ARTICLES

Accessible information in combined and sequential quantum measurements on a binary-state signal

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The accessible information obtained in combined and sequential quantum measurements is calculated for a binary pure quantum state signal. It is shown that the accessible information obtained in the combined measurement is equal to that obtained in the sequential measurement. The result indicates that the conjecture by Peres and Wootters [Phys. Rev. Lett. **66**, 1119 (1991)] is not valid, at least for a binary quantum state signal. Furthermore, the result is compared with that obtained by Brody and Meister [Phys. Rev. Lett. **76**, 1 (1996)]. [S1050-2947(97)06601-8]

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I. INTRODUCTION

It is well known in quantum mechanics that we cannot completely distinguish between two different nonorthogonal quantum states. This fact indicates that detection errors in quantum measurement for such quantum states are inevitable. Therefore it is important to consider quantum measurement that predicts, as accurate as possible, which quantum state a physical system takes. This problem is not only of fundamental interest in quantum mechanics, but also of essential importance in optical communication and quantum information theory. A performance of quantum measurement is evaluated by the average probability of error or by the mutual information. Quantum measurement that minimizes the average probability of error or that maximizes the mutual information is referred to as optimum quantum measurement. The maximum value of the mutual information is called the accessible information.

Consider a quantum communication system in which information is transmitted in terms of two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$. When the quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ are nonorthogonal, detection error occurs, with finite probability, in such a binary communication system. To reduce the detection error and to transmit the information more accurately, we can send the information with the two identical states. In this case, the quantum state of the signal that carries information is given by $|\Psi_1\rangle=|\psi_1\rangle\otimes|\psi_1\rangle$ or $|\Psi_2\rangle=|\psi_2\rangle\otimes|\psi_2\rangle$. Thus to obtain the information, the receiver should perform quantum measurement on the signal whose quantum state is $|\Psi_1\rangle$ or $|\Psi_2\rangle$. When we perform the quantum measurement on such a signal, there are two ways; one is a combined measurement in which the two component states are treated as one composite quantum state, and the other is a sequential measurement in which one of the two component states is first measured and then the other component state is measured by using the result of the first measurement. It is interesting to consider which measurement, combined or sequential, is more effective for the signal detection.

Peres and Wootters considered the detection process for three spin-$\frac{1}{2}$ states, $|\phi_1\rangle=|\uparrow\rangle$, $|\phi_2\rangle=\frac{1}{2}|\uparrow\rangle+\sqrt{3}/2|\downarrow\rangle$, and $|\phi_3\rangle=\frac{1}{2}|\uparrow\rangle-\sqrt{3}/2|\downarrow\rangle$, which are linearly dependent ($|\phi_1\rangle=|\phi_2\rangle+|\phi_3\rangle$), where $|\uparrow\rangle$ and $|\downarrow\rangle$ stand for spin-up and spin-down states. They conjectured that the combined measurement gives more information than the sequential measurement [1]. Subsequently, Massar and Popescu have shown that the combined measurement is more effective than the sequential measurement [2]. However, they have used the evaluation quantity, which is neither the mutual information nor the average probability of error. Recently, for a binary signal consisting of the spin-$\frac{1}{2}$ particles, Brody and Meister have proven that the minimum value of the average probability of error in the combined measurement is equal to that in the sequential measurement [3]. This result seems to contradict the conjecture by Peres and Wootters. They have considered that such a contradiction is due to the difference of the optimizations of quantum measurements; Peres and Wootters considered the quantum measurement maximizing the mutual information while Brody and Meister considered the quantum measurement minimizing the average probability of error.

Therefore the purpose of this paper is to investigate the accessible information obtained in the combined and sequential quantum measurements on a binary pure quantum state signal. Our result shows that both measurements give the same amount of information. Thus the conjecture by Peres and Wootters [1] is not valid, at least, for a binary quantum state signal, though the general validity of the conjecture still remains an open problem. Furthermore our result means that Brody and Meister's reasoning is not true [3]. In Sec. II, we investigate a property of an optimum quantum measurement in which we can obtain the accessible information. Using the results, we consider the combined and sequential quantum measurements in Sec. III and Sec. IV. We give concluding remarks in Sec. V.

II. INFORMATION-OPTIMUM MEASUREMENT

In this section, we consider the accessible information obtained in quantum measurement for a binary pure quantum state signal, where the statistical operators of the signal are given by $\hat{\rho}_1=|\psi_1\rangle\langle\psi_1|$ and $\hat{\rho}_2=|\psi_2\rangle\langle\psi_2|$. We assume that the quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ are nonorthogonal and linearly independent, and we set $\langle\psi_1|\psi_2\rangle=\kappa e^{i\varphi}$ ($\kappa>0$). Then the Hilbert space of the signal becomes a two-dimensional space given by $\mathcal{H}_s=\{c_1|\psi_1\rangle+c_2|\psi_2\rangle|c_1,c_2\in\mathcal{C}\}$, where $\mathcal{C}$ stands for the field of complex numbers. The quantum measurement of the binary signal detection is described in terms of positive operator-valued measures (called quantum detection operators) [4,5], $\hat{\Pi}_1$ and $\hat{\Pi}_2$, which satisfy the relations, $\hat{\Pi}_1+\hat{\Pi}_2=\hat{I}$ and $\hat{\Pi}_j\geqslant 0$, where $\hat{I}$ is an identity operator defined on the Hilbert space $\mathcal{H}_s$. The quantity $P_c(j|k)=\mathrm{Tr}[\hat{\Pi}_j\hat{\rho}_k]$ is the conditional probability that the measurement outcome exhibits the quantum state $|\psi_j\rangle$ when the quantum state $|\psi_k\rangle$ has been actually received, where Tr is the trace operation over the Hilbert space $\mathcal{H}_s$. Let ξ_j be the prior probability of the quantum state $|\psi_j\rangle$, which is normalized as $\xi_1+\xi_2=1$. Then the average probability of error in the quantum measurement is given by $P_e=P_c(1|2)\xi_2+P_c(2|1)\xi_1$, and the mutual information [6] is calculated as

$$I=\sum_{j=1,2}\sum_{k=1,2}P_c(j|k)\xi_k\ln\left[\frac{P_c(j|k)}{\sum_{m=1,2}P_c(j|m)\xi_m}\right]. \quad (1)$$

Furthermore the output probability of the quantum state $|\psi_j\rangle$ is given by $P_{\mathrm{out}}(j)=\Sigma_{k=1,2}P_c(j|k)\xi_k$. After the measurement whose outcome indicates the quantum state $|\psi_k\rangle$, using the Bayes theorem [6], we can obtain the posterior probability of the quantum state $|\psi_j\rangle$ as $P_p(j|k)=P_c(k|j)\xi_j/P_{\mathrm{out}}(k)$. Then the mutual information is expressed as

$$I=H_{\mathrm{initial}}-H_{\mathrm{final}}, \quad (2)$$

where H_{initial} and H_{final} are the initial and expected final entropies calculated as

$$H_{\mathrm{initial}}=-\xi_1\ln\xi_1-\xi_2\ln\xi_2=H_{\mathrm{bin}}(\xi_1), \quad (3)$$

$$H_{\mathrm{final}}=-\sum_{j=1,2}P_{\mathrm{out}}(j)\left[\sum_{k=1,2}P_p(k|j)\ln P_p(k|j)\right]. \quad (4)$$

In Eq. (3), $H_{\mathrm{bin}}(x)$ is the binary entropy function [7],

$$H_{\mathrm{bin}}(x)=-x\ln x-(1-x)\ln(1-x). \quad (5)$$

In this paper, we measure information in nats.

Suppose that the quantum measurement described by the detection operators $\hat{\Pi}_1$ and $\hat{\Pi}_2$ maximizes the mutual information I. Then the detection operators $\hat{\Pi}_1$ and $\hat{\Pi}_2$ must satisfy the necessary condition obtained by Holevo [8],

$$(\hat{F}_j-\hat{\Gamma})\hat{\Pi}_j=0 \quad (j=1,2), \quad (6)$$

$$\hat{\Gamma}=\sum_{j=1,2}\hat{F}_j\hat{\Pi}_j=\sum_{j=1,2}\hat{\Pi}_j\hat{F}_j, \quad (7)$$

where the operator $\hat{F}_j$ is given by

$$\hat{F}_j=\sum_{k=1,2}\hat{\rho}_k\xi_k\ln\left[\frac{P_c(j|k)}{\sum_{m=1,2}P_c(j|m)\xi_m}\right]. \quad (8)$$

Using Eqs. (6) and (7), we can show that $\hat{\Pi}_1$ and $\hat{\Pi}_2$ become projection operators defined on the two-dimensional Hilbert space $\mathcal{H}_s$. This is proved as follows [9]. First the resolution of identity $\hat{\Pi}_1+\hat{\Pi}_2=\hat{I}$ gives the relation $\hat{\Pi}_1\hat{\Pi}_2=\hat{\Pi}_2\hat{\Pi}_2$. Then using Eqs. (6) and (7), we obtain the relation

$$(\hat{F}_1-\hat{F}_2)\hat{\Pi}_1\hat{\Pi}_2=(\hat{F}_1-\hat{F}_2)\hat{\Pi}_2\hat{\Pi}_1=0, \quad (9)$$

where the operator $\hat{F}_1-\hat{F}_2$ is calculated from Eq. (8) as

$$\hat{F}_1-\hat{F}_2=\pi_1\hat{\rho}_1-\pi_2\hat{\rho}_2=|\psi_1\rangle\pi_1\langle\psi_1|-|\psi_2\rangle\pi_2\langle\psi_2|, \quad (10)$$

with

$$\pi_1=\xi_1\ln\left[\frac{P_c(1|1)P_{\mathrm{out}}(2)}{P_c(2|1)P_{\mathrm{out}}(1)}\right], \quad \pi_2=\xi_2\ln\left[\frac{P_c(2|2)P_{\mathrm{out}}(1)}{P_c(1|2)P_{\mathrm{out}}(2)}\right]. \quad (11)$$

Since $\hat{F}_1-\hat{F}_2$ is a Hermitian operator defined on the two-dimensional Hilbert space $\mathcal{H}_s$, this operator has two eigenstates $|\hat{\psi}_1\rangle$ and $|\hat{\psi}_2\rangle$ with eigenvalues μ_1 and μ_2. Using the spectral decomposition of the operator $\hat{F}_1-\hat{F}_2$ and the orthogonality of the eigenstates, Eq. (9) becomes $\mu_j\langle\hat{\psi}_j|\hat{\Pi}_1\hat{\Pi}_2=\mu_j\langle\hat{\psi}_j|\hat{\Pi}_2\hat{\Pi}_1=0$ ($j=1,2$). If $\mu_1\mu_2\neq 0$, the completeness of the eigenstates gives the relation $\hat{\Pi}_1\hat{\Pi}_2=\hat{\Pi}_2\hat{\Pi}_1=0$, or equivalently, $\hat{\Pi}_j^2=\hat{\Pi}_j$ ($j=1,2$). Thus we have found that $\hat{\Pi}_1$ and $\hat{\Pi}_2$ are one-dimensional projection operators defined on the Hilbert space $\mathcal{H}_s$ unless the operator $\hat{F}_1-\hat{F}_2$ has zero eigenvalue.

Since the detection operators $\hat{\Pi}_1$ and $\hat{\Pi}_2$ that maximize the mutual information are one-dimensional projection operators, we can set $\hat{\Pi}_j=|\omega_j\rangle\langle\omega_j|$ ($j=1,2$), where $\{|\omega_1\rangle,|\omega_2\rangle\}$ is a complete orthonormal system in the Hilbert space $\mathcal{H}_s$. On the other hand, since the signal quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly independent, we can construct a complete orthonormal system $\{|\phi_1\rangle,|\phi_2\rangle\}$ as

$$|\phi_1\rangle=\frac{|\psi_1\rangle+e^{-i\varphi}|\psi_2\rangle}{\sqrt{2(1+\kappa)}}, \quad |\phi_2\rangle=\frac{|\psi_2\rangle-e^{i\varphi}|\psi_1\rangle}{\sqrt{2(1-\kappa)}}. \quad (12)$$

Since $\{|\omega_1\rangle,|\omega_2\rangle\}$ and $\{|\phi_1\rangle,|\phi_2\rangle\}$ are complete orthonormal systems in the same Hilbert space, there must be a unitary transformation as follows:

$$\begin{pmatrix}|\omega_1\rangle\\|\omega_2\rangle\end{pmatrix}=\begin{pmatrix}e^{i\alpha/2}\cos(\beta/2) & e^{-i\alpha/2}\sin(\beta/2)\\-e^{i\alpha/2}\sin(\beta/2) & e^{-i\alpha/2}\cos(\beta/2)\end{pmatrix}\begin{pmatrix}|\phi_1\rangle\\|\phi_2\rangle\end{pmatrix}, \quad (13)$$

where the parameters α and β are determined by the requirement that the mutual information should be maximized. Such a requirement is that $\partial I/\partial\alpha=\partial I/\partial\beta=0$ and the Hesse matrix calculated from I with respect to α and β is negative definite. After some calculation, we obtain the parameters α and β [10],

$$\alpha=\varphi,\quad \sin\beta=\mp\sqrt{\frac{1-\kappa^2}{1-4\xi_1\xi_2\kappa^2}},$$
$$\cos\beta=\pm\frac{(\xi_1-\xi_2)\kappa}{\sqrt{1-4\xi_1\xi_2\kappa^2}}. \tag{14}$$

Using Eqs. (12) and (13), we can express the optimum detection operators $\hat{\Pi}_1$ and $\hat{\Pi}_2$ in terms of the signal quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$,

$$\hat{\Pi}_1=\frac{1}{2}\left(\frac{1-\kappa\cos\beta}{1-\kappa^2}-\frac{\sin\beta}{\sqrt{1-\kappa^2}}\right)|\psi_1\rangle\langle\psi_1| + \frac{1}{2}\left(\frac{1-\kappa\cos\beta}{1-\kappa^2}+\frac{\sin\beta}{\sqrt{1-\kappa^2}}\right)|\psi_2\rangle\langle\psi_2| - \frac{\kappa-\cos\beta}{2(1-\kappa^2)} \times(e^{i\varphi}|\psi_1\rangle\langle\psi_2|+e^{-i\varphi}|\psi_2\rangle\langle\psi_1|), \tag{15}$$

$$\hat{\Pi}_2=\frac{1}{2}\left(\frac{1+\kappa\cos\beta}{1-\kappa^2}+\frac{\sin\beta}{\sqrt{1-\kappa^2}}\right)|\psi_1\rangle\langle\psi_1| + \frac{1}{2}\left(\frac{1+\kappa\cos\beta}{1-\kappa^2}-\frac{\sin\beta}{\sqrt{1-\kappa^2}}\right)|\psi_2\rangle\langle\psi_2| - \frac{\kappa+\cos\beta}{2(1-\kappa^2)} \times(e^{i\varphi}|\psi_1\rangle\langle\psi_2|+e^{-i\varphi}|\psi_2\rangle\langle\psi_1|). \tag{16}$$

It is easy to check that $\hat{\Pi}_1+\hat{\Pi}_2$ is an identity operator defined on the Hilbert space $\mathcal{H}_s$ of the signal, namely, $(\hat{\Pi}_1+\hat{\Pi}_2)|\psi\rangle=|\psi\rangle$ for any $|\psi\rangle\in\mathcal{H}_s$.

Therefore we finally obtain the accessible information $I_{\rm opt}$,

$$I_{\rm opt}=\frac{1}{2}\sum_{j=1,2}\xi_j(1+P_j)\ln\left(\frac{1+P_j}{1+Q_j}\right) \times+\frac{1}{2}\sum_{j=1,2}\xi_j(1-P_j)\ln\left(\frac{1-P_j}{1-Q_j}\right), \tag{17}$$

where the parameters P_j and Q_j are given by

$$P_j=\frac{1-2(1-\xi_j)\kappa^2}{\sqrt{1-4\xi_1\xi_2\kappa^2}},\quad Q_j=\frac{2\xi_j-1}{\sqrt{1-4\xi_1\xi_2\kappa^2}}, \tag{18}$$

which satisfy the following relations:

$$\xi_1P_1-\xi_2P_2=Q_1=-Q_2,\quad \xi_1P_1+\xi_2P_2=\sqrt{1-4\xi_1\xi_2\kappa^2}, \tag{19}$$

$$\frac{(1+P_1)(1-Q_1)}{(1-P_1)(1+Q_1)}=\frac{(1+P_2)(1-Q_2)}{(1-P_2)(1+Q_2)}=\frac{1+\sqrt{1-4\xi_1\xi_2\kappa^2}}{1-\sqrt{1-4\xi_1\xi_2\kappa^2}}, \tag{20}$$

$$\frac{(1+P_1)(1-P_2)}{(1+Q_1)(1-Q_2)}=\frac{(1-P_1)(1+P_2)}{(1-Q_1)(1+Q_2)}=\kappa^2, \tag{21}$$

$$\frac{1-P_1^2}{1-Q_1^2}=\kappa^2\frac{\xi_2}{\xi_1},\quad \frac{1-P_2^2}{1-Q_2^2}=\kappa^2\frac{\xi_1}{\xi_2}. \tag{22}$$

These relations are used to show that the accessible information in the sequential measurement is equal to that in the combined measurement.

Furthermore it is easy to see that the average probability of error becomes $P_e=\frac{1}{2}(1\pm\sqrt{1-4\xi_1\xi_2\kappa^2})$ in the quantum measurement described by the detection operators given by Eqs. (15) and (16). It is important to note that $P_{\rm opt}=\frac{1}{2}(1-\sqrt{1-4\xi_1\xi_2\kappa^2})$ is the minimum value of the average probability of error obtained by means of the quantum detection theory [4]. When we use this minimum value, the accessible information $I_{\rm opt}$ can be expressed as

$$I_{\rm opt}=H_{\rm bin}(\xi_1)-H_{\rm bin}(P_{\rm opt}), \tag{23}$$

where the function $H_{\rm bin}(x)$ is given by Eq. (5). In this expression, the first term on the right-hand side is the Shannon information of the signal, and the second term represents the information loss due to the detection error. In the optimum quantum measurement, the conditional probability $P_c(j|k)$ can be expressed as

$$P_c(1|1)=1-P_c(2|1)=\tfrac{1}{2}(1+P_1), \tag{24}$$

$$P_c(2|2)=1-P_c(1|2)=\tfrac{1}{2}(1+P_2), \tag{25}$$

and the output probability $P_{\rm out}(j)$ is given by

$$P_{\rm out}(1)=1-P_{\rm out}(2)=\tfrac{1}{2}(1+Q_1). \tag{26}$$

Moreover, after the measurement whose outcome indicates the quantum state $|\psi_k\rangle$, using the Bayes theorem, we obtain the posterior probability $P_p(j|k)$ of the quantum state $|\psi_j\rangle$ as

$$P_p(1|1)=1-P_p(2|1)=\xi_1\frac{1+P_1}{1+Q_1}, \tag{27}$$

$$P_p(2|2)=1-P_p(1|2)=\xi_2\frac{1+P_2}{1+Q_2}. \tag{28}$$

These results are used when we investigate the combined and sequential quantum measurements.

III. COMBINED QUANTUM MEASUREMENT

We now consider a quantum state signal that consists of the two particles, where the quantum states of the signal are given by $|\Psi_1\rangle=|\psi_1\rangle\otimes|\psi_1\rangle$ and $|\Psi_2\rangle=|\psi_2\rangle\otimes|\psi_2\rangle$ with $|\langle\Psi_1|\Psi_2\rangle|=\kappa^2$. In this section, we consider the combined measurement that maximizes the mutual information. In the combined measurement, we perform the quantum measurement on one composite particle instead of the individual particles to know the quantum state $|\Psi_j\rangle$, but not $|\psi_j\rangle$. Since the quantum states of the composite particle are $|\Psi_1\rangle$ and $|\Psi_2\rangle$, which are linearly independent, the Hilbert space of the signal becomes a two-dimensional space defined by $\mathcal{H}_s=\{c_1|\Psi_1\rangle+c_2|\Psi_2\rangle|c_1,c_2\in\mathcal{C}\}$. Therefore the accessible information I_C in the combined measurement is obtained by replacing κ^2 with κ^4 in Eqs. (17) and (18),

$$I_C = H_{\rm bin}(\xi_1) - H_{\rm bin}(\mathcal{P}), \tag{29}$$

with

$$\mathcal{P} = \frac{1}{2}(1 - \sqrt{1 - 4\xi_1\xi_2\kappa^4}). \tag{30}$$

Furthermore in the combined measurement that gives the accessible information I_C, the average probability of error P_C is calculated to be $P_C = \mathcal{P}$, which is the minimum value.

IV. SEQUENTIAL QUANTUM MEASUREMENT

In this section, we consider the sequential quantum measurement for the signal that takes the quantum state $|\Psi_j\rangle = |\psi_j\rangle \otimes |\psi_j\rangle$ $(j=1,2)$. In this measurement, the two particles are measured in sequence. Thus the measurement consists of the two steps. The first is to perform the measurement on one of the two particles such that the mutual information should be maximized. By using the result of the first measurement, the second measurement is carried out on the other particle such that the maximum value of the mutual information is obtained.

After the first measurement, we can obtain the posterior probability $P_p^{(1)}(j|k)$ of the quantum state $|\psi_j\rangle$ from Eqs. (18), (27), and (28),

$$P_p^{(1)}(1|1) = 1 - P_p^{(1)}(2|1) = \xi_1 \frac{1 - 2\xi_2\kappa^2 + \sqrt{1 - 4\xi_1\xi_2\kappa^2}}{\xi_1 - \xi_2 + \sqrt{1 - 4\xi_1\xi_2\kappa^2}}, \tag{31}$$

$$P_p^{(1)}(2|2) = 1 - P_p^{(1)}(1|2) = \xi_2 \frac{1 - 2\xi_1\kappa^2 + \sqrt{1 - 4\xi_1\xi_2\kappa^2}}{\xi_2 - \xi_1 + \sqrt{1 - 4\xi_1\xi_2\kappa^2}}, \tag{32}$$

and the output probability in the first measurement is obtained from Eqs. (18) and (26),

$$P_{\rm out}^{(1)}(1) = 1 - P_{\rm out}^{(1)}(2) = \frac{1}{2}\left[1 + \frac{\xi_1 - \xi_2}{\sqrt{1 - 4\xi_1\xi_2\kappa^2}}\right]. \tag{33}$$

When we perform the second measurement, we use the posterior probability after the first measurement as the prior probability of the quantum state in the second measurement. That is, when the outcome of the first measurement exhibits the quantum state $|\psi_j\rangle$, we set the prior probability in the second measurement as $\xi_k = P_p^{(1)}(k|j) \equiv \xi_k(j)$. Thus when the second measurement maximizes the mutual information, the conditional probability becomes

$$P_c^{(2)}(1|1;j) = 1 - P_c^{(2)}(2|1;j) = \tfrac{1}{2}[1 + P_1(j)], \tag{34}$$

$$P_c^{(2)}(2|2;j) = 1 - P_c^{(2)}(1|2;j) = \tfrac{1}{2}[1 + P_2(j)], \tag{35}$$

where we have used Eqs. (24) and (25). The parameters $P_1(j)$ and $P_2(j)$ are given by

$$P_1(j) = \frac{1 - 2\xi_2(j)\kappa^2}{\sqrt{1 - 4\xi_1\xi_2\kappa^4}}, \quad P_2(j) = \frac{1 - 2\xi_1(j)\kappa^2}{\sqrt{1 - 4\xi_1\xi_2\kappa^4}}. \tag{36}$$

Here we have used the relation $\xi_1(j)\xi_2(j) = \xi_1\xi_2\kappa^2$. The output probability in the second measurement becomes

$$P_{\rm out}^{(2)}(1;j) = 1 - P_{\rm out}^{(2)}(2;j) = \tfrac{1}{2}[1 + Q_1(j)], \tag{37}$$

with

$$Q_1(j) = \xi_1(j)P_1(j) - \xi_2(j)P_2(j). \tag{38}$$

Furthermore the posterior probability after the second measurement is obtained by using the Bayes theorem,

$$P_p^{(2)}(1|1;j) = 1 - P_p^{(2)}(2|1;j) = \xi_1(j)\frac{1 + P_1(j)}{1 + Q_1(j)}, \tag{39}$$

$$P_p^{(2)}(2|2;j) = 1 - P_p^{(2)}(1|2;j) = \xi_2(j)\frac{1 + P_2(j)}{1 + Q_2(j)}, \tag{40}$$

where we set $Q_2(j) = -Q_1(j)$. In Eqs. (34)–(40), the parameter j means that the outcome of the first measurement exhibits the quantum state $|\psi_j\rangle$.

Using Eqs. (4), (37), (39), and (40), we can calculate the expected final entropy after the second measurement as

$$H_{\rm final} = -\sum_{j=1,2} P_{\rm out}^{(1)}(j)\left\{\sum_{k=1,2} P_{\rm out}^{(2)}(k;j) \times \left[\sum_{m=1,2} P_p^{(2)}(m|k;j)\ln P_p^{(2)}(m|k;j)\right]\right\}. \tag{41}$$

Substituting Eqs. (37), (39), and (40) into this equation, we can obtain the accessible information I_S obtained in the sequential measurement from Eq. (2),

$$I_S = I_1 + I_2, \tag{42}$$

with

$$I_1 = H_{\rm bin}(\xi_1) + \sum_{j=1,2} P_{\rm out}^{(1)}(j)\sum_{k=1,2}\xi_k(j)\ln\xi_k(j), \tag{43}$$

$$I_2 = \frac{1}{2}\sum_{j=1,2} P_{\rm out}^{(1)}(j)\sum_{k=1,2}\xi_k(j)[1 + P_k(j)]\ln\left[\frac{1 + P_k(j)}{1 + Q_k(j)}\right] + \frac{1}{2}\sum_{j=1,2} P_{\rm out}^{(1)}(j)\sum_{k=1,2}\xi_k(j)[1 - P_k(j)]\ln\left[\frac{1 - P_k(j)}{1 - Q_k(j)}\right]. \tag{44}$$

It should be noted here that for the parameters $P_k(j)$, $Q_k(j)$, and $\xi_k(j)$, the relations obtained by replacing P_k, Q_k, and ξ_k with $P_k(j)$, $Q_k(j)$, and $\xi_k(j)$, in Eqs. (19)–(22) are established. Using these relations and $\xi_1(j)\xi_2(j) = \xi_1\xi_2\kappa^2$, we can calculate Eq. (44) as

$$I_2 = -\sum_{j=1,2} P_{\rm out}^{(1)}(j)\sum_{k=1,2}\xi_k(j)\ln\xi_k(j) - H_{\rm bin}(\mathcal{P}), \tag{45}$$

where the quantity $\mathcal{P}$ is given by Eq. (30). Thus we finally obtain the following expression of the accessible information in the sequential measurement,

$$I_S = H_{\rm bin}(\xi_1) - H_{\rm bin}(\mathcal{P}). \tag{46}$$

Comparing this with Eq. (29), we find the equality

$$I_C = I_S. \tag{47}$$

Therefore it is found that the accessible information obtained in the sequential measurement is equal to that obtained in the combined measurement.

The minimum value of the average probability of error in the sequential measurement is calculated as

$$P_S = \sum_{j=1,2} P_{\rm out}^{(1)}(j)[P_c^{(2)}(2|1;j)\xi_1(j) + P_c^{(2)}(1|2;j)\xi_2(j)]. \tag{48}$$

Substituting Eqs. (33)–(37) into this equation, we can obtain $P_S = \mathcal{P}$. Thus we find the equality, $P_S = P_C$, which indicates that the minimum value of the average probability of error in the combined measurement is equal to that in the sequential measurement. When the quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ are the spin-$\frac{1}{2}$ quantum states, this result is equivalent to that obtained by Brody and Meister [3].

V. CONCLUDING REMARKS

The main result of this paper is that for a binary pure quantum state signal, the accessible information obtained in the combined measurement is shown to be equal to that obtained in the sequential measurement; namely, $I_C = I_S$. This result indicates that the conjecture by Peres and Wootters [1] is not valid, at least for a binary quantum state signal. The general validity of their conjecture remains an open problem. They considered the linearly dependent three quantum states $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ of a spin-$\frac{1}{2}$ particle. In such a special case, the combined measurement could give more information than the sequential measurement. Furthermore, our result shows that Brody and Meister's reasoning is not correct. They considered the minimization of the average probability of error and showed that the combined and sequential measurements gave the same average probability of error. Then they have inferred that the difference from Peres and Wootters' result is due to the difference between the minimization of the average probability of error and the maximization of the mutual information. As seen from our result, this is not the case. For a binary pure quantum state signal, an optimum quantum measurement maximizing the mutual information is equal to that minimizing the average probability of error [9].

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