Simultaneously forbidden resonances in the Autler-Townes effect with a modulated pump

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It was shown recently that systems subject to a strong modulating interaction can exhibit a new property in their response to a probe field. Under certain conditions an infinite number of resonances are simultaneously forbidden [V. N. Smelyanskiy, G. W. Ford, and R. S. Conti, Phys. Rev. A **53**, 2598 (1996)]. In the present paper we investigate this effect in the case of a three-state system in which a strong pump field with a periodic frequency modulation Ω couples a pair of excited levels while the complex Autler-Townes spectrum is probed via a weak field that connects one of the coupled states to the ground state. Under certain conditions a half-infinite comb of spectral lines, spaced by Ω, simultaneously disappear from the Autler-Townes spectrum. These lines are positioned above or below a unique edge frequency, which is that of the probe transition in the absence of the strong field. It is shown that the aforementioned effect results from a special factorization property of the corresponding Floquet Hamiltonian that describes the Autler-Townes spectrum. Detailed analysis of this property is presented. In particular, it is found that the subset of the parameter space of the system where the factorization occurs consists of an infinite number of quasiperiodic manifolds. These manifolds exhibit some universal features related to the degeneracy of the dressed states. The line shapes of the probe resonances near the degeneracy points are derived. The intensities of the probe resonances are investigated in the limit of Ω small compared with the modulation depth and the strength of the pump field. In the latter case, effects related to the avoided crossings of the dressed-state levels are considered. The Floquet Hamiltonian that describes the Autler-Townes spectrum in the case considered, *effectively* corresponds to a special model of a periodically driven system (with period $2\pi/\Omega$) in which an external perturbation has the form of an operator projecting onto a single-quantum state. We generalize this model to the case of an N-level periodically driven system where the simultaneous vanishing of a half-infinite number of the dressed-state Fourier harmonics are analyzed. Possible experimental tests of the effect are suggested. [S1050-2947(97)04303-5]

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I. INTRODUCTION

One of the most important aspects of nonlinear matter-radiation interactions is related to the fact that the action of a strong resonance field alters the properties of a physical system in an essential way. This topic has been very thoroughly investigated and many interesting effects have been observed. Among these are the ac Stark effect [1], the Autler and Townes effect [2], gain without inversion [3,4], electromagnetically induced transparency [5,6], etc.

One of the basic concepts in this area, the spectrum of a driven system, can be understood from the consideration of a pair of quantum energy levels resonantly coupled via a strong monochromatic field of frequency ω [7]. This field will induce low-frequency oscillations in the system, termed "Rabi flopping," with frequency $r \ll \omega$. Considered in the rotating-wave approximation the time-varying wave function of the system will oscillate at four distinct frequencies, which can be chosen, for example, as $\pm r/2$ and $\omega \pm r/2$. This doubling of oscillation frequencies compared with the undriven system is effectively the splitting of each of the energy levels into a doublet. The splitting of each doublet is r, while the spacing between the two doublets is the driving frequency ω. Such a *spectrum* for a resonantly driven system is frequently described in terms of a pair of dressed states, which are combined states of an electromagnetic field and a quantum system [8]. Each dressed state corresponds to a pair of levels spaced with the energy of a photon of the driving field.

This spectrum can be investigated experimentally using a widely adopted experimental method originated in the seminal work of Autler and Townes [2]. In this method an additional weak probe field that connects one of the two strongly coupled energy levels to some third level is used. As a result of the strong-field level splitting described above, the spectral peak in the probe absorption spectrum will also split into a doublet of lines [2]. This splitting, called the Autler-Townes effect, is one of the most direct manifestations of the spectrum of dressed states. Numerous observations of this effect were made for various wavelengths in gases [9,10], laser-cooled atoms [11], and the solid state [12], in both steady-state and transient [13] regimes, and for three [9] or more levels involved in the effect [10], including the case of continuum-continuum splitting [14].

Theoretical investigation of this effect has included time-independent probe absorption and emission spectra in the three-level configuration [15], as well as multilevel systems and time-dependent absorption spectra [16]. The general case of multiple strong fields connecting different pairs of levels has been treated in Refs. [6,17].

Recently the phenomenon of electromagnetically induced transparency [5,6] has attracted considerable interest. This effect is closely related to the Autler-Townes effect when the medium is made transparent to a certain probe frequency as a result of destructive interference of the split components of the probe transition [18]. At the same time a resonantly enhanced third-order susceptibility can be obtained [5].

All the work mentioned above was restricted to the use of monochromatic fields, which is not to say that sufficient attention has been paid to the Autler-Townes pump-probe scheme with a *modulated* pump field. In this regard consider fields with a periodic frequency modulation (FM). Use of such fields, for example, is the basis for ultrasensitive absorption spectroscopy [19–21] where the resonant information is put at a modulation frequency (and its harmonics) that is high compared to the predominantly low-frequency noise of lasers. A periodically modulated pump field provides an interesting tool for Autler-Townes-type spectroscopy due to the large number of sidebands and additional parameters that can be sensitively controlled over wide ranges.

In this paper we consider a new object for Autler-Townes-type spectroscopy, a quantum system dressed by a strong resonant pump field of the form

$$E(t)=\mathcal{E}\exp[-i\omega t-i\phi(t)]+\text{c.c.},\qquad \phi(t)=M\sin(\Omega t). \tag{1}$$

Here $\mathcal{E}$ is a field amplitude, ω is the carrier frequency of the field, M is the modulation index, and Ω is the modulation frequency. Such a field is an essentially polychromatic field with a carrier wave at the frequency ω and an infinite number of sidebands $\omega+n\Omega$ $(n=\pm1,\pm2,\dots)$.

If both the modulation index M and the field amplitude $\mathcal{E}$ are sufficiently large then the Fourier harmonics of the field will be strongly mixed by a nonperturbative nonlinearity in the saturated quantum transition [22]. Due to the periodic character of the modulation the spectrum of dressed states will consist, not of doublets as in the unmodulated case, but rather of infinite *ladders* of sublevels with the spacing between sublevels corresponding to the modulation frequency. Such a spectrum exhibits nontrivial nonlinear features [23,24].

The effect of the interaction of a two-level system with a periodic FM field has been intensively investigated earlier [19,27–30] and more recently [22,23,25,26,31–33]. Periodic modulation of the field produces additional sidebands in the scattered spectrum [31] and leads to a periodic modulation of the absorption coefficient and fluorescence signal [22]. The strong field and strong modulation produce additional resonances in the spectra [22], a nonlinear phenomenon that has also been intensively studied, but not yet completely understood, in the related field of magnetic resonance [23,25].

All of the above-mentioned results are restricted to the case with two quantum levels. Following the approach of Autler-Townes spectroscopy that involves a third level and an additional (probe) field, allows one to investigate more observables and to deal directly with the spectrum of a system dressed by a periodically modulated pump. Because the type of nonlinearity in this case is more complex than that for the case of the usual dressed states, new effects can be expected to appear [24].

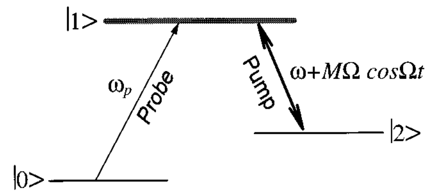

FIG. 1. Three-level pump-probe scheme in the λ configuration with a frequency modulated pump.

The organization of this paper is as follows: In Sec. II we first consider a two-level system subject to a strong pump field of the form (1) with (near) resonant carrier frequency. We then describe its dressed-state spectrum and the infinite set of resonances induced by a weak probe that connects one of the two strongly coupled levels to a third level. In Sec. III we consider the factorization property of the effective (Floquet) Hamiltonian that describes the dressed-state spectrum of the system and a related effect of simultaneously forbidden resonances. This is the principal result of the paper. Then in Sec. IV we investigate this effect analytically and numerically, using the method of continued fractions. Section V deals with topological properties of the manifolds in the parameter space of the system on which the effect of simultaneously forbidden resonances takes place. In this section also the effect of a degeneracy of the dressed-state levels will be considered. In Sec. VI we investigate the limit of small modulation frequency. Section VII concludes with a discussion and summary.

II. THE LINEAR RESPONSE TO A PROBE FIELD

We consider a three-level system in the Λ configuration as shown in Fig. 1. A strong FM field of the form (1) is applied between an excited state $|1\rangle$ and an intermediate state $|2\rangle$. A weak probe field

$$E_p(t)=\mathcal{E}_p e^{-i\omega_p t}+\text{c.c.} \tag{2}$$

resonantly excites the system from the ground state $|0\rangle$ to the state $|1\rangle$. We will be interested in the linear response of the system to the probe field and will calculate the probe absorption spectrum and the induced polarization for the probe transition. The equations for the time-varying probability amplitudes of a single atom are

$$i\frac{d}{dt}\begin{pmatrix}\Psi_0\\ \Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}-\omega_0 & -\frac{1}{\hbar}d_{01}E_p(t) & 0\\ -\frac{1}{\hbar}d_{10}E_p(t) & -\frac{i}{2}\gamma_1 & -\frac{1}{\hbar}d_{12}E(t)\\ 0 & -\frac{1}{\hbar}d_{21}E(t) & -\omega_2-\frac{i}{2}\gamma_2\end{pmatrix}\begin{pmatrix}\Psi_0\\ \Psi_1\\ \Psi_2\end{pmatrix}, \tag{3}$$

where $-\omega_0$ and $-\omega_2$ are the energies of the states $|0\rangle$ and $|2\rangle$, respectively, the energy of the state $|1\rangle$ is set to zero. The relaxation widths of the levels are $\hbar\gamma_{1,2}$, and d_{01} and d_{12} are dipole matrix elements (considered real). In order to find a linear response to a probe field the effect of the probe upon the ground-state population must be neglected. Therefore we write

$$\Psi_0(t)=e^{i\omega_0 t}. \tag{4}$$

We now transform to the frame rotating with the instantaneous frequency of the coupling field

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}\psi_1\\ e^{i\omega t+i\phi(t)}\psi_2\end{pmatrix}. \tag{5}$$

After neglecting counterrotating terms in the new frame [valid if $|\dot{\phi}(t)|\ll\omega$] and taking into account Eq. (4), we write the equations for the slowly varying amplitudes $\psi_{1,2}(t)$

$$i\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}-\dfrac{i\gamma_1}{2} & r\\ r & \Delta+\dot{\phi}(t)-\dfrac{i\gamma_2}{2}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}+r_p\begin{pmatrix}e^{-i(\omega_p-\omega_0)t}\\ 0\end{pmatrix}, \tag{6}$$

where

$$r=-\frac{1}{\hbar}d_{12}\mathcal{E},\ \ r_p=-\frac{1}{\hbar}d_{01}\mathcal{E}_p, \tag{7}$$

and

$$\Delta=\omega-\omega_2. \tag{8}$$

The quantity Δ in Eq. (8) is the detuning from resonance of the carrier frequency of the coupling field. Equation (6) is valid for $t\ll\gamma_1/r_p^2$ and $\gamma_{1,2}\gg r_p$. The first term on the right-hand side (rhs) of Eq. (6) describes the resonant coupling between the states $|1\rangle$ and $|2\rangle$ including the effect of dissipation, while the second term in Eq. (6) corresponds to the weak probe.

In this paper we will be interested in the limit of weak dissipation

$$r,\ \ \Omega\gg\gamma_{1,2}. \tag{9}$$

It is advantageous in this limiting case to use, rather than the stationary states $|1\rangle$ and $|2\rangle$, the dynamical (field-dressed) states, which incorporate the effect of a strong coupling exactly. These can be obtained by solving Eq. (6) while neglecting the small terms proportional to $\gamma_{1,2}$ and r_p. The Schrödinger equation for the time-varying amplitudes of the dressed states $\Phi_{1,2}(t)$ then takes the form

$$i\frac{d}{dt}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=\begin{pmatrix}0 & r\\ r & \Delta+M\Omega\cos(\Omega t)\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}. \tag{10}$$

Here we have used the explicit form of the phase modulation $\phi(t)$ [Eq. (1)].

Because the Hamiltonian in Eq. (10) depends upon time periodically with a period $2\pi/\Omega$ one can write two linearly independent solutions of Eq. (10) in a form that follows from the Floquet theorem [2,34]

$$\Phi_i^{\alpha}(t)=e^{-i\lambda_\alpha t}\sum_{n=-\infty}^{\infty}\Phi_{i,n}^{\alpha}e^{-in\Omega t}\quad(\alpha=\pm,\ \ i=1,2), \tag{11}$$

where the index α distinguishes the two solutions. The parameters $\lambda_\pm$ (rather, $\hbar\lambda_\pm$) are called the quasienergies [35,36]. As seen from the form of the dressed states in Eq. (11) the choice of quasienergies is ambiguous within integer multiples of Ω. Each of the dressed states (11) is associated with a ladder of quasienergy ''levels''

$$\lambda_\alpha+n\Omega,\quad \alpha=\pm,\ \ n=0,\pm1,\ldots. \tag{12}$$

There will be two such ladders that correspond to the two dressed states. It follows from Eq. (10) that these ladders are complimentary to each other

$$\lambda_+ + \lambda_- = \Delta,\ \ \mathrm{mod}(\Omega). \tag{13}$$

The harmonics of the dressed states $\Phi_{1,n}^{\alpha}$ $\Phi_{2,n}^{\alpha}$ in Eq. (11) as well as the quasienergies λ_α can be found as the solution of a matrix eigenvalue problem for a certain infinite matrix Hamiltonian [35]. This problem will be treated in Sec. III. Here all harmonics $\Phi_{1,n}^{\alpha}$ and the quasienergies are assumed to be known for both dressed states (11). These dressed states will be used as the basis in Eq. (6) rather than using $|1\rangle$, $|2\rangle$. If we set

$$\psi_i(t)=C_+(t)\Phi_i^{+}(t)+C_-(t)\Phi_i^{-}(t),\quad i=1,2 \tag{14}$$

then, using Eqs. (6), (10) we can derive the equations for the amplitudes $C_\pm(t)$

$$i\dot{C}_\alpha=-\frac{i}{2}\sum_{\beta=\pm}\gamma_{\alpha\beta}(t)C_\beta+r_p\Phi_i^{\alpha}(t)^*e^{-i(\omega_p-\omega_0)t},\quad \alpha=\pm \tag{15}$$

where $\gamma_{\alpha\beta}(t)$ are coefficients associated with dissipation

$$\gamma_{\alpha\beta}(t)=\sum_{k=1,2}\gamma_k\Phi_k^{\alpha}(t)^*\Phi_k^{\beta}(t),\quad \alpha,\beta=\pm. \tag{16}$$

It follows from the form of the dressed-state amplitudes $\Phi_{1,2}^{\alpha}(t)$ [Eq. (11)] that the coefficients $\gamma_{\alpha\beta}(t)$ oscillate at the frequencies $\lambda_\alpha-\lambda_\beta+p\Omega$ $(p=0,\pm1,\ldots)$, whereas the inhomogeneous terms in Eq. (15), associated with the probe field, oscillate at the frequencies $\omega_p-\omega_0-\lambda_\alpha+n\Omega$ $(\alpha=\pm;\ n=0,\pm1,\ldots)$.

Under the condition (9) the intervals between quasienergy levels are much greater than the relaxation widths of the levels γ_1 and γ_2

$$|\lambda_+-\lambda_-+m\Omega|\gg\gamma_{1,2}\qquad \text{for any integer } m. \tag{17}$$

It follows then from Eq. (15) that the probe field will produce resonance in the system whenever its frequency satisfies one of the conditions

$$\omega_p \approx \nu_{\alpha,n} \equiv \omega_0 + \lambda_\alpha + n\Omega, \tag{18}$$

where $\alpha=\pm$, $n=0,\pm1,\pm2,\ldots$. There will be two "combs" of the probe resonance frequencies $\nu_{\pm,n}$ corresponding to various intervals between the ground-state level $-\omega_0$ and levels $\lambda_\alpha+n\Omega$ of the quasienergy ladders for two dressed states ($\alpha=\pm$). Near the resonance (18) a steady-state solution of Eq. (15) has the form

$$C_\alpha(t) \approx r_p \Phi^\alpha_{1,n} \frac{e^{-i\delta t}}{\delta + i\Gamma_\alpha/2}, \quad C_{-\alpha} \approx 0 \tag{19}$$

(in what follows we will use the subscript $-\alpha$ to identify the other dressed state relative to that identified by α, e.g., if $\alpha\to+$ then $-\alpha\to-$ and vice versa). In Eq. (19) the "field-dressed" damping coefficient Γ_α is

$$\Gamma_\alpha = \sum_{k=1,2} \gamma_k \sum_{n=-\infty}^{+\infty} |\Phi^\alpha_{k,n}|^2 \tag{20}$$

and the small probe detuning δ is

$$\delta = \omega_p - \nu_{\alpha,n}, \quad |\delta| \ll \Omega. \tag{21}$$

Note that $C_\alpha(t)$ is a nondiagonal element of the density matrix between the ground state $|0\rangle$ and one of the dressed states. Near the resonance (18) the probe field resonantly excites the system to this dressed state from the ground state and the amplitude $C_\alpha(t)$ is enhanced. The amplitude $C_{-\alpha}(t)$ corresponding to the other dressed state is much smaller ($\sim r_p/\Omega$) provided that the condition (17) holds. This is not true near the degeneracy points where two ladders of quasienergy nearly coincide (this effect will be considered in Sec. V).

Based on the solution (19) one can find the polarization $p(t)$ induced in an optically thin medium by the probe field near the resonance (18)

$$p(t) = \mathcal{E}_p \sum_{m=-\infty}^{+\infty} \chi^\alpha_{n,m}(\delta) e^{-i(\nu_{\alpha,m}+\delta)t} + \text{c.c.}, \tag{22}$$

where

$$\chi^\alpha_{n,m}(\delta) = -\frac{N d_{01}^2}{\hbar(\delta + i\Gamma_\alpha/2)} \Phi^\alpha_{1,n}\Phi^\alpha_{1,m}. \tag{23}$$

Here N is the atom density and the detuning δ is given in Eq. (21). As seen from Eqs. (22), (23) the probe-absorption curve near each resonance (18) has a Lorentzian shape with a width determined by the field-dressed damping parameter Γ_α. There will be two infinite sets of resonances (18) corresponding to two quasienergy ladders and they will be well resolved under condition (9). It follows from Eq. (22) that each resonance (18) is characterized by an absorption cross section proportional to $|\Phi^\alpha_{1,n}|^2$ (cf. Ref. [2]).

As seen from Eq. (22), a monochromatic probe field with frequency close to $\nu_{\alpha,n}$ gives rise to a stimulated emission at an infinite comb of frequencies $\nu_{\alpha,m}+\delta$ with corresponding amplitudes of individual harmonics proportional to $\phi^\alpha_{1,n}\phi^\alpha_{1,m}$. Those harmonics correspond to all possible transitions from the quasienergy levels $\lambda_\alpha+m\Omega$ of the resonantly excited dressed state back to the ground state.

Thus, in the limit of weak dissipation [Eq. (17)], the harmonics of the dressed states $\Phi^\alpha_{1,n}$ provide the essential information about the probe-absorption resonances and stimulated emission of the probe. These quantities will be of central interest here.

III. FACTORIZATION PROPERTY OF THE FLOQUET HAMILTONIAN

In the preceding section the linear response to a probe field [Eqs. (22), (23)] was calculated in terms of the quasienergies $\lambda_\pm$ and the harmonics of the dressed-state amplitudes $\Phi^\pm_{1,2}(t)$, satisfying Eq. (10). The Hamiltonian in Eq. (10) is special, in that it *effectively* describes a two-level system subject to a strong harmonic perturbation with frequency Ω that is coupled to a system via a single-quantum state $|1\rangle$ (that is, described by a single diagonal matrix element). To our knowledge, this form of the interaction has not been encountered before in studies of two-level driven systems. It leads to a very special property of the dressed-state harmonics $\Phi^\pm_{1,n}$ and $\Phi^\pm_{2,n}$ that will be considered here.

To describe this property we use the approach developed by Shirley [35]. After substituting expressions (11) for $\Phi^\alpha_i(t)$ ($i=1,2$) into Eq. (10) we obtain an infinite set of matrix recursion relations for $\Phi^\alpha_{i,n}$ ($i=1,2$). They can be written in the form of a matrix eigenvalue problem for the λ's

$$\begin{pmatrix} -n\Omega & r \\ r & \Delta - n\Omega \end{pmatrix}\begin{pmatrix}\Phi_{1,n}\\ \Phi_{2,n}\end{pmatrix} + \mu\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}\Phi_{1,n-1}\\ \Phi_{2,n-1}\end{pmatrix} + \begin{pmatrix}\Phi_{1,n+1}\\ \Phi_{2,n+1}\end{pmatrix}\right] = \lambda\begin{pmatrix}\Phi_{1,n}\\ \Phi_{2,n}\end{pmatrix}, \quad \mu \equiv M\Omega/2. \tag{24}$$

Here the integer n ranges from $-\infty$ to $+\infty$ and superscript α is omitted for simplicity. The operator whose eigenvalues are λ's is an infinite-dimensional Hermitian matrix (Floquet Hamiltonian)

$$\sum_{m=-\infty}^{\infty} \sum_{j=1,2} \mathcal{H}^{j,m}_{i,n} \Phi_{j,m} = \lambda \Phi_{i,n}, \tag{25}$$

with rows and columns identified by pairs of indices, e.g., (i,n) and (j,m), where the indices take on the values $i=1,2$ and $n=0,\pm1,\ldots$. The eigenvectors are infinite-dimensional column vectors. If one orders their elements $\Phi_{i,n}$ so that i runs over its values before each change in n then the Floquet Hamiltonian $\mathcal{H}$ is

$$\begin{array}{c} \\ \cdot \\ (1,k+1) \\ (2,k+1) \\ (1,k) \\ (2,k) \\ (1,k-1) \\ (2,k-1) \\ \cdot \end{array}
\begin{array}{c} \begin{array}{cccccccc} \cdot & (1,k+1) & (2,k+1) & (1,k) & (2,k) & (1,k-1) & (2,k-1) & \cdot \end{array} \\
\left(\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -(1,k+1) & r & 0 & 0 & 0 & 0 & \cdot \\
\cdot & r & \Delta-(k+1)\Omega & 0 & \mu & 0 & 0 & \cdot \\
\cdot & 0 & 0 & -k\Omega & r & 0 & 0 & \cdot \\
\cdot & 0 & \mu & r & \Delta-r\Omega & 0 & \mu & \cdot \\
\cdot & 0 & 0 & 0 & 0 & -(k-1)\Omega & r & \cdot \\
\cdot & 0 & 0 & 0 & \mu & r & \Delta-(k-1)\Omega & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right) \end{array} \tag{26}$$

The eigenvalues of $\mathcal{H}$ (quasienergies) can be found as roots of the characteristic equation $\det[\mathcal{H}-\mathbf{I}\lambda]=0$, where $\mathbf{I}$ is an identity matrix.

The Floquet Hamiltonian has a periodic structure with only the coefficient of Ω in the diagonal elements varying from block to block. This structure endows the eigenvalues and eigenvectors with periodic properties. If λ is an eigenvalue, then so also is $\lambda+p\Omega$ for any integer p. The quasienergies form two infinite sets described by Eqs. (12), (13). The elements $\Phi_{i,n}^{\alpha,p}$ of the eigenvector with the quasienergy $\lambda_\alpha+p\Omega$ are related to the elements $\Phi_{i,n}^{\alpha}$ of the eigenvector with quasienergy λ_α via the equality $\Phi_{i,n+p}^{\alpha}=\Phi_{i,n}^{\alpha,p}$ Thus, the difference between eigenvectors within the same set is a matter of assignment of integer n in $\Phi_{i,n}^{\alpha}$. It corresponds to the ambiguity in the choice of quasienergy in Eq. (11) and has no physical significance. It suffices, therefore, to investigate any two branches of quasienergies $\lambda_\pm$ from different sets and their eigenvectors.

Let us assume that the quasienergy λ satisfies the following condition for a certain integer k:

$$\lambda+k\Omega=0. \tag{27}$$

In this case, as seen from Eq. (26), the matrix $\mathcal{H}-\lambda\mathbf{I}$ has the column $(1,k)$ and row $(1,k)$ with only one nonzero element $(=r)$. Therefore the determinant of $\mathcal{H}-\lambda\mathbf{I}$ can be factorized

$$\det[\mathcal{H}-\lambda\mathbf{I}]=-r^2\det[\mathcal{H}_k^{(+)}-\lambda\mathbf{I}]\det[\mathcal{H}_k^{(-)}-\lambda\mathbf{I}]. \tag{28}$$

Here matrices $\mathcal{H}_k^{(-)}$ and $\mathcal{H}_k^{(+)}$ are half-infinite diagonal blocks of $\mathcal{H}$ that correspond to the values of index n ranging from $-\infty$ to $k-1$ and from $k+1$ to $+\infty$, respectively. The matrices $\mathcal{H}_k^{(-)}-\lambda\mathbf{I}$ and $\mathcal{H}_k^{(+)}-\lambda\mathbf{I}$ determine the coefficients of the half-infinite systems of equations in Eq. (25) for $\Phi_{i,n}$ corresponding to $n<k$ and $n>k$, respectively.

This special factorization property of the Floquet Hamiltonian (28) is a manifestation of the fact that under condition (27) each of the half-infinite systems of equations in (25) is closed. Indeed, one can see from the form of $\mathcal{H}$ [Eq. (26)] that the coupling between those systems occurs via the matrix elements $\mathcal{H}_{2,k+1}^{2,k}=\mathcal{H}_{2,k}^{2,k+1}=r$. Therefore those systems will be closed whenever

$$\Phi_{2,k}=0. \tag{29}$$

On the other hand, the component $\Phi_{2,k}$ *always* vanishes under condition (27) because in this case the column $(1,k)$ of the matrix $\mathcal{H}-\lambda\mathbf{I}$ contains only one nonzero matrix element, $\mathcal{H}_{1,k}^{2,k}$ [precisely the same condition leads to Eq. (28)].

It follows from Eq. (28), that under condition (27), the characteristic equation for λ is reduced to one of the two conditions

$$\det[\mathcal{H}_k^{(+)}+k\Omega]=0 \text{ or } \det[\mathcal{H}_k^{(-)}+k\Omega]=0, \tag{30}$$

which are the conditions of existence for nontrivial bounded solutions of the closed half-infinite systems of equations (see discussion above). However, in general, these conditions cannot be satisfied simultaneously because the matrices $\mathcal{H}_k^{(\pm)}+k\Omega$ are not similar and their eigenvalues do not coincide. Therefore, under condition (27)

$$\Phi_{1,n}=\Phi_{2,n}=0, \quad \text{for all } n<k \text{ or } n>k, \tag{31}$$

depending on which of the determinants in Eq. (30) vanishes. This property has an immediate effect on the polarizability of the system induced by the resonant probe field [Eq. (22)]. If the parameters of the coupling field $E(t)$ are adjusted so that one of the quasienergies, λ_+ or λ_-, satisfies condition (27) then an infinite number of probe-absorption resonances will be simultaneously forbidden [see Eqs. (22), (31)]. According to Eqs. (18), (27), and (31), the values of the resonance frequencies ν_m for the forbidden resonances are

$$\nu_m=\omega_0+m\Omega, \tag{32}$$

where integer m varies from $-\infty$ to -1 or from 1 to $+\infty$. Therefore, the positions of these resonances correspond to an infinite sequence of equally spaced probe frequencies, $m\Omega+\omega_0$, below $(m<0)$ or above $(m>0)$ the unique frequency edge ω_0. It is clear that the same half-infinite sequence of frequencies will be absent in the spectrum of stimulated emission of the probe [cf. Eqs. (22), (23)] when the frequency ω_p is resonant with the allowed transition. We refer to this effect as simultaneously forbidden resonances (SFR) ([24]).

Note that this effect appears to be a direct consequence of the special factorization property (28) that the Floquet Hamiltonian possesses when its eigenvalues are integer multiples of Ω. One would expect that this factorization is a manifestation of a certain kind of a symmetry of the Hamiltonian. We have not found this symmetry explicitly, however, we did find that, for certain choices for the parameters, the factorization leads to a degeneracy of quasienergy levels (see Sec. V).

IV. THE SOLUTION IN TERMS OF CONTINUED FRACTIONS

We will now investigate the appearance of simultaneous forbidden resonances by solving the matrix eigenvalue problem (24). One of the methods to obtain its exact, nonperturbative solution is based on the use of infinite continued fractions [37]. This method is especially suitable here because the terms proportional to the modulation index M in Eqs. (24) involve a projection operator. It is, therefore, possible to express the harmonics $\Phi_{2,n}$ in terms of the harmonics $\Phi_{1,n}$ with the same n

$$\Phi_{2,n}=\frac{\lambda+n\Omega}{r}\Phi_{1,n}, \tag{33}$$

and a *scalar* three-term recursion relation for the harmonics $\Phi_{1,n}$ can be obtained after substituting Eq. (33) into Eq. (24)

$$d(\lambda_n)\Phi_{1,n}+\mu\lambda_{n+1}\Phi_{1,n+1}+\mu\lambda_{n-1}\Phi_{1,n-1}=0, \tag{34}$$

where

$$\lambda_n=\lambda+n\Omega, \quad d(x)=(\Delta-x)x+r^2. \tag{35}$$

The method to be used here for solving Eq. (34) is similar to one used in Ref. [2]. The ratios of the Φ's are expressed as infinite continued fractions by a standard method detailed in Appendix A. Then the following equations that relate $\Phi_{1,n\pm1}$ to $\Phi_{1,n}$ can be written:

$$\Phi_{1,n+1}=-\mu\lambda_n[d(\lambda_{n+1})-Y^{(+)}(\lambda_{n+1})]^{-1}\Phi_{1,n}, \tag{36}$$

$$\Phi_{1,n-1}=-\mu\lambda_n[d(\lambda_{n-1})-Y^{(-)}(\lambda_{n-1})]^{-1}\Phi_{1,n}, \tag{37}$$

where

$$Y^{(\pm)}(\lambda_n)=\frac{\mu^2\lambda_n\lambda_{n\pm1}}{d(\lambda_{n\pm1})-Y^{(\pm)}(\lambda_{n\pm1})}. \tag{38}$$

The coefficients $Y^{(\pm)}(\lambda_n)$ can be expressed in terms of infinite continued fractions, using a backward recursion for $Y^{(-)}$ and a forward recursion for $Y^{(+)}$. Furthermore, Eqs. (34)–(38) are used to obtain the algebraic equation for λ. One divides both parts of the recursion relation (34) by $\Phi_{1,n}$ and then substitutes the ratios $\Phi_{1,n\pm1}/\Phi_{1,n}$ from Eqs. (36), (37). This yields

$$d(\lambda_n)-Y^{(+)}(\lambda_n)-Y^{(-)}(\lambda_n)=0. \tag{39}$$

The parameters λ_n and $d(x)$ are defined in Eq. (35) and the $Y^{(\pm)}(\lambda_n)$ are defined as continued fractions in Eq. (38). It follows from the structure of the $Y^{(\pm)}(\lambda_n)$ that the form of Eq. (39) does not depend on the choice of n. Using Eqs. (38), (39) one can readily verify the periodic properties of the quasienergies λ_n discussed above (i.e., if λ_n satisfies this equation, so does $\lambda_n+p\Omega$).

Equations (33), (35–39) constitute the basis for the numerical solution of the matrix eigenvalue problem (24). One first solves the algebraic equation (39), using definitions (35), (38), and selects any two quasienergies $\lambda_\pm$ (12), (13) corresponding to different dressed states. Then for each quasienergy $\lambda_\pm$ one uses a recursion in Eqs. (36), (37) together with Eq. (33) to express the harmonics $\Phi_{1,n},\Phi_{2,n}$ at various values of n solely in terms of a single component $\Phi_{1,0}$. The component $\Phi_{1,0}$ can be found then from the normalization condition.

Before proceeding with numerical work we describe the SFR effect analytically, using Eqs. (33)–(39). This effect appears whenever the condition (27) is satisfied. That is, whenever a rung of one of the quasienergy ladders coincides with zero. Assume that this condition is satisfied for some integer k. It can be written as $\lambda_k=0$, using the notation (35). In this case the numerators in the expressions for $\Phi_{1,k\pm1}$ Eqs. (36) and (37) vanish. However, as can be shown, both harmonics $\Phi_{1,k\pm1}$ do not vanish simultaneously under the condition (27) (otherwise, $\Phi_{1,n}\equiv\Phi_{2,n}\equiv0$ for all n, a case which is not of physical interest). Thus, at least one of the denominators in Eqs. (36), (37) must vanish under the condition (27) and this leads to one of the following equations:

$$d(\Omega)=Y^{(+)}(\Omega), \tag{40}$$

or

$$d(-\Omega)=Y^{(-)}(\Omega). \tag{41}$$

It can be shown that Eq. (40) corresponds to the vanishing of $\det[\mathcal{H}_k^{(+)}+k\Omega]$ while Eq. (41) corresponds to the vanishing of $\det[\mathcal{H}_k^{(-)}+k\Omega]=0$ [cf. Eqs. (30)]. In general, both equations cannot be simultaneously satisfied (see below). Assume, for example, that Eq. (40) is satisfied while Eq. (41) is not. It follows then from Eqs. (37) that $\Phi_{1,k-1}=0$, and using the backward recursion in this equation and also Eq. (33), one obtains

$$\Phi_{2,n},\Phi_{1,n}=0 \quad \text{for } n<k. \tag{42}$$

If, on the contrary, Eq. (41) is satisfied and Eq. (40) is not, then, using the forward recursion in Eq. (36) and also Eq. (33), one obtains

$$\Phi_{2,n},\Phi_{1,n}=0 \quad \text{for } n>k. \tag{43}$$

The SFR corresponding to Eqs. (40), (42) can be called lower in the sense that harmonics of the dressed state with $n<k$ vanish. Conversely, the SFR corresponding to Eqs. (41), (43) can be called upper. The picture is that the parameters of the system are adjusted so that one of the rungs of one of the ladders coincides with the bare level of state 1 ($\lambda_k=0$). At this point all the harmonics corresponding to the rungs above ($n>k$) or below ($n<k$) the given rung vanish. Note that in either case the condition $\lambda_k=0$ leads to $\Phi_{2,k}=0$ [see Eq. (33)].

In Fig. 2(a) two branches of quasienergies $\lambda_\pm(M)$ for the two different dressed states are plotted as functions of the modulation index M. Due to the periodic property of the quasienergies [Eq. (12)] these branches produce the two ladders of quasienergy curves $\lambda_\pm(M)+n\Omega$ $(n=0,\pm1,\ldots)$.

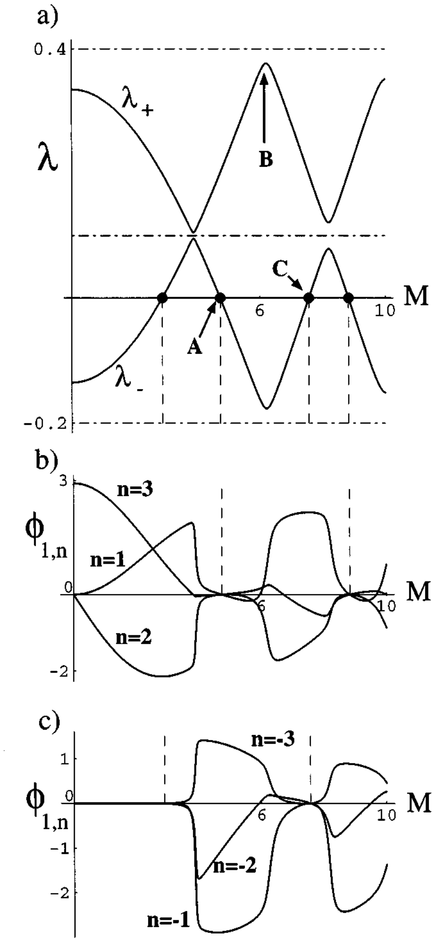

FIG. 2. (a) Quasienergies $\lambda_\pm$ vs M for fixed values of the coupling field strength $r=1.2$, detuning from resonance $\Delta=0.8$, and modulation frequency $\Omega=0.6$. Crossings of λ_- with the zero energy marked with bold dots correspond to $M\approx2.9$, 4.7, 7.6, 8.8. For the dressed state with quasienergy λ_- plots of $\Phi_{1,n}$ vs M are shown in (b) and (c). Each harmonic is labeled by its n on the plots. Note that the harmonic $\Phi_{1,3}$ does not vanish at $M\to0$ (in this limit $\lambda_-+3\Omega$ approaches the "bare" dressed-state energy in the absence of modulation.

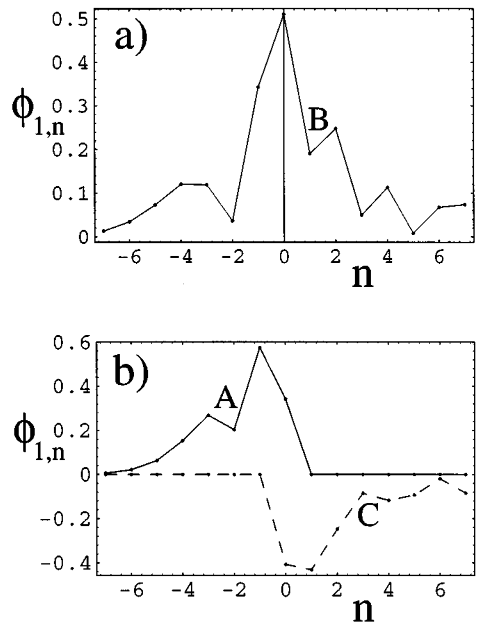

FIG. 3. Dots in each plot correspond to the values of the harmonics $\Phi_{1,n}$ of the dressed state with quasienergy λ_- for different integers n. Values of r,Ω, and Δ are the same as in Fig. 2. The plot in (a) corresponds to $M=6.2$ which lies near the second avoided crossing in Fig. 2(a) (marked with the letter B). Dots that lie on solid and dashed lines in (b), labeled with A and C, respectively, are plotted for the values of M that correspond to the points of SFR from Fig. 2 with the same labels. At the point A all positive harmonics vanish whereas at the point C all negative harmonics vanish.

The boundaries of the interval of periodicity with the width Ω (the zone) are indicated by dot-dashed lines. It follows from Eq. (13) that the quasienergy curves $\lambda_\pm(M)$ are positioned symmetrically relative to the zone center $\Delta/2+p\Omega$ [the zone center in Fig. 2(a) corresponds to the integer $p=-1$ and is also indicated by the dot-dashed line]. Different quasienergy curves, in general, do not cross each other and therefore each of them is confined within a half zone with a width $\Omega/2$. The quasienergy curves repeatedly approach the zone boundaries and the gaps in the avoided crossings grow with modulation index M. The horizontal axis corresponds to zero energy and coincides with the position of the energy level of the state $|1\rangle$ (cf. Fig. 1). There is only one quasienergy curve (labeled λ_-) that crosses the horizontal axis. The crossing points are indicated in Fig. 2(a) by bold dots. At those points the condition (27) is satisfied for $k=0$.

The harmonics $\Phi_{1,n}$ of the dressed state with quasienergy λ_- are shown in Figs. 2(b) ($n=1,2,3$) and 2(c) ($n=-1,-2,-3$). Note that all the harmonics in either Fig. 2(b) or 2(c), but not both, vanish alternately at the crossing points (points of SFR), which are indicated by vertical dashed lines to connect with Fig. 2(a). At the first and third points (lower SFR) Eq. (40) is satisfied and, according to the discussion above, all transitions to the quasienergy levels $n\Omega$ below zero are simultaneously forbidden. At the second and fourth points (upper SFR) Eq. (41) is satisfied and all transitions to the quasienergy levels $n\Omega$ above zero are simultaneously forbidden.

It can be immediately seen by comparison of Figs. 2(a) and 2(b), 2(c) that the points of SFR lie inside of the regions where the harmonics $\Phi_{1,n}$ of one sign of n dominate over the harmonics of the opposite sign of n and those regions are separated by the avoided crossing regions. In contrast, in the vicinity of the avoided-crossings the harmonics of both signs are, in general present. For example, in Fig. 3(a) the dots correspond to the values of $\Phi_{1,n}$ for different integer n's calculated at the fixed value of M corresponding to the second avoided crossing in Fig. 2(a) (it is marked with the letter B). For comparison, the plots of $\Phi_{1,n}$ in Fig. 3(b) correspond to the values of M at the second and third points of SFR [points A and C in Fig. 2(a)] which are positioned before and after the avoided crossing, correspondingly.

It turns out that in order to understand qualitatively this behavior of the dressed state harmonics one needs to consider the low-frequency limit $\Omega\ll r$, as will be done in Sec. VI. Here we remark that the Figs. 2(a), 2(b), 2(c) correspond

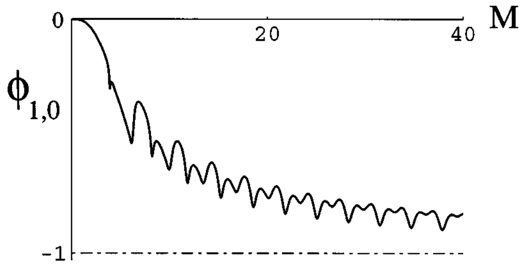

FIG. 4. Zeroth harmonic $\Phi_{0,n}$ for the dressed state with quasienergy λ_- vs M.

to the frequency $\Omega=0.5r$, while for larger $\Omega\gtrsim r$ there will be no such distinct and comparatively wide ''regions of SFR'' as in Figs. 2(b), 2(c) (cf. [24]).

The component $\Phi_{1,0}$ is plotted in Fig. 4. According to the discussion above this component does not vanish at the SFR points. In the limit $M\gg r$ it asymptotically approaches unity while all the other harmonics $\Phi_{1,n}$ with $n\neq 0$ decrease to zero [cf. Figs. 2(b), 2(c)]. This can be understood from the Schrödinger Eq. (10). In the limit of strong modulation the coupling field between the states $|1\rangle$ and $|2\rangle$ (term proportional to r) becomes increasingly off resonance and its effect will be diminished. In this case the quasienergy for one of the dressed states approaches zero [λ_+ in Fig. 2(a)] and the corresponding amplitudes have the limits $\Phi_1(t)\to 1$, $\Phi_2(t)\to 0$.

In addition to the points of SFR there are also cases in which the harmonics $\Phi_{2,n},\Phi_{1,n}$ vanish at single values of integer n rather than at an infinite number of its values as in the SFR effect. For example, only the component $\Phi_{1,3}$ shown in Fig. 2(b) has zeros at the values $M\approx 3.82$ and 6.71 and 9.98 while all other harmonics do not. This effect is considered in Appendix B.

V. FACTORIZATION MANIFOLDS

The conditions for SFR (40), (41) impose certain relations between the parameters r, M, Δ, and Ω. As can be readily verified these equations involve three-independent dimensionless parameters r/Ω, M, Δ/Ω, and the points of SFR form two manifolds in the 3D space of these parameters which we will now investigate (we shall call them factorization manifolds).

Using Eq. (24) one can derive the following transformation properties of the harmonics of the two dressed states and their corresponding quasienergies:

$$\Phi^{+}_{1,n}(-\Delta)=(-1)^n\Phi^{-}_{1,-n}(\Delta),$$

$$\Phi^{+}_{2,n}(-\Delta)=(-1)^{n+1}\Phi^{-}_{2,-n}(\Delta),\quad \lambda_+(-\Delta)=-\lambda_-(\Delta) \tag{44}$$

(here the $\pm$ signs designate the different dressed states). On the other hand, it is seen from the explicit forms of $d(x)$ and $Y^{(\pm)}(\lambda_n)$ [Eqs. (35,38)], that the Eqs. (40), (41) that describe the factorization manifolds can be transformed one into the other by the replacement $\Delta\to-\Delta$. Thus, if the harmonics $\Phi_{i,n}$ for one of the dressed states obey Eq. (42) with a certain integer k then, after the transformation $\Delta\to-\Delta$ the harmonics of the other dressed state will obey the Eq. (43) with k replaced by $-k$. Therefore it suffices to investigate just one of the equations, (40) or (41).

In order to gain some intuition about the structure of the manifold of SFR we first consider the case of weak modulation

$$M\ll 1, \tag{45}$$

where the results can be obtained analytically. It is convenient to choose the quasienergy branches $\lambda_\pm$ so that they go continuously into the ''energies'' of the dressed states in the absence of modulation [the latter can be obtained from Eq. (10) with $M=0$]:

$$\lambda_\pm\to\epsilon_\pm=\Delta/2\pm(r^2+\Delta^2/4)^{1/2}\quad (M\to 0). \tag{46}$$

In this case the ladders of the quasienergy levels (12) can be approximately written as $\epsilon_\pm+n\Omega$ and the condition for SFR (27) amounts to one of the conditions

$$\epsilon_+ + k\Omega\approx 0,\quad k<0, \tag{47}$$

$$\epsilon_- + l\Omega\approx 0,\quad l>0. \tag{48}$$

Note that, the zero harmonics $\Phi_{1,0},\Phi_{2,0}$ do not vanish under the condition (27) and, in general, dominate in the limit $M\ll 1$. Nonzero harmonics $\Phi_{1,n},\Phi_{2,n}\sim M^{|n|}$ and are formed by all possible elementary processes of net absorption (for $n<0$) or emission ($n>0$) of $|n|$ ''quanta'' of frequency Ω. It can be shown, using a diagrammatic approach, that under the condition (47) all processes with net absorption of more than $|k|$ quanta Ω are forbidden [Eq. (42)], while under the condition (48) the processes with net emission of l quanta are forbidden [Eq. (43)].

The Eq. (40) can be rearranged and then simplified in the limit $M\ll 1$, using the explicit form of the continued fraction $Y^{(+)}(\Omega)$ [Eq. (38)] and keeping only leading terms in M^2. Finally, for small M it takes the form

$$d(-k\Omega)=\frac{M^2\Omega^4}{4}\left(\frac{k(k-1)}{d((1-k)\Omega)}+\frac{k(k+1)}{d(-(1+k)\Omega)}\right). \tag{49}$$

In the limit $M\to 0$ this equation corresponds to Eq. (47), as can be seen if one takes into account that the function $d(x)$ in Eq. (49) can be written in the form

$$d(x)=(\epsilon_+-x)(x-\epsilon_-)$$

[cf. Eqs. (35), (46)]. Solving Eq. (49) for Δ one obtains to the first order in M^2

$$\Delta/\Omega=-k+\frac{1}{k}(r/\Omega)^2+M^2\frac{(r/\Omega)^2}{2k\left\{\left[k+\frac{1}{k}(r/\Omega)^2\right]^2-1\right\}},$$

$$k=-1,-2,\ldots. \tag{50}$$

For a fixed Ω these relations define the system of nearly parabolic surfaces in the space of parameters r/Ω, M, and Δ/Ω, identified by the various integer values of k.

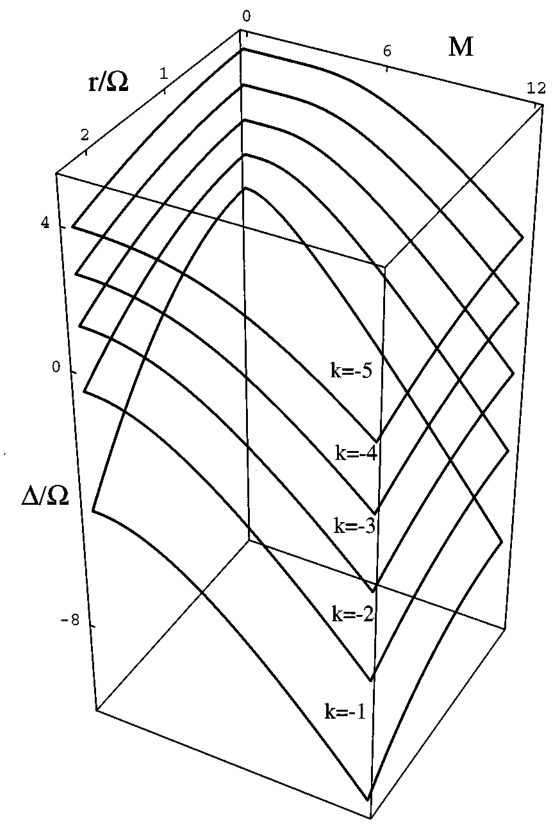

FIG. 5. One of the two factorization manifolds defined by Eq. (40) in the space of the dimensionless parameters r/Ω, M, and Δ/Ω. All surfaces are enumerated with the integer $k=-1,-2,\ldots$ and points on the surface with a given k correspond to one and the same value of the quasienergy $\lambda_+(r,M,\Delta)=-k\Omega>0$ that reduces to the ''bare'' energy of the dressed state $\epsilon_+=-k\Omega$ at $M\to 0$. At the kth surface all of the harmonics $\Phi_{1,n}$, $\Phi_{2,n}$ with $n<k$ vanish simultaneously. Note that, according to Eqs. (46), (47) the kth surface crosses the axis Δ/Ω at the integer points $\Delta/\Omega=-k>0$. The other factorization manifold defined by the Eq. (41) can be obtained simply by reflection in the plane $\Delta=0$ [cf. Eq. (44) and the subsequent discussion].

The approximation (49), (50) fails when the modulation is not weak ($M\gtrsim 1$). Essentially, the expressions (50) give the asymptotic forms, at $M\ll 1$, of a set of nonlinear surfaces that represent the solution of Eq. (40). This equation was solved numerically, and the results are shown in Fig. 5. Each surface is represented by its intersections with two pairs of planes orthogonal to the M axis and to the r/Ω axis.

The intersections with the $M=0$ plane are the parabolic curves described by Eq. (50). Points on each surface correspond to a fixed value of the positive quasienergy $\lambda_+(r,M,\Delta)=-k\Omega$ which connects smoothly to the dressed state energy $\epsilon_+=-k\Omega$ (46), (47) at $M\to 0$. Thus, each surface can be identified by the value of integer k. The system of surfaces is semi-infinite, its lowest sheet corresponds to $k=-1$ and the sheets with $k=-2,-3,\ldots$ are stacked above. At the kth surface the quasienergy ladder of $\alpha=+$ dressed state has a level $\lambda_+ + k\Omega$ that crosses zero energy and all harmonics $\Phi_{i,n}$ vanish for $n<k$. Thus the surfaces with different values of k correspond to the *different* levels of the quasienergy ladder crossing zero energy [38].

The manifold corresponding to the other type of SFR [Eqs. (41), (43)] can be obtained from the above manifold by *reflection* in the plane $\Delta=0$, as follows from Eq. (44) and the subsequent discussion. This manifold also consists of a semi-infinite system of surfaces, each corresponding to a particular value of negative quasienergy $\lambda_-(r,M,\Delta)=-l\Omega$ ($l>0$) that is traceable to the dressed-state energy $\epsilon_-=-l\Omega$ (46), (48) at $M=0$. At these surfaces we have $\Phi_{i,n}=0$ for $n>l$.

Degeneracy of the dressed states

If one plots together the two factorization manifolds they will form a ''honeycomb'' structure. Cross sections of this structure at different values of the parameter r are shown in Figs. 6(a), 6(b). The downward curves correspond to lower SFR (40), (42) and upward curves correspond to upper SFR (41), (43).

Any two surfaces belonging to the different factorization manifolds intersect along a certain line and each such line corresponds to a point in the cross sections shown in Figs. 6(a), 6(b) (these points lie at the crossing of downward and upward curves). Note that the occurrence of the intersection lines becomes possible because points on the factorization manifolds correspond to the roots of two different algebraic equations, (40) and (41).

The remarkable feature of the intersection lines is that they always correspond to the field detuning Δ being an integer multiple of the modulation frequency Ω [cf. Figs. 6(a), 6(b)]

$$\Delta=n\Omega,\quad n=0,\pm 1,\pm 2,\ldots. \tag{51}$$

The intersection at $\Delta=0$ is immediately seen from Eqs. (40), (41) which coincide with each other in this case [cf. Eqs. (35), (38)]. However those equations do not coincide at $\Delta\neq 0$ and the intersections corresponding to $n\neq 0$ are nontrivial. This result is nonperturbative because it holds in the range of M and r where they cannot be considered small [cf. Figs. 6(a), 6(b)].

The physical importance of the intersection lines is related to the fact that at those lines the quasienergy levels belonging to different dressed states cross each other. Indeed, at points on the intersection line the two different dressed states [the one satisfying condition (42) and the other satisfying (43)] correspond to the same ladder of quasienergy levels, since $\lambda_+(r,M,\Delta)-\lambda_-(r,M,\Delta)=(l-k)\Omega$.

This effect of degeneracy between the dressed states can be explained from the factorization property of the Floquet Hamiltonian (28). If the parameters of the system are fixed at some point of one of the intersection lines of the factorization manifolds then the ''wave functions'' of the two dressed states, $\Phi^+_{i,n}$ and $\Phi^-_{i,n}$ obey the ''Schrödinger equations'' (25) with two different Floquet Hamiltonians, $\mathcal{H}^{(+)}$ and $\mathcal{H}^{(-)}$, correspondingly. Those Hamiltonians act in the different subspaces of positive and negative harmonics, which are not coupled to each other. Therefore, the quasienergies do not repel each other and crossing is allowed.

The reason that the degeneracy of the dressed states always occurs at integer values of Δ/Ω [Eq. (28)] is as follows. At points on the intersection line between the two sur-

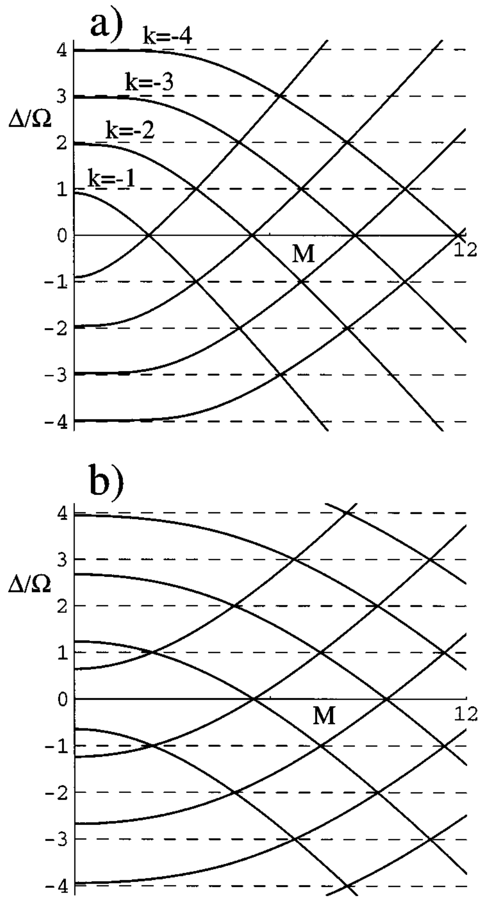

FIG. 6. Cross sections of the factorization manifolds corresponding to fixed values of the field strength r/Ω. In (a) $r/\Omega=0.3$, in (b) $r/\Omega=2.3$. Downward curves enumerated with k correspond to the surfaces of the manifold shown in Fig. 5. These curves form an infinite system bounded from below. The system of upward curves is obtained by reflection in $\Delta/\Omega=0$ and correspond to the other manifold. The infinite set of these curves is bounded from above. Dashed lines indicate the crossings of the curves belonging to different manifolds (degeneracy points) that occur at integer values of Δ/Ω. Curves in (a) (small r/Ω) nearly reproduce the dependences of zeros of Bessel functions on their indices [cf. Eq. (53)]. When r/Ω grows the downward curves move up whereas the upward curves move down [cf. Eq. (50)]. At $r/\Omega>1$ the initial points of the curves at the Δ/Ω axis begin to interlace. This case corresponds to (b).

faces of the factorization manifolds identified by the integer numbers k and l the following equations are satisfied simultaneously:

$$\lambda_+ + k\Omega=0, \quad \lambda_- + l\Omega=0, \quad \text{where } k<0, \quad l>0 \tag{52}$$

and, therefore, not only is the difference between the quasienergies $\lambda_\pm$ an integer (the degeneracy) but their sum is also an integer: $\lambda_+ + \lambda_- = -(k+l)\Omega$. However, according to Eq. (13) the sum of the quasienergies $\lambda_+ + \lambda_-$ *always* equals Δ mod Ω. Therefore, in this case the detuning Δ must be a multiple of Ω.

In the limit of small $r\ll\Omega$ the above property can be related to an obvious property of the Bessel functions. In this limit Eqs. (40), (41) can be reduced to the following:

$$M\frac{J_{\mp\nu}(M)}{J_{\mp\nu+1}(M)}=0, \quad \nu\equiv\Delta/\Omega \tag{53}$$

where the upper sign corresponds to Eq. (40) and the lower to Eq. (41). Thus the system of curves in Fig. 5 lying at the intersection of the manifold of SFR and the plane $r=0$ correspond to Bessel zeros: $j_{-\Delta/\Omega,n}=M$ $(n=1,2,\dots)$. The curves corresponding to the other manifold [Eq. (41)] are given by the equations: $j_{\Delta/\Omega,n}=M$ $(n=1,2,\dots)$. Because the zeros of Bessel functions of integer order $J_{\pm n}(M)$ are the same for any n it follows then that the curves from both manifolds cross at the points where

$$\Delta=p\Omega, \quad M=j_{p,k} \quad (p=0,\pm1,\dots;k=1,2,\dots). \tag{54}$$

The above effect of the degeneracy of the dressed states has an interesting experimentally observable manifestation. Indeed, when quasienergies nearly cross, the dressed states are strongly mixed via dissipation and *both* dressed-state amplitudes $C_\pm(t)$ [Eq. (15)] will be resonantly enhanced [the off-diagonal dissipative coefficient $\gamma_{+-}(t)$ in Eq. (15) is slowly varying near the degeneracy points and therefore both equations are coupled to each other]. In this situation the probe response will be modified as compared to that given by Eqs. (22), (23). For convenience we now define the quasienergies of the dressed states $\lambda_\pm$ within the reduced zone scheme [analogous to that in Fig. 2(a)] so that the crossing corresponds exactly to $\lambda_\pm\equiv0$. Assume now that the dressed states $\Phi^+$and Φ^- correspond to the semi-infinite ladders of levels $n\Omega$ with $n\geqslant0$ and $n\leqslant0$, respectively. It follows from the above discussion that there will be a unique comb of resonance frequencies at the crossing point $\omega_0+n\Omega$ but the line shapes of the resonances with $n=0$ and $n\neq0$ are different.

If the probe frequency is $\omega_p\approx n\Omega+\omega_0$ with $n\neq0$, then only one of the dressed states is excited directly by the probe and the dissipative coupling between the dressed states occurs via the degenerate quasienergy level $\lambda_\pm\equiv0$ (the only common level of the effectively ''semi-infinite'' ladders). It can be shown, using the formulas of Sec. II that near the crossing point the absorption rate per atom has the following non-Lorentzian form:

$$Q(\omega_p)=\mu(\phi^{\alpha}_{1,n})^2 \times\frac{(\delta_\beta)^2\Gamma_\alpha+\Gamma_\beta(\Gamma_+\Gamma_- - \Gamma_c^2)/4}{[\delta_+\delta_- - (\Gamma_+\Gamma_- - \Gamma_c^2)/4]^2+(\Gamma_+\delta_- + \Gamma_-\delta_+)^2/4}, \tag{55}$$

$$\alpha\equiv\mathrm{sgn}(n), \quad \beta\equiv\mathrm{sgn}(-n), \quad \delta_\pm=\omega_p-\omega_0-\lambda_\pm-n\Omega, \tag{56}$$

$$n\neq0$$

where $|\delta_+|\sim|\delta_-|$ are the probe detunings, the coefficient $\mu=(\mathcal{E}_p d_{01})^2\omega_p/\hbar$, the damping coefficients $\Gamma_\pm$ are given in Eq. (20), and the coefficient $\Gamma_c=\gamma_1\Phi^+_{1,0}\Phi^-_{1,0}$ determines the dissipative coupling between the dressed states [note that the ''bare'' damping constant γ_2 is dropped out because according to Eq. (33) at the degeneracy points $\Phi^{\pm}_{2,0}\equiv 0$]. It is immediately seen from the explicit forms of $\Gamma_{\pm}$ and Γ_c that

$$\Gamma_+\Gamma_- - \Gamma_c^2 > 0$$

and, therefore, $Q(\omega_p)$ is always positive. Note also that, unlike the nondegenerate case, the line shapes of the resonances depend on the sign of n.

Away from the degeneracy point the expression for the resonance probe absorption (55) corresponds to the results obtained in Sec. II [cf. Eq. (22)]. Exactly at the degeneracy points, it reduces to a form

$$Q(\omega_p)=\mu\frac{A_1(\delta)^2+A_2(\Gamma^2-\Gamma_c^2)/4}{\delta^4+\delta^2(\Gamma^2+\Gamma_c^2)/2+(\Gamma^2-\Gamma_c^2)^2/16}, \quad (57)$$

$$A_1=A_2=\Gamma(\phi^{\alpha}_{1,n})^2$$

where $\delta=\omega_p-\omega_0-n\Omega$ and $\Gamma=\Gamma_+=\Gamma_-$.

In the case of $n=0$ the probe frequency is $\omega_p\approx\omega_0$, and both dressed states will be excited simultaneously because neither of the harmonics $\Phi^{\pm}_{k,0}$ vanish. Exactly at the degeneracy points the expression for the absorption rate takes the form of Eq. (57) with, however, different values for the coefficients A_1 and A_2

$$A_{1,2}=\Gamma((\Phi^+_{1,0})^2+(\Phi^-_{1,0})^2)\pm 2\Gamma_c\Phi^+_{1,0}\Phi^-_{1,0} \quad (58)$$

[note that according to Eq. (44) at the degeneracy points $\Phi^+_{1,0}\equiv\Phi^-_{1,0}$].

VI. THE LIMIT OF A VERY STRONG-COUPLING FIELD

We now will consider the case in which the strength of the strong-field coupling r is much greater than the modulation frequency Ω. In this case the dressed-state harmonics $\Phi_{i,n}$ reveal some interesting behavior as functions of n. Under the condition

$$\Omega\ll r \quad (59)$$

the time dependence of the Hamiltonian for the dressed-state amplitudes $\Phi_{1,2}(t)$ in Eq. (10) can be regarded as slow. After solving this equation in the adiabatic approximation the amplitudes of the two dressed states take the form

$$\Phi^{\alpha}(t)\approx\chi^{\alpha}_i(t)\exp\left(-i\int^t E_{\alpha}(\tau)d\tau\right) \quad (i=1,2;\ \ \alpha=\pm) \quad (60)$$

where the amplitudes $\chi^{\pm}_{1,2}(t)$ are the instantaneous eigenvectors of the Hamiltonian in Eq. (10) and $E_{\pm}(t)$ are its instantaneous eigenvalues

$$E_{\alpha}(t)=(\Delta+M\Omega\ \cos\Omega t)/2 + \alpha\sqrt{(\Delta+M\Omega\ \cos\Omega t)^2/4+r^2} \quad (\alpha=\pm). \quad (61)$$

Each adiabatic solution (60) corresponds to a ladder of levels (12). In the following we will be interested in the limit of strong modulation $M\gg 1$. Under this condition and condition (59), the characteristic number of harmonics $\Phi^{\alpha}_{i,n}$ [Eq. (11)] will be large. It is convenient then to identify the quasienergy levels as well as $\Phi^{\alpha}_{i,n}$ for various n by the values of the "quasidiscrete" variable $z=\lambda_{\alpha}+n\Omega$ with the quasienergies $\lambda_{\alpha}\cong\langle E_{\alpha}(t)\rangle_t$ (mod Ω) where the time average is over the period of modulation $2\pi/\omega$. With this the harmonics of the dressed states can be written as

$$\Phi^{\alpha}_{i,n}\equiv\Phi^{\alpha}_i(z)=\langle\exp(izt)\Phi^{\alpha}_i(t)\rangle.$$

The integrand functions here are highly oscillatory, provided that $|E_{\alpha}(t)|\gg\Omega$, and therefore the integrals can be evaluated by the method of steepest descent. The stationary-phase conditions for the two dressed states are

$$E_{\alpha}(t_s)=z \quad (\alpha=\pm). \quad (62)$$

The energy $E_{\alpha}(t)$ periodically varies with time within a certain zone [cf. Eq. (61)] and whenever z is inside of this zone,

$$(\Delta-M\Omega)/2+\alpha\sqrt{(\Delta-M\Omega)^2/4+r^2} < z < (\Delta+M\Omega)/2+\alpha\sqrt{(\Delta+M\Omega)^2/4+r^2} \quad (\alpha=\pm), \quad (63)$$

the roots t_s of Eq. (62) are real and the harmonics of the corresponding dressed state $\Phi^{\alpha}_i(z)$ reveal oscillatory behavior as a function of z, e.g.,

$$\Phi^{\alpha}_1(z)\cong\alpha\left(\frac{2}{\pi M\sin\Omega t}\right)^{1/2}\cos\left(\int_0^{t_s}[E_{\alpha}(t)-z]dt-\pi/4\right),$$

$$t_s=\frac{1}{\Omega}\ \arccos\left(\frac{z^2-r^2-\Delta z}{zM\Omega}\right). \quad (64)$$

On the other hand, for the values of z outside of the zone (63) the roots t_s of Eq. (62) are complex and the harmonics $\Phi^{\alpha}_i(z)$ decay exponentially away from the zone boundaries. For example, for z in the range

$$0<\alpha z<(\alpha\Delta-M\Omega)/2+\sqrt{(\alpha\Delta-M\Omega)^2/4+r^2} \quad (65)$$

the corresponding asymptotic expression for $\Phi^{\alpha}_i(z)$ is [39]

$$\Phi^{\alpha}_1(z)\cong\frac{\alpha(-\alpha)^n}{(2\pi M\sinh\Omega\tau_s)^{1/2}} \times\exp\left(-\int_0^{\tau_s}dt[(\alpha\Delta-M\cosh\Omega t)/2 + \sqrt{(\alpha\Delta-M\Omega\cosh\Omega t)^2/4+r^2}-\alpha z]\right), \quad (66)$$

$$\tau_s=\frac{1}{\Omega}\ \cosh^{-1}\left(\frac{r^2-z^2+\Delta z}{\alpha z\Omega M}\right). \quad (67)$$

The root of Eq. (62) corresponding to the asymptotic (66) is $t_s=-i\tau_s+\pi/\Omega$ for the case $\alpha=+$ and $t_s=i\tau_s$ for $\alpha=-$. It is seen from Eq. (66) that the asymptotic behavior of $\Phi^{\alpha}_1(z)$ in Eq. (66) has a singularity at $z=0$. In the vicinity of this point we have $\Phi^{\alpha}_1(z)\sim\tau^{-1/2}\to 0$. This reflects the fact

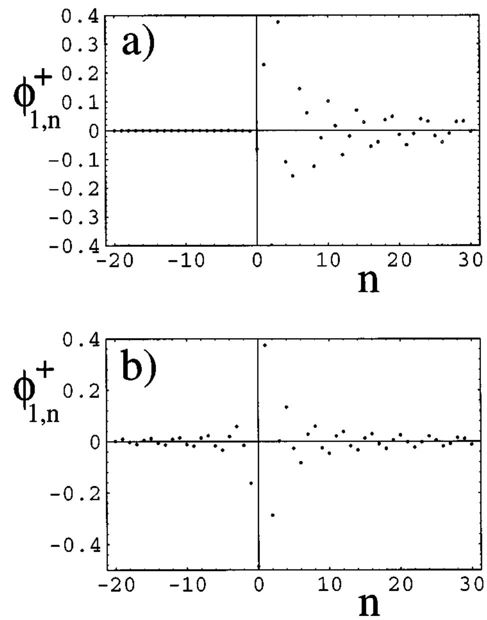

FIG. 7. (a) Dressed-state harmonics $\Phi^+_{1,n}$ vs n. The boundary of the zone (63) is near zero ($\approx 1.67\Omega$); however, the negative components are almost completely suppressed (e.g., $\Phi^+_{1,-1} \approx 0.0001$). Note that the above values of the parameters *do not correspond* to points of SFR. (b) The same r and Ω but larger $M=200$. The number of negative harmonics is strongly increased compared to (a).

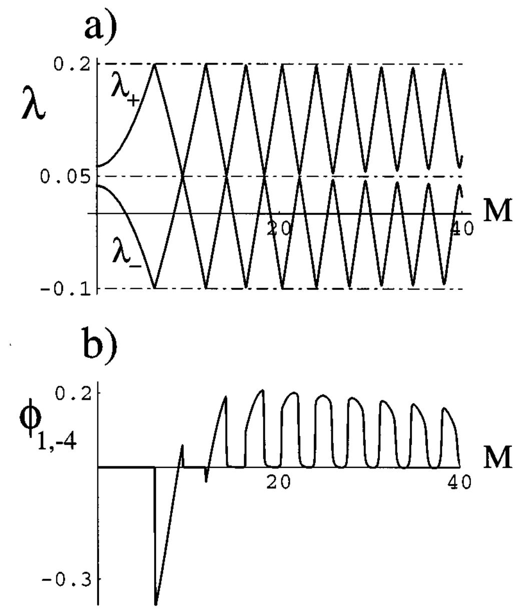

FIG. 8. (a) Quasienergies $\lambda_\pm$ vs M. $\Omega=0.3$ $r=1.5$ $\Delta=0.4$. The dot-dashed lines indicate the zone boundaries and zone center. (b). The dressed-state harmonic $\Phi^-_{1,4}$ vs M (the same values of Δ, r and Ω).

that the stationary-phase condition [Eq. (62)] has *no* roots in the domain of real $\alpha z<0$. We do not consider here the problem of the continuation of the asymptotic of $\Phi^\alpha_1(z)$ through the vicinity of $z=0$.

However, one can draw an important conclusion based on Eqs. (63), (66). If M is not very large, that is, $M \ll r^2/\Omega^2$, the boundaries of zone (63) are much farther from the origin compared to the scale of Ω and therefore the asymptotic (66) breaks down only ''deeply'' within region (65) where the harmonics $\Phi^\alpha_{1,2}(z)$ are already negligibly small. Under this condition the harmonics of *different* dressed states $\Phi^\pm_{1,2}(z)$ as functions of z *do not overlap*. They are narrowly localized within corresponding distinct zones of oscillatory behavior (63) that are positioned on the opposite sides of the origin $z=0$ and decay exponentially steeply away from the zone boundaries [40].

Numerical calculation of the dressed-state harmonics $\Phi^\alpha_1(z)$ based on the Floquet theory [Eqs. (33), (35)–(39)] show that even for relatively large values of $M \lesssim r^2/\Omega^2$ when the boundaries of zones (63) approach $z=0$ ($\sim\Omega$) the harmonics $\Phi^\alpha_i(z)$ are still suppressed in the region $\alpha z<0$. The apparent reason is the absence of stationary points of the integrand in $\langle \exp(izt)\Phi^\alpha_i(t)\rangle$ because Eq. (62) does not have roots for real $\alpha z<0$. Therefore, the value of the above integral in this domain is strongly suppressed. This case corresponds to Fig. 7(a) where the harmonics $\Phi^+_{1,n}$ are shown with solid dots for various integer values of n. Here n is assigned so that $n=0$ corresponds to the discrete value of z closest to zero. The zone boundary is only at a distance $\approx 1.67\Omega$ from zero $[r^2/(\Omega^2 M) \approx 1.67]$, however, negative harmonics are almost completely suppressed (e.g. $\Phi^+_{1,-1} \approx 0.0001$). In Fig. 7(b) parameter M is taken to be nearly 3 times larger $[r^2/(M\Omega^2) \approx 0.5]$ and a large number of negative harmonics appear.

We now establish the relation between the adiabatic approximation (60) and the Floquet approach considered in previous sections. Consider the plot of quasienergy curves $\lambda_\pm(M)$ in Fig. 8(a) that is obtained numerically by solving Eq. (39) (ratio $r/\Omega=5$). As the modulation index M varies far from the avoided crossings each quasienergy $\lambda_\alpha(M)$ within a multiple of Ω is alternately equal to one of the functions $U_\pm(M)=\langle E_\pm(t)\rangle_t$, while the corresponding dressed state is determined in turn by one of the adiabatic solutions Eq. (60). Avoided crossings correspond to the solutions of the equation $U_+(M)-U_-(M)=m\Omega$, where m is an integer. Thus, in the vicinity of the avoided crossings the dressed state is a superposition of both adiabatic states

$$\Phi^\alpha_i(t)=\sum_{\beta=\pm} C_{\alpha\beta}\chi^\beta_i(t)\exp\left(-i\int^t E_\beta(\tau)d\tau\right) \qquad (68)$$

[cf. definition (11)]. It follows then from the discussion above that away from the avoided-crossing regions the dressed state will have either positive or negative harmonics, whereas at the avoided crossing it will have both. It can be shown that the separation of the quasienergy curves at the avoided crossing (their repulsion) is proportional to the amplitude of transition between the adiabatic states. It is easy to estimate, using the Landau-Zener theory [41] and Eq. (61) that this repulsion is small for values of $M \lesssim r^2/\Omega^2$. In such a case the individual harmonics of the dressed states will undergo sharp ''steplike'' behavior when one changes the parameters of the system.

This effect is seen in Fig. 8(b), where the harmonics $\Phi^-_{1,-4}$ of the dressed state corresponding to the $\lambda_-(M)$

quasienergy curve is plotted as a function of M. According to the discussion above, in the intervals of M where $\lambda_-(M)\cong\langle E_+(t)\rangle_t$ this component nearly vanishes. The effect is very profound when $M\lesssim r^2/\Omega^2=25$. For larger values of M the repulsion of the quasienergy curves increases, the dependence of $\Phi_{1,-4}$ on M becomes smoother, and the regions where it takes on small values become narrower, shrinking down to the points of SFR. The reason for this is the following. The minimum separation between the eigenvalues $E_\pm(t)$ is $r\gg\Omega$, however, as M grows the eigenvalues change rapidly and when $M>r^2/\Omega^2$ the adiabatic approximation in Eq. (10) breaks down within each period of modulation. Because of the resulting switching between the adiabatic solutions (cf. Ref. [33]), the dressed states, in general, have a large number of harmonics of both signs. This case corresponds to Fig. 7(b) where the harmonics $\Phi^+_{1,n}$ are plotted vs n.

VII. THE *N*-LEVEL SYSTEM WITH MONOCHROMATIC PUMP

The example considered thus far in this paper corresponds *effectively* to a special model of a periodically driven system in which an external perturbation has the form of a projection operator onto a single-quantum state [state $|2\rangle$ in Eq. (10)]. The phenomenon of simultaneously forbidden resonances is an intrinsic property of such a model (see also [24]). Indeed, consider now an N-level system with the Hamiltonian

$$\mathcal{H}(t)=\sum_{i=1}^{N}(\mathcal{E}_i|i\rangle\langle i|)+\sum_{i=1}^{N-1}(r_{iN}|i\rangle\langle N|+r_{Ni}|N\rangle\langle i|)+2A\cos\Omega t|N\rangle\langle N|. \tag{69}$$

A monochromatic field with an amplitude A and frequency Ω is coupled directly only to a state $|N\rangle$, which is connected to the other states by the matrix elements r_{iN}. Again, we will investigate the spectrum of quasienergies in the system, using a probe channel consisting of a weak field connection to some other (probe) state $|0\rangle$. The resonance response to the weak field is described by the harmonics of the dressed-state amplitudes [cf. Eq. (11)]

$$\sum_{i=1}^{N}\Phi_i^{\alpha}(t)|i\rangle,\quad \alpha=1,2,\ldots,N$$

where the index α enumerates the dressed states. There will be N ladders of quasienergy levels corresponding to those states [cf. Eq. (12)]. Applying the arguments similar to those considered in the Sec. II one obtains the recursion equations for the harmonics corresponding to a given dressed state [cf. Eq. (24)]

$$(\mathcal{E}_N-n\Omega)\Phi_{N,n}+\sum_{i=1}^{N-1}r_{Ni}\Phi_{i,n}+A(\Phi_{N,n-1}+\Phi_{N,n+1})=g\Phi_{N,n}, \tag{70}$$

$$r_{iN}\Phi_{N,n}+(\mathcal{E}_i-n\Omega)\Phi_{i,n}=g\Phi_{i,n}\quad (i=1,2,\ldots,N-1) \tag{71}$$

(for simplicity, the index α is omitted). It follows from Eqs. (71) that harmonics $\Phi_{i,n}$ with $i\neq N$ can be expressed in terms of the component $\Phi_{N,n}$ [cf. Eq. (33)]

$$\Phi_{i,n}=-\frac{r_{iN}}{\mathcal{E}_i-g-n\Omega}\Phi_{N,n},\quad i=1,2,\ldots,N-1. \tag{72}$$

Using these relations and Eq. (70) one derives the *scalar* recursion equation for the $\Phi_{N,n}$ which can be written in the following form:

$$\psi_n=c_{n+1}\psi_{n+1}+c_{n-1}\psi_{n-1},\quad \Phi_{N,n}\equiv c_n\psi_n, \tag{73}$$

where the coefficients c_n are

$$c_n=A\left(g+n\Omega-\mathcal{E}_N+\sum_{i=1}^{N-1}\frac{r_{Ni}r_{iN}}{\mathcal{E}_i-g-n\Omega}\right)^{-1}. \tag{74}$$

Based on the results of Sec. IV and Appendices A and B one can obtain the explicit solution for the harmonics of the dressed states and an algebraic equation for the spectrum of quasienergies in terms of infinite continued fractions.

The significant property of the above N-level model is that it can be described by a single equation (73) that corresponds to the processes of emission and absorption of photons in the state $|N\rangle$. One could naively think of ψ_n as being proportional to the amplitude for being in the state $|N\rangle$ with n photons. The coefficients c_n could then be regarded as amplitudes of those transitions in which the number of photons n in the state $|N\rangle$ changes by 1. It is clear then that the dependence of ψ_n upon n will be strongly affected if the coefficient $c_n=0$ for some $n=k$. Applying similar arguments to Eq. (73) as were applied to Eq. (34) one can obtain the result that, in this case, $\psi_n\equiv0$ for all $n>k$ or $n<k$. As seen from Eq. (74), the condition $c_k=0$ is satisfied whenever the quasienergy $g+k\Omega$ coincides with one of the diagonal matrix elements $\mathcal{E}_j$

$$\mathcal{E}_j=g+k\Omega\quad j\neq N \tag{75}$$

[Eq. (75) is a generalization of condition (27) derived above for the case of a three-level system]. It immediately follows from Eqs. (71) that in this case the dressed-state harmonics are

$$\Phi_{i,n}=0,\quad \text{for } n>k\quad (n<k)\quad \text{and } i=1,2,\ldots,N-1.$$

The harmonics corresponding to $n=k$

$$\Phi_{i,k}=0,\quad (i\neq j)\quad \Phi_{j,k}\neq0.$$

This model represents a strongly driven system in which an exact result can be obtained. It can also be understood from the factorization property considered in Sec. III. The generalization of the analysis for the N-level case is straightforward.

VIII. SUMMARY AND DISCUSSION

In this paper we investigated the dynamics of the two-level system subject to a strong resonant field with a periodic frequency modulation using an additional ''probe'' channel that connects this system to some other level (chosen here as a ground level). The new observables introduced via this channel are off-diagonal elements of the density matrix between the ground state $|0\rangle$ and resonantly coupled states $|1\rangle$ and $|2\rangle$. The time evolution of those observables in the limit of weak dissipation is formally described by the Schrödinger equation (10) for the two-level system driven via a strong modulating interaction. The specific form of the Hamiltonian of this system with only one level being modulated gives rise to the factorization (27), (28) of the determinant of the associated Floquet Hamiltonian. As a result the dressed state with zero quasienergy is special, either its positive or negative harmonics are zero.

The above property has an immediate experimental manifestation when the dissipation is weak and the widths of the probe absorption and emission lines are narrower than their separations [condition (9)]. In this case the strong FM pump periodically modulates the probe-induced dipole moment [Eq. (22)] much more rapidly than the characteristic relaxation times in the system. Therefore the effect of such high-frequency modulation on the medium will not be reduced to just modulation of its refraction index and absorption coefficient. Rather, this modulation creates a sort of ''temporal grating'' (quasienergy ladders) that resonantly scatters the monochromatic probe field with frequencies $\omega_p = \nu_{\pm,n} + \delta$ [cf. Eq. (21)]. In general, there will be two infinite combs of the probe transition frequencies. However, at certain values of the parameters one of the combs becomes *semi-infinite* with a unique edge. We refer to this effect as simultaneously forbidden resonances. According to Eq. (23), when this effect occurs one and the same half-infinite sequence of resonant frequencies will be absent in spectra of both probe absorption and emission. In the latter case the probe frequency should be at resonance with one of the allowed transitions.

It is of interest to consider the line shape of the probe response under the condition for SFR for an experimentally suitable choice of parameters. We will consider the case of probe-absorption. To find the full probe-absorption curve we solved numerically Eq. (6) for the amplitudes $\psi_1(t)$, $\psi_2(t)$. In the stationary regime the amplitudes contain time-periodic multiples, $\psi_j(t) = \exp[i(\omega_0 - \omega_p)t]\Sigma_{n=-\infty}^{n=\infty}\psi_{j,n}\exp(-in\Omega t)$ and the probe absorption rate is proportional to Im $\psi_{1,0}$. The amplitudes $\psi_{j,n}$ satisfy a set of recurrent relations which were solved using the method of continued fractions (we do not describe the details here). The probe-absorption rate per atom can be written in the form

$$Q(\omega_p) = 2\omega_p \left|\frac{\mathcal{E}_p d_{01}}{\hbar\Omega}\right|^2 a(\omega_p/\Omega), \tag{76}$$

where $a(\omega_p/\Omega)$ is a dimensionless function of ω_p/Ω. This function is plotted in Fig. 9 for the choice of the parameters r, M, Δ, Ω that correspond to one of the points of SFR. We fixed the relaxation width $\hbar\gamma_1$ of the upper level (state $|1\rangle$ in Fig. 1) such that $\gamma_1/\Omega = 0.03$. It is seen that the set of absorption resonances are well resolved, and, in particular, even the weakest resonances are not buried under the nonresonant background from the stronger lines. There are two combs of probe frequencies in the set. One of the combs corresponds to the frequencies $\omega_p \approx \omega_0 + \Omega(n+0.5)$, $n = \pm 1, \pm 2, \ldots$, which occur both above and below ω_0. There are 12 resonances shown in this comb. However the other comb corresponds to the frequencies $\omega_p \approx \omega_0 + n\Omega$, $n = 0, -1, -2, \ldots$, and all resonances with $n > 0$ are forbidden (the corresponding dressed state has its positive harmonics equal to zero). There are six resonances seen in this comb. We stress that the position of the comb of frequencies exhibiting the SFR effect is fixed by the central point at $\omega_p = \omega_0$, whereas the position of the other comb depends on the choice of parameters.

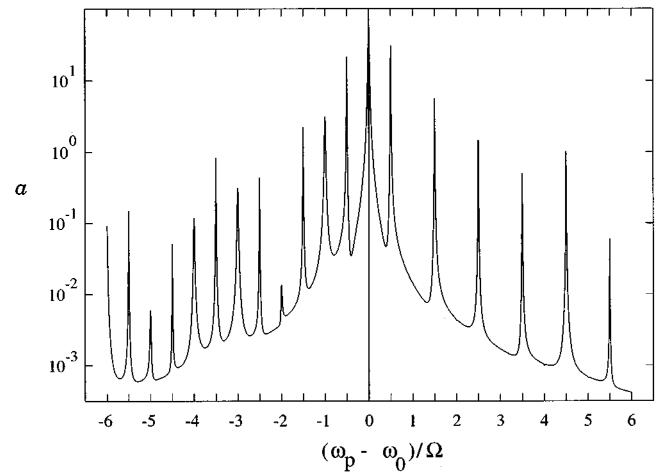

FIG. 9. The reduced absorption rate per atom a is plotted vs $(\omega_p - \omega_0)/\Omega$ for the following values of the parameters: $r = 1.0$, $\Omega = 1.0$, $\Delta = 0.5$, $M = 8.963$, $\gamma_1 = 0.03$, $\gamma_2 = 0$. Two combs of resonant frequencies ω_p appear and are shifted with respect to each other by $\Delta/2 = 0.5$. Note the absence in one of the combs of all resonances above the central resonance. This is the SFR effect.

The spectrum showing the SFR effect exhibited in Fig. 9 can be realized in a straightforward experiment, choosing an atomic beam as the medium (to minimize Doppler broadening). If the atomic density in the beam is $N \sim 10^{10}$ cm^{-3} and one assumes $\gamma/(2\pi) \lesssim 10$ MHz [$\gamma \equiv \max(\gamma_1, \gamma_2)$] one can use high modulation frequencies $\Omega/(2\pi)$, of the order of several hundreds of MHz so that the ratio Ω/γ can be made large (in Fig. 9 it is ≈ 33). Using for the power density of the pump laser $\sim$ 100 W/cm^2, the corresponding value of $r/(2\pi)$ is a few hundred MHz, of the same order as $\Omega/(2\pi)$. This corresponds to the parameters chosen for Fig. 9. The value of the modulation index M in this figure corresponds to a modulation depth of $8.9\Omega/(2\pi)$ which will be of the order of 2 GHz. The reciprocal absorption lengths for various frequencies of the probe can be expressed in terms of the dimensionless coefficient a in Eq. (76) (near the probe resonances it is also of the order of $a_0^{-1}|\Phi_{1,n}^{\alpha}|^2$ cm^{-1}, where a_0 is the absorption length in the absence of modulation). With the above choice of parameters, a beam of width 1 mm will absorb on the order of 10% of the probe at the central resonance. Thus with a sensitivity approaching the level of 0.01% the structure of the lines in Fig. 9 could be seen, revealing the absence of the forbidden resonances discussed above. We note that when the atom density in the beam increases the medium becomes optically thick and the line shape should be modified. However, the absorption in the vicinity of the forbidden resonances remains strongly suppressed provided the condition $\Omega/\gamma \gg 1$ is satisfied. This happens because the matrix elements for the corresponding probe transitions ($\propto|\Phi^{\alpha}_{1,n}|$) are identically zero.

In this paper the values of the system parameters where the SFR effect occurs (corresponding to the factorization of the Floquet Hamiltonian) were investigated. These values form two manifolds in the parameter space of the system (honeycomb). The shapes of the factorization manifolds reflects the periodic property of the quasienergy ladders and the strong nonlinearity in the system.

We further have shown that the factorization property and the related vanishing of a half-infinite number of dressed state harmonics appear in the general context of a periodically driven N-level system in which an external perturbation has the form of an operator projecting onto a single-quantum state (state $|2\rangle$ in our case). The above effect is an intrinsic property of this model for which the particular case of $N=2$ was considered in detail. It is worth noting that this model with more than two levels can be realized in the framework of the Autler-Townes scheme if we couple either of the states $|1\rangle$ or $|2\rangle$ to some other state $|3\rangle$ by a strong monochromatic resonant field. Then either of the strongly coupled levels should be connected by a weak probe to the level $|0\rangle$. In this case the envelopes of the off-diagonal elements of the density matrix that describe the probe response will obey the Schrödinger equation with the Hamiltonian (69) with $N=3$ (in principle more than one additional field and state can be involved).

The other consequence of the factorization is the possibility of a degeneracy of the dressed states. At the degeneracy points quasienergy ladders for both states coincide and there is a unique comb of equally spaced probe resonance frequencies $\omega_0+n\Omega$. The shape of the resonance curves is non-Lorentzian in this case and depends on the sign of n.

The problem of a degeneracy of quasienergy levels in the two-level system with a modulating interaction has been intensively studied elsewhere, primarily in the context of the spin-1/2 magnetic-resonance problem (see [23,26] and references therein). The static field spin-precession frequency corresponds to $2(\Delta^2/4+r^2)^{1/2}$ in our notation [cf. Eq. (10)] whereas the rf field strength corresponds to the modulation depth $M\Omega$ and the rf frequency corresponds to Ω (the quantity r corresponds to the component of the static field normal to the rf field polarization). Some limiting cases were considered explicitly, in particular, Eq. (54) was discussed in [23]. However, it is the factorization property (28) and the structure of the factorization manifolds (Sec. V) which give a crucial insight for the understanding of this phenomenon—the degeneracy points occur at the intersection of the factorization.

The factorization explains another effect that appears in the context of the magnetic-resonance problem with spin 1/2. This effect, referred to as Haroche-like resonances (see [23,26], and references therein) is related to the suppression of the time-averaged resonance rf absorption signal at certain values of the system parameters. The absorption of the weak rf is resonantly enhanced when the frequency satisfies one of the conditions [cf. Eq. (46)]

$$p\Omega=\epsilon_+-\epsilon-, \quad (p=1,2,\ldots). \tag{77}$$

Under these conditions the positions of the ''bare'' quasienergy ladders coincide. In general, the ladders will split due to the resonance interaction with the rf. Assume now that apart from the above resonance condition the factorization condition is also satisfied. The latter will amount to both Eqs. (47), (48) for some integers $k<0$ and $l>0$, $(l-k=p)$. In order to make a resonance transition between the ''up'' and ''down'' spin states (states $|1\rangle$ and $|2\rangle$ in our case) at least p quanta of rf are needed. However, according to the discussion in Sec. V the factorization leads to the forbidding of the transitions between the two different subspaces of the Floquet Hamiltonian, i.e., the transitions with net absorption of more than $|k|$ quanta or net emission of more than l quanta. It is clear then, that at the factorization points rf resonance will be suppressed. It can be easily seen from Eqs. (77), (47), (48) that for $r\neq 0$ the suppression of the rf resonances occurs for $p>1$ and $\Delta=\pm(2n-p)\Omega$ where $n=0,1,\ldots,p-1$.

The important point is that the degeneracy of the quasienergy levels and the above described Haroche-like (forbidden) resonances occur at particular points of the more general factorization manifolds (Sec. V). The physical effect that occurs at all points of these manifolds is that of SFR, however, it cannot be observed in the context of a two-level system and requires a probe channel connecting to a third level.

ACKNOWLEDGMENTS

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APPENDIX A

Here, we describe the standard technique for obtaining a continued-fraction solution for a three-term recursion relation

$$A_n\phi_n+B_{n+1}\phi_{n+1}+B_{n-1}\phi_{n-1}=0. \tag{A1}$$

We first define ''backward'' and ''forward'' coefficients

$$X_n^{(\pm)}=\frac{\phi_{n\pm1}}{\phi_n}. \tag{A2}$$

Then after dividing Eq. (A1) by ϕ_n and using the definitions in Eq. (A2) we obtain the recursion relations for the backward and forward coefficients

$$A_n+B_{n+1}X_n^{(+)}+B_{n-1}(X_{n-1}^{(+)})^{-1}=0, \tag{A3}$$

$$A_n+B_{n+1}(X_{n+1}^{(-)})^{-1}+B_{n-1}X_n^{(-)}=0. \tag{A4}$$

We now formally solve Eq. (A3) for $X_{n-1}^{(+)}$ and Eq. (A4) for $X_{n+1}^{(-)}$

$$X_{n-1}^{(+)}=-B_{n-1}(A_n+B_{n+1}X_n^{(+)})^{-1}, \tag{A5}$$

$$X_{n+1}^{(-)}=-B_{n+1}(A_n+B_{n-1}X_n^{(-)})^{-1}. \tag{A6}$$

The replacements $n\rightarrow n+1$ in Eq. (A5) and $n\rightarrow n-1$ in Eq. (A6) yield

$$X_n^{(\pm)}=-B_n(A_{n\pm1}+B_{n\pm2}X_{n\pm1}^{(\pm)})^{-1}. \tag{A7}$$

Applying a forward recursion in the equation for $X_n^{(+)}$ and a backward recursion in the equation for $X_n^{(-)}$ these quantities are represented as infinite continued fractions [Eqs. (A7) serve as their definitions]. Finally, we use the expressions (A7) in Eq. (A1) which give us the relation between the coefficients of this equation

$$A_n + B_{n+1} X_n^{(+)} + B_{n-1} X_n^{(-)} = 0 \tag{A8}$$

(it can be immediately verified that this is one and the same equation for any value of the integer n). Equation (A8) is a condition of existence of unique bounded solutions of Eq. (A1). If this condition is satisfied then one can use a recursion in Eqs. (A2) to find a solution of Eq. (A1) with an accuracy to an arbitrary multiple. The correspondence between A_n, B_n, ϕ_n, and the quantities introduced in Sec. IV is clear from the comparison of Eqs. (A1), (34).

APPENDIX B

Assume, at first, that neither of quantities λ_n equal zero. Then, using Eqs. (36), (38), one can write the ratios $\Phi_{1,n+1}/\Phi_{1,n}$ for two adjacent values of n

$$\frac{\Phi_{1,n+1}}{\Phi_{1,n}} = -\frac{1}{\mu \lambda_{n+1}} Y^{(+)}(\lambda_n), \tag{B1}$$

$$\frac{\Phi_{1,n}}{\Phi_{1,n-1}} = -\mu \frac{\lambda_{n-1}}{d(\lambda_n)} - Y^{(+)}(\lambda_n). \tag{B2}$$

Furthermore, it is seen from Eqs. (38) that $Y^{(+)}(\lambda_n)$ approaches infinity as

$$d(\lambda_{n+1}) - Y^{(+)}(\lambda_{n+1}) = 0. \tag{B3}$$

In such a case the component $\Phi_{1,n}$ in Eqs. (B1), (B2) vanishes while the harmonics $\Phi_{1,n\pm 1}$ do not. It can be shown, based on Eq. (39) (written for $n-1$), that the quasienergy λ must satisfy the following equation simultaneously with Eq. (B3):

$$d(\lambda_{n-1}) - Y^{(-)}(\lambda_{n-1}) = 0. \tag{B4}$$

Equations (B3) and (B4) together constitute a condition for $\Phi_{1,n}=0$ [as well as $\Phi_{2,n}=0$, according to Eq. (33)] and give a relation between the three dimensionless parameters of the system Δ/Ω, M, r/Ω for this condition to be satisfied.

We now turn to the condition (27), that is, $\lambda_k = 0$ for some integer k. According to the discussion in Sec. III, this condition leads, in general, to *one* of the equations, (40) or (41). It can be seen, that under the condition (27) Eqs. (B3) and (B4) formally coincide at $n=k$ with Eqs. (40) and (41), respectively. However, due to the fact that $\lambda_k = 0$ neither of the latter equations leads to the vanishing of the component $\Phi_{1,k}$ [cf. Eqs. (36), (37)].

Consider now the harmonics $\Phi_{1,n}$ with $n \neq k$. In the case of the SFR corresponding to Eqs. (40), (42) the condition (B3) cannot be satisfied for $n=k+1$, because $Y^{(+)}(\Omega)$ does not diverge [cf. Eqs. (38), (40)]. Therefore the component $\Phi_{1,k+1}$ never equals zero in this case. This also can be seen directly from Eqs. (34), (35)

$$\frac{\Phi_{1,k+1}}{\Phi_{1,k}} = -\frac{r^2}{\mu \Omega} \neq 0. \tag{B5}$$

On the other hand, single harmonics with $n > k+1$ can vanish and this is described by Eqs. (B3), (B4) with $n > k+1$. The other type of SFR (41), (43) can be analyzed using similar arguments.

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[37] This method was extensively used in the literature in the study of atom-field periodic interactions (see, for example, [2,22,26,42,43]).

[38] Strictly speaking, the quasienergies $\lambda_{\pm}(M,\Delta,r)$ introduced according to the rule (46), have discontinuities at $M\neq 0$. They occur when Δ and r satisfy the resonance conditions $\epsilon_{+}-\epsilon_{-}=p\Omega$ $(p=1,2,\ldots)$. Indeed, when the parameters of the system are adjusted to satisfy the resonant condition the quasienergy curves $\lambda_{+}=\lambda_{+}(M)$ and $\lambda_{-}=\lambda_{-}(M)+p\Omega$ cross at $M=0$ but, in general, *do not cross* each other at any other value of M. However, the continued functions $\lambda_{\pm}(M,\Delta,r)$ can be defined by Eq. (46) *within* the manifold of SFR. The reason is the following. It is seen from Eqs. (27), (46) that at $M=0$ the resonance conditions are satisfied on the manifold of SFR only when the ratio Δ/Ω takes integer values. On the other hand as will be shown *all* points of the manifold of SFR with integer values of Δ/Ω are the points of quasienergy level crossings.

[39] In the vicinity of the zone boundaries $(\sim\Omega)$ the steepest descent method fails and the manifestation of this is the divergence of the prefactors in the asymptotic expressions (64), (66) at the zone boundaries. This fact is in close analogy with WKB theory.

[40] This behavior is similar to that of a wave function in the quasiclassical double-well potential when energy levels from the different wells are far away from each other and the wave function corresponding to a given level is localized inside of one of the wells. A different type of behavior occurs in the case of resonance when levels from the different wells are close and due to resonance tunneling the wave function has maxima in both wells. This situation corresponds to the avoided crossing in our case [see Figs. 8(a), 8(b), and related discussion in the text].

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