|g_{13}|^2(S_{33}-S_{11})+|g_{14}|^2(S_{44}-S_{11})] \\ &\quad +N_2[|g_{23}|^2(S_{33}-S_{22})+|g_{24}|^2(S_{44}-S_{22})] \\ &\quad +[(g_{13}g_{14}^*N_1+g_{23}g_{24}^*N_2)S_{34}+\text{H.c.}] \\ &\quad -[(g_{13}^*g_{23}+g_{14}^*g_{24})\sqrt{N_1}\sqrt{N_2}S_{12}+\text{H.c.}]\}. \end{aligned} \tag{34}$$

As has been pointed out in [5], the effective interaction between two lower levels is zero-photon: there is neither gain nor loss of photons but any transition from one level to another is connected by the exchange of photons between modes. The Hamiltonian above shows that between the two upper levels we have a zero-photon coupling in an even stricter sense: there are Rabi oscillations in the subsystem consisting of the states $|3\rangle$ and $|4\rangle$, but there is no exchange of photons between the two modes.

E. Case 5: $\delta_1-\delta_2=O(1)$, $\delta_3=O(\epsilon^2)$

Now $\mu=\min(|\Delta_1|,|\Delta_2|)$, and the other variables are defined as before. In this case we obtain an effective Hamiltonian which contains just the effective interaction between the two upper levels, without any coupling between the lower ones:

$$\begin{aligned} H_{\text{eff}} &= -\sum_{i=1}^{3}\Delta_iS_{ii}+\frac{1}{\Delta_1}[|g_{13}|^2N_1(S_{33}-S_{11}) \\ &\quad +|g_{14}|^2N_1(S_{44}-S_{11})]+\frac{1}{\Delta_2}[|g_{23}|^2N_2(S_{33}-S_{22}) \\ &\quad +g_{24}^2N_2(S_{44}-S_{22})] \\ &\quad +\left[\left(\frac{g_{13}g_{14}^*N_1}{\Delta_1}+\frac{g_{23}g_{24}^*N_2}{\Delta_2}\right)S_{34}+\text{H.c.}\right]. \end{aligned} \tag{35}$$

This Hamiltonian provides a two-mode generalization of that proposed by [16]. We will use it in the next section to produce a kind of Schrödinger-cat state in the cavity.

In this section we have applied a variant of the method of multiple scales to extract effective interactions in a ''Λ-type'' four-level system with weak coupling. The *notation* has been greatly simplified by the previous linearization of our system, but we would like to stress that MMS would work equally well even if we had no idea about how to linearize the initial Hamiltonian. Let us also notice that it is very systematic and universal, and provides corrections to the evolution operators defined by the effective Hamiltonians, and thus we believe that it is superior with respect to an adiabatic elimination or a Foldy-Wouthuysen transformation as applied in [6].

IV. SCHRÖDINGER-CAT STATES AND POPULATION DYNAMICS IN A RESTRICTED TWO-MODE TWO-PHOTON INTERACTION

In this section we consider the possible appearance of some special states of the cavity fields from a superposition of the products of simple coherent states with large mean numbers of photons. By a turn of phrase, such states are called ''Schrödinger-cat'' states. They are quite important from a fundamental point of view as their existence is a reply to Einstein's and Schrödinger's objections against quantum theory, based on the fact that it is very difficult to find any coherent superposition of quantum states on a macroscopic level. In fact, on the one hand, we can now produce such coherent superpositions in high-Q cavities and in optical traps, but on the other, there is a satisfactory explanation of the difficulty in observing such superpositions in terms of the decoherence introduced by the coupling with the reservoir which is always present.

One of the most important nonclassical features of the generalized Jaynes-Cummings models is the presence of the so-called collapses and revivals in the dynamics of the populations of atomic states. Usually, however, one cannot find a closed formula to express these dynamics. Asymptotic and numerical methods (e.g., see [1] and [17]) show that when time increases, collapses and revivals usually appear less and less regularly and then finally in an erratic way, due to the strong overlap between neighboring revivals. Phoenix and Knight [16] have found a system which exhibits perfectly regular dynamics of collapses and revivals. The system described by our effective Hamiltonian in Case 5 is another example with this feature. In this section we will assume—without loss of generality—that the g_{ij}'s are real. Let us begin with Schrödinger-cat-type states. We will assume—just in order to obtain a closed-form expression—that $g_{13}=g_{14}$, $g_{23}=g_{24}$, and $\Delta_3=0$. This means that the upper levels are perfectly degenerate: not only are their energies (without interactions) equal; the Stark shifts are also the same.

Let us rewrite the Hamiltonian containing the effective coupling between the levels $|3\rangle$ and $|4\rangle$ as

$$H_{\mathrm{eff}}=E_1+E_2+(\omega_1+\omega_2)(S_{33}+S_{44})+\omega_1 n_1+\omega_2 n_2+H_I, \tag{36}$$

where $n_i=a_i^\dagger a_i$, $i=1,2$, and H_I is given by

$$H_I=-\frac{2g_{13}^2}{\Delta_1}N_1S_{11}-\frac{2g_{23}^2}{\Delta_2}N_2S_{22}+\left(\frac{g_{13}^2}{\Delta_1}N_1+\frac{g_{23}^2}{\Delta_2}N_2\right)(S_{33}+S_{44}+S_{34}+S_{43}). \tag{37}$$

Products of the excitation number operators and S_{jk}'s can be written in terms of the photon number operators: $N_iS_{jk}=(n_i+1)S_{jk}$, $i=1,2$, $j,k=3,4$. We will write λ_i instead of g_{i3}^2/Δ_i.

Let us suppose that initially the cavity fields are in coherent states and the atom is in a coherent superposition of the upper states:

$$|\Psi(0)\rangle=(c_3|3\rangle+c_4|4\rangle)|\alpha\rangle|\beta\rangle. \tag{38}$$

Let us denote the time of flight of the atom through the cavity by t_0 (we will identify this time with the time of the atom-cavity interaction). After a time t_0+t_1, we perform a measurement to project the wave functions onto one of the states $\langle 3|$ or $\langle 4|$; because the atomic energies in these two states are the same, we propose to use a Stern-Gerlach type of experiment if the upper levels differ in their m_J value—atoms in the state $|3\rangle$ will fly to a different spatial region from those in the state $|4\rangle$ and we can then measure the atomic energy in an ionization chamber to project on the states $|3\rangle$ or $|4\rangle$. Let us suppose that we have found the atom in the state $|4\rangle$. Then the field in the cavity will be in the state

$$\begin{aligned}|\Psi\rangle_{\mathrm{field}}=&\tfrac{1}{2}\exp(-i(\omega_1+\omega_2)(t_0+t_1))\\&\times\exp(-i(\omega_1n_1+\omega_2n_2)(t_0+t_1))[(c_4-c_3)|\alpha\rangle|\beta\rangle\\&+(c_4+c_3)e^{-2i(\lambda_1+\lambda_2)t_0}|\alpha e^{-2i\lambda_1t_0}\rangle|\beta e^{-2i\lambda_2t_0}\rangle].\end{aligned} \tag{39}$$

If we choose λ_i and t_0 appropriately, the expression in the square bracket above can be equal, e.g., to $(c_4-c_3|\alpha\rangle|\beta\rangle+(c_4+c_3)|-\alpha\rangle|-\beta\rangle$, taking a shape characteristic to cavity Schrödinger cats. Let us note that, unlike in the usual dispersive interaction of the Jaynes-Cummings atom with a cavity mode, it is not actually necessary to prepare the atom in a superposition of two states. It can be prepared in the state $|3\rangle$ or in the state $|4\rangle$, or in any superposition of these states (provided that $c_3\neq c_4$ and $c_3\neq -c_4$), and we can perform projection onto either of the states $\langle 3|$ or $\langle 4|$—in any case, a Schrödinger-cat state will arise, without any additional $\pi/2$ pulses applied to the atom before measurement. On the other hand, if there is no degeneracy in the two upper levels and the interactions are purely dispersive (the effective Hamiltonian is as in Case 1 of the previous section), we can prepare the atom in a superposition of, say, the two upper levels before it enters the cavity and then apply an additional pulse after it leaves the cavity, as in [3]. The subsequent measurement of atomic energy will produce two-mode Schrödinger cats.

Let us now consider the population dynamics for the case of the exact degeneracy of the levels $|3\rangle$ and $|4\rangle$; that is, the assumptions about g_{ij} are the same as above in this section. Let the system be initially prepared as

$$|\Psi(0)\rangle=|4\rangle|\alpha\rangle|\beta\rangle.$$

Then by straightforward calculation we find that

$$\langle S_{44}(t)\rangle=\frac{1}{2}\{1+\mathrm{Re}(\exp[2i(\lambda_1+\lambda_2)t+\overline{n_1}(e^{2i\lambda_1t}-1)+\overline{n_2}(e^{2i\lambda_2t}-1)])\}. \tag{40}$$

Thus the population dynamics in our system can be expressed by a closed-form formula. Overlap between neighboring revivals does not appear as a consequence of the fact that in our effective Hamiltonian the coupling contains the actual field intensities rather than their square roots. Nevertheless the population dynamics are, in general, only quasiperiodic in time, since in the time dependence of $\langle S_{44}\rangle$ we get all frequencies of the type $k\lambda_1+l\lambda_2$, where k,l are arbitrary integers. Contributions of various frequencies are weighted by the products of Bessel and modified Bessel functions. Thus, the behavior of $\langle S_{44}(t)\rangle$ can be both quite regular and also fairly bizarre, depending on the relation between the λ's and n_i's, as well as on the scale of time of the observation.

V. STABILIZATION PROPERTIES OF THE SYSTEM AND SPECTRA OF THE TRANSMITTED LIGHT AND SPONTANEOUS EMISSION

In the preceding sections we have analyzed the system under the assumption that the cavity is perfect. Before we start a simple analysis of the trapping and stabilization properties of the system, let us first briefly discuss the possibility of including losses in the present formalism—the method of multiple scales. The following difficulties arise. The most rigorous approach would require taking into account both our atom with the cavity and one or several reservoirs as all one closed system and then applying MMS. Unfortunately, proceeding this way we encounter quite a fundamental problem: any realistic model of a reservoir will contain infinitely many quasiresonant modes for which we would not be able to even write down the solvability conditions in an unambiguous way. This same problem has been recognized by the authors of [19], where an algorithm has been constructed to perturbatively solve the Heisenberg equations of motion, based on the Lindstedt-Poincaré perturbative method. One might overcome this difficulty by giving up with the description via the time-evolution operator or the Heisenberg equations of motion for the total system and applying a Markovian master equation for the density operator. In this case we may proceed as before (provided, of course, that a small parameter exists). Instead of operating with $\exp(-ih_0T_0)$, etc., we would have to do this with an appropriate exponential of a zero-order Liouvillian, $\exp(-il_0T_0)$, where $l_0(\cdot)=[h_0,(\cdot)]$. The main difficulty with this procedure is associated with the complicated form of the dissipator for the Markovian master equations, especially in the case of nonzero temperature. We believe, however, that MMS combined with numerical calculations could give excellent results for complicated systems like four-level atoms interacting with cavity modes. In this section we will use yet another simplified approach, restricting ourselves to the Wigner-Weisskopf approximation. That is to say, we will use the pure-state representation and adiabatically eliminate the reservoir degrees of freedom, thus allowing only one-photon transitions.

In a recent paper [18], Zhu and Scully have shown that a strong suppression of spontaneous emission and considerable modification of the spontaneous emission spectra can be observed for a four-level atom in a vacuum, due to the quantum interference. The necessary ingredient for such an observation is a particular relation between the transition dipole moments for transitions from the highest and the second highest levels to the lowest one. Also, the system has to be coherently pumped by an external classical field. In connection with this, there arises the following question: Is it possible to obtain an analogous result for an atom interacting with a lossy cavity? That is, is it possible to suppress leakage of photons from the cavity? It is shown below that it is indeed possible in a simple four-level system and a two-mode cavity, the same as considered in the preceding section with the addition of cavity losses. We will first analyze the case where only the field losses are taken into account and the time of flight of the atom through the cavity is sufficiently small when compared with the inverse of the spontaneous decay rate so that the spontaneous emission can be neglected.

The system Hamiltonian is given by Eq. (1). We add to it the Hamiltonians representing bath and system-bath couplings:

$$\Delta H=\sum_{j=1}^{2}\sum_{k}\omega_k b_{kj}^{\dagger}b_{kj}+\sum_{j=1}^{2}\sum_{k}(\xi_{kj}a_j^{\dagger}b_{kj}+\xi_{kj}^{*}b_{kj}^{\dagger}a_j). \tag{41}$$

We assume that reservoirs for modes ''1'' and ''2'' are uncorrelated. The wave function of the system is

$$\begin{aligned}|\Psi(t)\rangle={}&A^{(4)}(t)|4\rangle_A|0\rangle_1|0\rangle_2|0\rangle_{R1}|0\rangle_{R2}+A^{(3)}(t)|3\rangle_A|0\rangle_1|0\rangle_2|0\rangle_{R1}|0\rangle_{R2}+A^{(2)}(t)|2\rangle_A|0\rangle_1|1\rangle_2|0\rangle_{R1}|0\rangle_{R2}\\&+A^{(1)}(t)|1\rangle_A|1\rangle_1|0\rangle_2|0\rangle_{R1}|0\rangle_{R2}+\sum_k B_k|2\rangle_A|0\rangle_1|0\rangle_2|0\rangle_{R1}|1k\rangle_{R2}+\sum_k C_k|1\rangle_A|0\rangle_1|0\rangle_2|1k\rangle_{R1}|0\rangle_{R2},\end{aligned} \tag{42}$$

where, e.g., the ket $|1\rangle_A|1\rangle_1|0\rangle_2|0\rangle_{R1}|0\rangle_{R2}$ represents the atom in the state $|1\rangle$, with one photon in the first mode and zero photons in the second mode and both reservoirs. After writing the Schrödinger equations and adiabatically eliminating the bath variables, we obtain the following simple system of linear differential equations:

$$i\frac{d}{dt}\overline{A^{(4)}}=g_{24}\overline{A^{(2)}}+g_{14}\overline{A^{(1)}}, \tag{43}$$

$$i\frac{d}{dt}\overline{A^{(3)}}=-\Delta_3\overline{A^{(3)}}+g_{23}\overline{A^{(2)}}+g_{13}\overline{A^{(1)}}, \tag{44}$$

$$i\frac{d}{dt}\overline{A^{(2)}}=-\Delta_2\overline{A^{(2)}}-iK_2\overline{A^{(2)}}+g_{24}^{*}\overline{A^{(4)}}+g_{23}^{*}\overline{A^{(3)}}, \tag{45}$$

$$i\frac{d}{dt}\overline{A^{(1)}}=-\Delta_1\overline{A^{(1)}}-iK_1\overline{A^{(1)}}+g_{14}^{*}\overline{A^{(4)}}+g_{13}^{*}\overline{A^{(3)}}, \tag{46}$$

where $\overline{A^{(i)}}=A^{(i)}\exp(-iE_4t)$ and

$$K_i=\pi\sum_k |\xi_{ki}|^2\delta(\omega_k-\omega_i).$$

A very simple analysis of Eqs. (43)–(46) leads to the following interesting conclusions.

If the coupling constants satisfy the relations

$$g_{13}g_{24}-g_{14}g_{23}=0,$$
$$\Delta_1=\Delta_2, \tag{47}$$

and $K_1=K_2$, the atom-field system will never reach the upper states, however strong the interactions, provided that initially the two upper states are not populated and that $A^{(2)}(0)=-(g_{1j}/g_{2j})A^{(1)}(0)$, $j=1$ or 2. Let us recall that $\Delta_1=\Delta_2$ means that $E_1+\omega_1=E_2+\omega_2$. Then the transmitted light possesses just one spectral line (instead of four) of width $K_1=K_2$.

On the other hand, if the relations

$$g_{13}g_{24}-g_{14}g_{23}=0,$$
$$\Delta_3=0, \tag{48}$$

hold, the system can be trapped in the two upper levels, provided that initially the lower ones are not populated and that $A^{(4)}(0)=-(g^*_{13}/g^*_{14})A^{(3)}(0)$. Under these conditions, the system will not radiate through the walls at all and also no photon will appear in the cavity, however strong the interactions and however large the damping constants.

The last two effects depend very strongly on the initial conditions, and in particular the preparation of the system in the lower levels with exactly one photon in each mode in the cavity seems to be rather unrealistic. However, the relation in Eq. (47) as well as that in Eq. (48) leads to some interesting effects which are independent of the initial conditions.

In fact, if Eq. (48) holds, one of the eigenvalues of the matrix of coefficients on the right-hand side of Eqs. (43)–(46) is zero. In general, for arbitrary (not too small) g_{ij}, Eq. (48) does not introduce degeneracy into the linear system of Eqs. (43)–(46)—there are four different eigenvalues. But the eigenvector corresponding to the zeroth eigenvalue has its first two components equal to zero, which causes the cancellation of one line from the spectrum of transmitted light, since the amplitudes $A^{(1)}$ and $A^{(2)}$ will oscillate with just three frequencies and three damping constants. They will finally approach zero. On the other hand, the amplitudes $A^{(3)}$ and $A^{(4)}$ will approach some stationary values which are not zero. Thus, even for very slow atoms, there is a considerable probability of finding them finally in one of the excited states and with no photons in the cavity. This is completely independent of either the strength of the atom-cavity interactions—provided that the relation in Eq. (48) is fulfilled—or of the strength of the cavity-reservoir interactions. However, the latter must, of course, be very small, otherwise we could not perform the Wigner-Weisskopf elimination. The stabilization effect is shown in Fig. 1, where we have plotted the populations of the two upper levels as functions of time—as is clearly seen, the populations approach their stationary values. This and all other figures

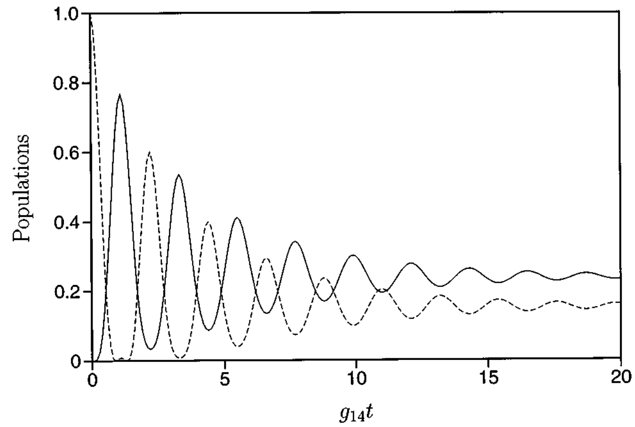

FIG. 1. The time evolution of the populations of the two upper levels is shown: $|4\rangle$ (dashed line) and $|3\rangle$ (solid line). The parameters are (in units of g_{14}) $\Delta_1=\Delta_2=0$, $K_1=0.5$, $K_2=0.4$, $g_{13}=0.8$, $g_{23}=1.6$, $g_{24}=2.0$, $\Delta_3=0$. Artificially large values of K_1 and K_2 have been chosen in order to make both the oscillations and the approach to the stationary state transparent.

have been plotted for the case of the following initial conditions: $A^{(4)}=1$, $A^{(i)}=0$, $i=1,2,3$.

Let us now suppose that the relations in Eq. (47) are approximately fulfilled and that initially the atom is prepared as an arbitrary superposition of the upper levels. Again, let us hasten to assert that in the system of Eqs. (43)–(46) there are still four different eigenvalues, i.e., *there is no degeneracy*. But one of the eigenvectors—that corresponding to the eigenvalue $-i\Delta_1-K_1$—has its third and fourth components equal to zero. This means that the system will oscillate with only three frequencies and three damping constants, hence only three lines appear in both spectra of the transmitted light. If the cavity is bad, the cancellation is fairly ''stable'' with respect to small deviations from Eq. (47) [see Figs. 2(c) and 3(c)]. This effect is of an interference nature: if Eqs. (47) are valid, there are specific phase relations in the system which do not allow one type of oscillation to be realized.

In Figs. 2 and 3, we have plotted the spectra of the transmitted light. There are two spectra since we have assumed that the reservoirs of two cavity modes are independent. Thus, $S_1(\omega)$ is the spectrum of transmitted light associated with the first mode while $S_2(\omega)$ is the spectrum associated with the second mode. They are defined as

$$S_1(\omega)\sim\frac{K_1}{\pi|\xi_{k1}|^2}\lim_{t\to\infty}|C(\omega_k,t)|^2, \tag{49}$$

$$S_2(\omega)\sim\frac{K_2}{\pi|\xi_{k2}|^2}\lim_{t\to\infty}|B(\omega_k,t)|^2. \tag{50}$$

In Figs. 2(a) and 3(a), there are shown four spectral lines for the case if Eq. (47) is not fulfilled. On the other hand, in Figs. 2(b), 2(c), 3(b), and 3(c), the spectra are plotted under the condition that Eq. (47) holds approximately. We see the evident cancellation of one spectral line.

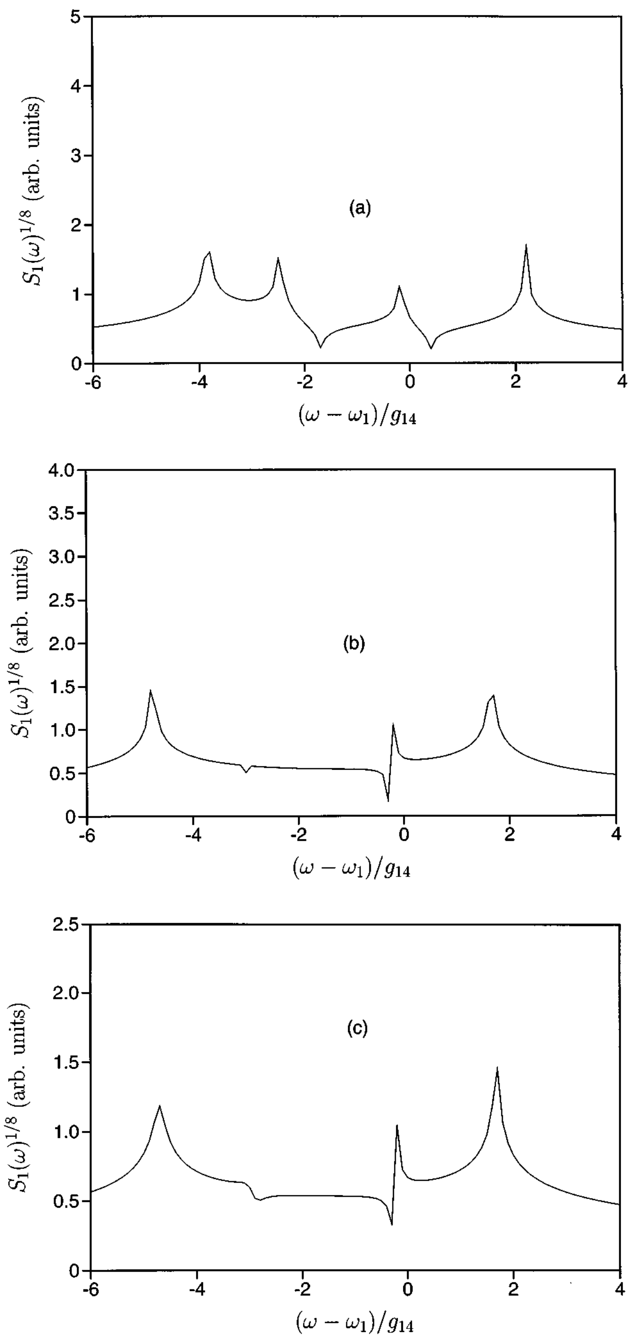

FIG. 2. Spectra of the transmitted light associated with the first mode in the absence of spontaneous emission are shown [(arb. units) are arbitrary units]: (a) $\Delta_1=3.0$, $\Delta_2=1.0$, $K_1=0.02$, $K_2=0.01$, $g_{13}=0.5$, $g_{23}=1.6$, $g_{24}=2.0$, $\Delta_3=0$ [the condition in Eq. (47) is not met]; (b) the same as in (a), but for $\Delta_2=3.001$, $g_{13}=0.801$ [the condition in Eq. (47) is approximately fulfilled]; (c) the same as in (a), but for $\Delta_2=2.9$, $g_{13}=0.81$, $K_1=0.15$, $K_2=0.1$ [there are larger deviations from Eq. (47), but the decay rates are also larger].

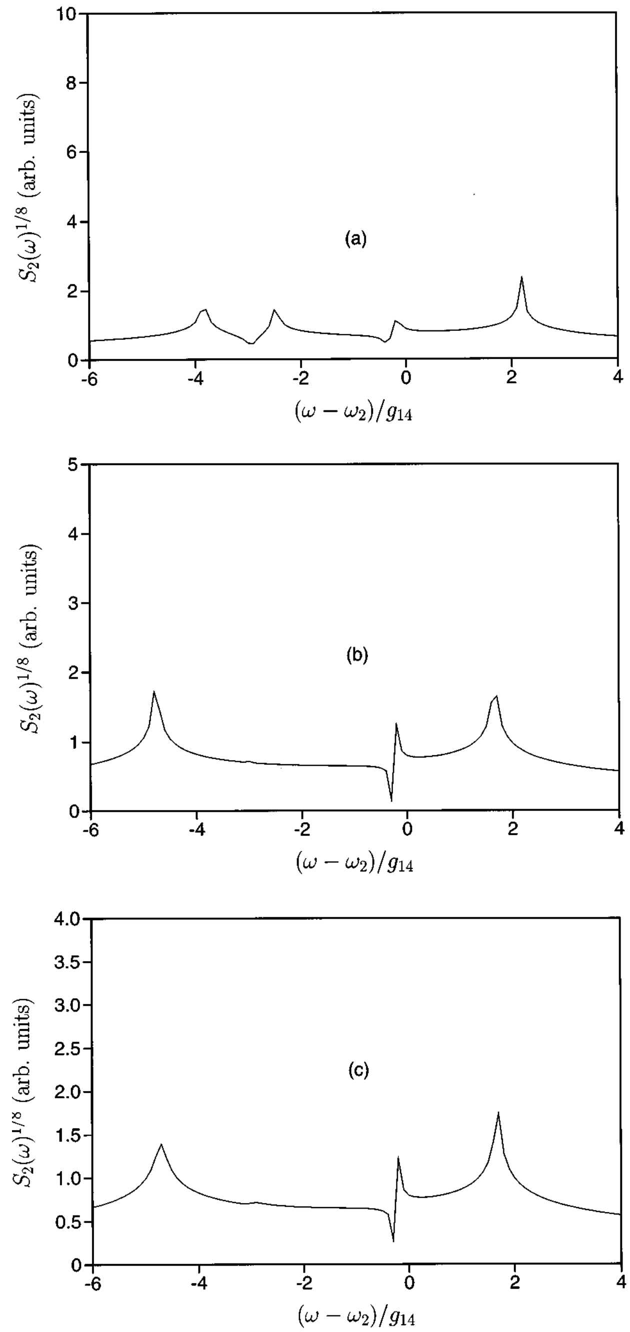

FIG. 3. This is the same as in Fig. 2, but the transmitted light is now associated with the second mode.

Let us now enrich the model by adding the possibility of spontaneous emission, which is especially important in the case of optical cavities. Again using the Wigner-Weisskopf approach, after elimination of the reservoirs' degrees of freedom we obtain

$$i\frac{d}{dt}\overline{A^{(4)}}=g_{24}\overline{A^{(2)}}+g_{14}\overline{A^{(1)}}-i(\Gamma_{41}+\Gamma_{42})\overline{A^{(4)}}-(p_1\sqrt{\Gamma_{41}\Gamma_{31}}+p_2\sqrt{\Gamma_{42}\Gamma_{32}})\overline{A^{(3)}}, \tag{51}$$

$$i\frac{d}{dt}\overline{A^{(3)}}=-\Delta_3\overline{A^{(3)}}+g_{23}\overline{A^{(2)}}+g_{13}\overline{A^{(1)}}-i(\Gamma_{31}+\Gamma_{32})\overline{A^{(3)}}-(p_1^*\sqrt{\Gamma_{41}\Gamma_{31}}+p_2^*\sqrt{\Gamma_{42}\Gamma_{32}})\overline{A^{(4)}}, \tag{52}$$

$$i\frac{d}{dt}\overline{A^{(2)}}=-\Delta_2\overline{A^{(2)}}-iK_2\overline{A^{(2)}}+g_{24}^*\overline{A^{(4)}}+g_{23}^*\overline{A^{(3)}}, \tag{53}$$

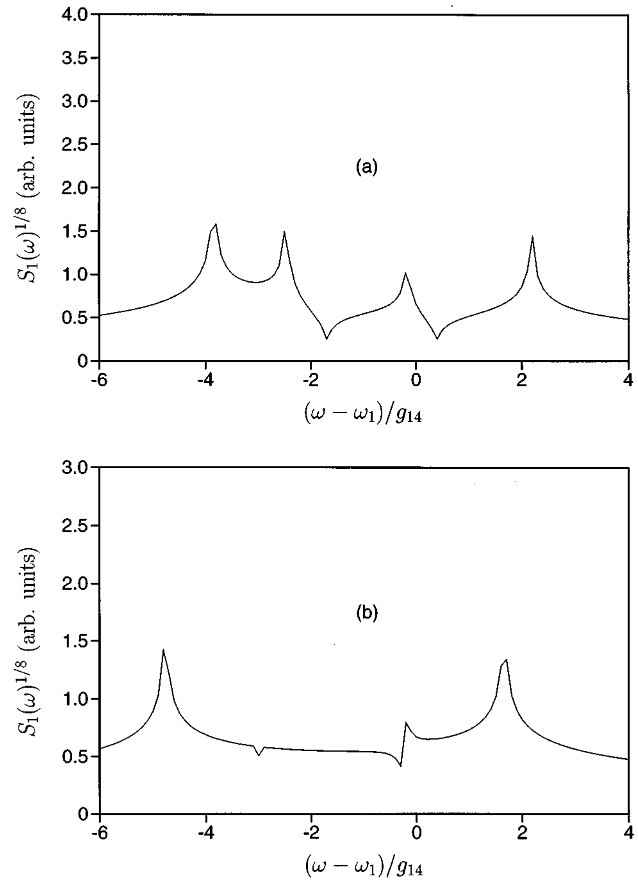

FIG. 4. Spectra of the transmitted light associated with the first mode but now in the presence of spontaneous emission are shown: (a) the same as in Fig. 2(a), but with $\Gamma_3=\Gamma_4=0.03$, $P=0$; (b) the same as in Fig. 2(b), but with $\Gamma_3=\Gamma_4=0.03$, $P=0$.

$$i\frac{d}{dt}\overline{A^{(1)}}=-\Delta_1\overline{A^{(1)}}-iK_1\overline{A^{(1)}}+g_{14}^*\overline{A^{(4)}}+g_{13}^*\overline{A^{(3)}}, \tag{54}$$

where Γ_{ji} denotes the (cavity-modified) spontaneous decay rates from level j to level i, while $p_i=\mu_{3i}^*\mu_{4i}/(|\mu_{3i}||\mu_{4i}|)$, where μ_{ji} are the transition dipole moments. In deriving Eqs. (51)–(54), we have had to assume that Δ_3 is very small when compared with both E_4-E_i and E_3-E_i, $i=1,2$ and we have ignored the Lamb shift contribution. Under the same conditions as before [Eq. (47) and $K_1=K_2$], the system can be trapped in the lower levels and thus will not exhibit any spontaneous emission at all. On the other hand, if the relation in Eq. (48) is fulfilled together with

$$i\Delta_3-\Gamma_3+\frac{g_{13}^*}{g_{14}^*}P^*=\frac{g_{14}^*}{g_{13}^*}P-\Gamma_4, \tag{55}$$

where $P=p_1\sqrt{\Gamma_{41}\Gamma_{31}}+p_2\sqrt{\Gamma_{42}\Gamma_{32}}$, $\Gamma_3=\Gamma_{31}+\Gamma_{32}$, $\Gamma_4=\Gamma_{41}+\Gamma_{42}$, one can get rid of the leakage of cavity photons—the system will be ''trapped'' in the two decaying upper levels. One may also achieve complete inhibition of radiation of any kind if additionally the following relation holds true:

$$\mathrm{Re}\left(\frac{g_{14}^*}{g_{13}^*}P-\Gamma_4\right)=0. \tag{56}$$

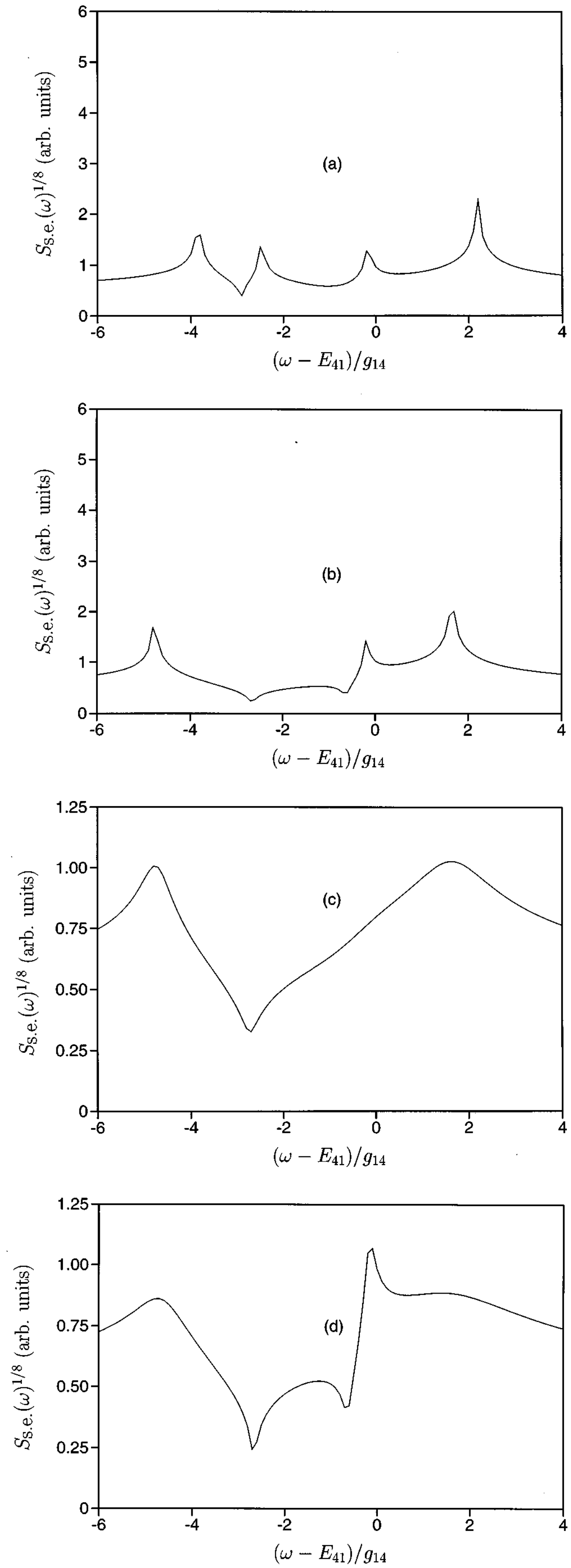

FIG. 5. Spectra of spontaneously emitted light from the cavity are shown: (a) The parameters are the same as in Fig. 4(a); (b) the parameters are the same as in Fig. 4(b); (c) this is the same as in (b), but with $\Gamma_3=\Gamma_4=1.0$, $P=0$; (d) this is the same as in (c), but with $P=0.95$.

Again, these effects depend very strongly on a special type of preparation of the system. Let us point out, however, that the cancellation effect in the spectra survives when spontaneous decay is present and it is visible in both the transmitted light and spontaneous emission spectra. In Fig. 4 we have plotted the spectrum of the transmitted light associated with the first mode (S_1). When the relation in Eq. (47) is not fulfilled [see Fig. 4(a)] we clearly see four lines. In Fig. 4(b) there are only three lines, since we have chosen parameters such that Eq. (47) does hold approximately. In Fig. 5 we show the spontaneous emission spectra, denoted by $S_{\text{s.e.}}(\omega)$ (it has been assumed that E_{21} is large when compared with the detunings and coupling constants and hence only part of the spectrum near $\omega = E_{41}$ is displayed). It consists again of four lines if the condition in Eq. 47 is not met [Fig. 5(a)] but only three lines [Fig. 5(b)] if it is fulfilled. In Figs. 5(c) and 5(d) the spontaneous emission spectra are shown for large values of the spontaneous decay rates. In Fig. 5(c) we have chosen $P=0$, while in Fig. 5(d) $P=0.95$. It is seen that the interference effects connected with the fact that there is a common reservoir for all the atomic operators does influence the spectra in our cavity case, too (cf. [18]). But we have found that these effects are actually important only for large values of the decay rates which are comparable with the coupling constants.

Thus, we have found quite a rich phenomenology of trapping and interference effects in the four-level Λ system. To close this section let us note that if in the system of Eqs. (43)–(46) there were small parameters besides those associated with the damping constants, we could perform our MMS analysis to predict the location and strength of each line. We would like to address this point in some future work.

VI. FINAL REMARKS

In this work we have analyzed a four-level Λ system consisting of four atomic levels and two modes of the electromagnetic field with special constraints imposed on the coupling constants. By linearizing the Hamiltonian, it has been shown that the system is exactly solvable: an explicit expression for the time-evolution operator can be given. In order to obtain a family of physically interesting effective Hamiltonians, the method of multiple scales has been applied. We have found that a particularly interesting example of the four-level Λ system arises if the two upper levels are degenerate. In fact, for this case, we have found an explicit, closed-form expression for the dynamics of the atomic populations, from which we infer that both regular and periodic, as well as only quasiperiodic, collapses and revivals can appear in the system. Additionally, the system can be used to produce the ''Schrödinger-cat'' states in the two-mode cavity via an experiment of the Stern-Gerlach type. We have also found the remarkable trapping properties of the system which can lead to the cancellation of radiation transmitted through the cavity walls, or the suppression of spontaneous emission, or both of these effects if some special relations between the coupling constants hold and the system is prepared in the appropriate initial conditions. Also, under some weaker and more realistic initial conditions, the system should exhibit the cancellation of one line in the transmitted light spectra and in spontaneous emission due to the interference effects.

In future work on the system, we plan to apply the method of multiple scales to the case of strong couplings and small detunings to obtain other families of effective Hamiltonians, and to investigate other nonclassical features like antibunching and squeezing as well as beating phenomena.

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