Semiclassical study of the laser transition

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A semiclassical, nonlinear theory of a single-mode laser is developed so that the transition around threshold can be studied. The field is expressed in the frequency domain which allows us to emphasize the role of the Fabry-Pérot cavity. A generalized Airy function is obtained for the laser line shape: it contains the source line shape and the empty cavity line shape. It describes the laser both above and below threshold, including the transition region where there is an abrupt increase of intensity and decrease of the laser linewidth. The general arguments are illustrated by detailed numerical calculations for the 3.39-μm line of the He-Ne laser. The intensity of the spontaneous emission source which plays a central role is computed from first principles. [S1050-2947(97)07601-4]

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I. INTRODUCTION

The laser line is known to be very narrow as compared to atomic emission lines; its shape can be described by a Lorentzian function whose width obeys a formula given by Schawlow and Townes [1] almost 40 years ago. This formula has been refined from different points of view, and basically applies to situations where the laser gain is either above or below threshold, with a factor of 2 which distinguishes the two cases. Far below threshold, the usual Airy function applies. It is the objective of this paper to give a description of the crossing of the laser from below to above threshold through a synthetic formulation of the line shape and intensity, valid whatever the gain, in the framework of the semiclassical theory of light-matter interaction.

A laser is an optical device in which three fundamental physical effects play important simultaneous roles: the active (pumped) medium provides the spontaneous emission (source) and the stimulated emission (amplification). The resonant cavity provides the feedback. A complete description of the laser should thus include these three basic effects together, i.e., one needs a formula or a set of equations which are able to simultaneously describe the source, the amplification and the resonant cavity. Up to now, laser line-shape theories have been essentially aimed at the description of the field far above threshold: two methods are used to obtain the line shape. The first is described in Verdeyen's textbook [2], for instance, and is based on an extension of the Airy function adapted to the laser. This is done in the frequency domain. The second, which is most popular, is based on Fokker-Planck [3] or Langevin [4,5] equations, and the use of Wiener-Kintchine's theorem which connects the time to the frequency domain. This last method leads to a very small width of the laser line (in fact a δ-Dirac distribution) and it is generally admitted that its measurable limit stems from quantum effects due to the phase diffusion of randomly emitted spontaneous photons. This view was also developed by Henry [6] under a different form, and the linewidth enhancement factor α introduced by him is always used in semiconductor lasers equations. A huge amount of literature has been devoted to the subject [7] but it seems that no semiclassical study exists which allows the calculation of the transition from the Airy to the laser line shape when the gain is increased from below to above threshold. The experiment that most directly relates to the study of this transition was performed by Güttner and co-workers [8,9], who measured the linewidth variation around threshold and verified the theories elaborated by Risken and co-workers [10–12] and Hampstead and Lax [3], essentially based on a quantum formalism for the field and describing its statistical properties. The state of this art was expressed in Mandel and Wolf's textbook [13]. After all these extensive studies, one can ask what urged us to attack the subject again. The basic reason is simply that in former formulas, it was not possible, in general, to recover the Airy function from the laser equations by removing the constants representing the active medium, nor to recover the active medium line-shape function by removing the effect of the cavity. An exception can be found in Ref. [14], where it was shown that the intrinsic linewidth of the laser broadens to the free-atom natural linewidth when the transmittivity of the mirrors increases. The above drawbacks probably occur because studies essentially focused on the description of the statistical properties of light in the time domain. Here we follow another path which leads to a synthetic formula, written in the frequency domain, based on semiclassical arguments and able to include simultaneously the three basic effects which comprise a laser. Once we obtained this generalized Airy formula from simple arguments, a challenge was to see whether this formula was able to be used quantitatively, and this is why considerable details are given in the following in an attempt to answer this question. The noise included in the laser light has different origins: (i) (the so-called ''technical noise'' originating from acoustic, thermal, or electromagnetic vibrations which affects the elements of the cavity; (ii) the noise brought by the pump; and (iii) the quantum noise coming from the spontaneous emission and from the leakage from the outside across the mirrors. Note that quantum and collisional noise are already phenomenologically partially included in the properties (line shape) of the emitting medium. In the following, we will be interested only in the way this line shape transforms

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when the medium is placed inside the cavity and when the gain is increased. Related subjects, such as the transformations of the technical noise or the temporal statistics of light, are not considered. However, it should be noted that it is very difficult to experimentally separate different types of noise, and the difficulty of the interpretation is increased when one becomes aware of their entanglement due to the nonlinear character of the laser equations. The generalized Airy function which is developed here can thus be considered as a tool which can also be used for other studies. A continuous description of the laser transition, i.e., the link between the properties of the empty or the passive cavity and the laser when the gain is progressively increased, can be obtained by keeping the three above-mentioned effects together. We will give the main ideas in Sec. II of this paper: a purely monochromatic, stationary source term is first considered inside the cavity, and the Airy function is written. This source corresponds to the ideal case of a noise-free source because its spectrum is a δ-like distribution in the frequency domain. This is the hypothesis usually taken in the case of an empty Fabry-Pérot cavity probed by an external incident beam of light whose spectrum is much narrower than that of the interferometer. Then the extension to the real case of the spontaneous emission with a wide spectrum is made: the source term and the amplification are calculated in Secs. III–V. The specific case of the single-mode He-Ne gas laser at 3.39 μm is developed in order to illustrate the arguments quantitatively. Here we will take into account the Zeeman substructure of the levels, but the paper is not devoted to vectorial aspects of light whose polarization will be kept linear. The mean-field approximation is used, and, in order to find the spontaneous emission rate, a balance is made between the number of spontaneously deexcited atoms and the number of spontaneous photons. No quantum representation of the field is used, but a close connection to quantum equations exists: for instance we arrive at some conclusions already given by Goldberg, Milonni, and Sundaram [14]. The difference from usual theories comes essentially from the calculation in the frequency domain, which permits a more precise description of the properties of the Fabry-Pérot cavity (which usually, is only represented by a quality factor Q, i.e., a Lorentzian) and those of the spontaneous source (which usually is only represented by a diffusion term D).

The intensity and the line-shape expressions are calculated in Sec. VI: we obtain a generalized Airy function which relates the spectral density of the laser light (i.e., its line shape) to that of the source and to the total saturating field intensity. Numerical results show, in the example studied, an evolution of the linewidth from 10 MHz below threshold to some tens of Hz above. Measurements of laser linewidths in connection with quantum effects, design of optimized lasers for metrology or telecommunications or studies of more complex lasers can benefit from this understanding.

II. BASIC ARGUMENTS

One aim of this study is to obtain a laser field equation valid below and above threshold, i.e., an equation from which one can recover the empty cavity properties by removing the active medium. Let us first consider an empty Fabry-Pérot cavity. Its length is l. r_1 and r_2 are the reflectances of the mirrors, and $\mathcal{S}$ the amplitude of the electric field corresponding to the optical source. For the moment the system is considered to be transversally infinite (we will take into account the transverse distribution later). Generally, $\mathcal{S}$ is related to an external field $\mathcal{E}_i$ which probes (or which is probed by) the cavity across one mirror: $\mathcal{S}=\sqrt{1-r_1^2}\mathcal{E}_i$. This field is considered to be monochromatic, i.e., its spectrum is much narrower than the cavity linewidth. The usual Airy function allows the calculation of the spectral density of the field inside the cavity, and the easiest way to obtain it is the so-called *round-trip method* [15,16] which we recall in Appendix A. The slowly varying part of the field $\mathcal{E}$ inside the cavity obeys

$$\frac{d\mathcal{E}}{dt}=-\frac{c}{2l}[1-e^{-L}e^{i\phi}]\mathcal{E}+\frac{c}{2l}\mathcal{S}, \tag{2.1}$$

where L stands for the losses and ϕ represents the round-trip cumulated phase; here,

$$\phi=-2\omega l/c. \tag{2.2}$$

In the stationary regime, the intensity: $I(\omega)=\mathcal{E}.\mathcal{E}^*$ can be written

$$I(\omega)=\frac{S}{(1-e^{-L})^2+4e^{-L}\sin^2\omega l/c}, \tag{2.3}$$

with $S=\mathcal{S}\mathcal{S}^*$, and this is the Airy function for the empty Fabry-Perot cavity. When doing this type of calculation, one has to keep in mind the time frequency relation: for instance, if we decompose the field in the frequency domain into slices having a width of 1 Hz, one needs at least 1 s to measure it, and thus Eq. (2.1) would be able to describe the evolution of the amplitude of this component on a time scale of several seconds which in fact, is considerably greater than the round trip time $2l/c$.

Now let us consider the same cavity filled with an amplifying medium able to emit light inside a frequency interval $\delta\omega$ narrow as compared to the resonance line of the cavity, i.e., ideally at a *single frequency* ω (noise-free source) and which is characterized by α, the complex saturated polarizability. The simplest form for α can be written, for a low saturating intensity I,

$$\alpha={}^0\alpha-\beta I=\alpha^r+i\alpha^i={}^0\alpha^r-\beta^r I+i[{}^0\alpha^i-\beta^i I], \tag{2.4}$$

where the superscripts r and i stand, respectively, for the real and imaginary parts. ${}^0\alpha$ is the linear part and β is the saturation coefficient. β can take the Gaussian transverse distribution of light into account by including a geometrical factor [17–19] (see Appendix B). The source is also saturated, and can be represented by $S=\mathcal{S}\mathcal{S}^*=S_0-aI$, where S_0 and a are real constants. This source corresponds to the photons spontaneously emitted during the round-trip time $2l/c$ and proportional to the saturated population of the upper level. When considering only *a saturating intensity averaged along the laser*, the round-trip method applies, and leads to the equation of evolution.

$$\frac{d\mathcal{E}}{dt}=-\frac{c}{2l}[1-e^{-L}e^{\omega l\alpha^i/c\epsilon_0}e^{i\phi}]\mathcal{E}+\frac{c}{2l}\mathcal{S}, \tag{2.5}$$

where, this time,

$$\phi = -\frac{\omega 2l}{c}[1+\alpha^r/2\epsilon_0]. \tag{2.6}$$

Here we have not included the Guoy phase shift due to the Gaussian transverse structure of the beam, as it is unessential to our calculation. Note in passing that a Lamb's-type equation for a single-mode laser can be recovered from Eq. (2.5) by expanding the exponential, choosing $\omega 2l/c = N2\pi$ and taking $S=0$,

$$\frac{d\mathcal{E}}{dt} = -\frac{c}{2l}\left[L + \frac{i\omega l\alpha}{c\epsilon_0}\right]\mathcal{E}. \tag{2.7}$$

The effect of the source and the properties of the resonant cavity are thus not included in this equation. The Airy function can be obtained from Eq. (2.5) in the stationary regime. Let us note here that the source term corresponds to the sum of the photons spontaneously emitted during the time $2l/c$ and interval $\delta\nu$. A random function should describe these events and should depend upon the laser size, a fact not included in usual laser theories. When the source and the polarizability are written with their developed forms, one obtains

$$I = \frac{S_0 - aI}{[1-e^{-L+\omega l({}^0\alpha^i - \beta^i I)/\epsilon_0 c}]^2 + 4e^{-L+\omega l({}^0\alpha^i - \beta^i I)/\epsilon_0 c}\sin^2[(\omega l/c)(1+({}^0\alpha^r - \beta^r I)/2\epsilon_0]}. \tag{2.8}$$

This formula is able to describe the intensity distribution when the frequency is scanned. Here the denominator can become very small but cannot cancel (this situation is physically impossible because it would imply that the intensity I would tend toward infinity). We have already given an account of this study in a recent conference [20]. Let us now compare the result given by formula (2.8) to the intensity given by Lamb's theory above threshold. Here we will also take the convention following which the threshold is defined for the linear gain which compensates exactly for losses. Let I_0 be the intensity at resonance (i.e., when the frequency is such that $\phi = N2\pi$. Equation (2.8) becomes

$$I_0 = \frac{S_0 - aI_0}{[1-e^{-L+\omega l({}^0\alpha^i - \beta^i I_0)/\epsilon_0 c}]^2}. \tag{2.9}$$

The term in the exponential is small (in the usual theory the saturated gain exactly compensates for the losses in the stationary regime). Thus it can be developed at first order, and I_0 is a solution of the third-order equation

$$I_0[L - \omega l/\epsilon_0 c({}^0\alpha^i - \beta^i I_0)]^2 - S_0 + aI_0 = 0. \tag{2.10}$$

The usual solution, which we will denote I_L, is obtained from the condition ''*saturated gain = losses*.'' It is written as

$$I_L = \frac{\omega l/\epsilon_0 c\,{}^0\alpha^i - L}{\omega l/\epsilon_0 c\beta^i}. \tag{2.11}$$

Taking the source term into account brings a small deviation from this solution. Let us take $I_0 = I_L + \delta I$. If one neglects the small term δI^3 and $a\delta I$ as compared to aI_L, one obtains, from Eq. (2.10),

$$I_L[\omega l/\epsilon_0 c\beta^i]^2\delta I^2 - S_0 + aI_L = 0, \tag{2.12}$$

and thus

$$\delta I = \pm\frac{\epsilon_0 c}{\omega l\beta^i}\left(\frac{S_0 - aI_L}{I_L}\right)^{1/2}. \tag{2.13}$$

Here the only stable solution is with the + sign, because otherwise the medium is still amplifying, which means that the intensity can still be increased. One sees that δI is positive, and thus the medium is not transparent, as in Lamb's theory. It is slightly absorbing, the compensation being the contribution from the source. This conclusion agrees with a result obtained in [14].

The simplified approach discussed above has been done to mimic the usual situation where a Fabry-Pérot cavity is probed by a monochromatic field. In fact, it hardly corresponds to reality in the sense that in general no emitting medium exists with an emission linewidth narrower than that of the laser. In order to compute a more realistic case, one thus has to take two effects into account.

(i) The source is the spontaneous emission of the medium itself: one should consider the entire emission band and not only a single frequency slice as above.

(ii) The saturating power which appears inside the complex gain has to be connected to the spectrum of power density. In other words, the intensity I which appears on the right hand side of Eq. (2.8) is no longer the same as that on the left-hand side.

In order to attain this goal, it is first necessary to decompose the field in the frequency domain. This will allow us to use the fundamental property of Fabry-Pérot cavities which is the fact that *the cumulated round-trip phase is frequency dependent*. Then, in order to make a quantitative calculation from first principles, one should compute the density-matrix elements of the system and deduce the source and amplification terms for each frequency. This will allow a calculation of the line shape. For this purpose, we have chosen to work here on the specific case of the single-mode He-Ne gas laser at 3.39 μm, because this line is a secondary standard of frequency whose properties have already been studied at length [7].

Sections III–V are now devoted to a calculation of the saturated source term and to the amplification. This is done for each component of the field *inside the laser emission band*. If the reader is not interested in these rather tedious derivations, he can jump to Sec. VI where he will find a generalization of Eq. (2.8), i.e., the spectral density for the laser, in Eq. (6.16).

III. DESCRIPTION OF THE FIELD AND DENSITY-MATRIX ELEMENTS

We want to know the expressions for the source term and for the amplification when the field is decomposed into slices $\delta\nu$ in the frequency domain inside the emission band. The linewidth of this band can vary very much: from 10 MHz, for instance, at low gain down to 1 Hz, or even less above threshold. Thus $\delta\nu$ has to be adapted to each case. As a consequence, the measurement time associated to $\delta\nu$ can vary considerably as well. We recall in Appendix C the connection between the frequency and time domains descriptions of the field. The real field corresponding to expansion (C4) in Appendix C will be used in the following; i.e., we will take a field such as

$$\vec{E}=\sum_q \vec{\mathcal{E}}_{fq}(t)e^{2i\pi\nu_q t-ikz}+\vec{\mathcal{E}}_{bq}(t)e^{2i\pi\nu_q t+ikz}+\text{c.c.}, \quad (3.1)$$

where we have explicitly identified the forward- and backward-traveling components with indices f and b. The modulus of these components will be approximated below by a mean value $\mathcal{E}_q$. Note that the field is decomposed into frequency slices which are narrow as compared to the laser line.

Now let us consider the amplifying medium itself. Many theoretical calculations have been made on the He-Ne laser [5], and here we will give only the main steps of the theory. The first is to compute the density-matrix elements to first order in the field intensity and in the context of the mean-field approximation. Then the source term whose intensity is proportional to the population of the upper level can be calculated. The amplification is then obtained from the optical coherences. The source term and the amplification have to be found in the interval $\delta\nu$ around each particular frequency ν_q as defined in Appendix C. For this purpose, we use a two-energy-level model in which the upper state $|u\rangle$ and the lower state $|l\rangle$ are characterized by angular quantum numbers $j_u=1$ and $j_l=2$.

The calculations of the density-matrix elements will be worked out in a vector basis which represent eigenstates of the system. They are labeled $|u,m\rangle$ or $|l,m\rangle$, where m is the magnetic quantum number. Because dipole moments are not the same between sublevels, one has to take this decomposition into account. As is generally the case, the field will be taken as linearly polarized. The density-matrix elements for a class of velocity v obey the general equation

$$i\hbar\frac{d\rho(v)}{dt}=[(H_0-\vec{\mu}\cdot\vec{E}),\rho(v)]_-+\text{p.t.} \quad (3.2)$$

$\rho(v)$ is the density operator, H_0 is the Hamiltonian for the atom alone and the perturbation, in the dipole approximation, is $-\vec{\mu}\cdot\vec{E}$. $\vec{\mu}$ is the dipole operator and $\vec{E}$ is the field. p.t. stands for the phenomenological terms which allow for the restriction of the complete system to a two-level model: this means, for instance, that the nonradiative deexcitation of the population of a level will be included in this symbol. The quantization axis is taken to be along the propagation axis, i.e., along the laser axis. The derivative operator $d/dt=(\delta/\delta t)+v(\delta/\delta z)$ is written in the laboratory frame of reference in order to take the velocity of atoms along z into account. Equations for the populations are developed as

$$i\hbar\frac{d\rho_{um,um}}{dt}=-\vec{E}\cdot\vec{\mu}_{um,lm\pm1}\rho_{lm\pm1,um}+\vec{E}\rho_{um,lm\pm1}\cdot\vec{\mu}_{lm\pm1,um}-i\hbar\gamma_u\rho_{um,um}+i\hbar\Lambda_u, \quad (3.3)$$

$$i\hbar\frac{d\rho_{lm,lm}}{dt}=-\vec{E}\cdot\vec{\mu}_{lm,um\pm1}\rho_{um\pm1,lm}+\vec{E}\rho_{lm,um\pm1}\vec{\mu}_{um\pm1,lm}-i\hbar\gamma_l\rho_{lm,lm}+i\hbar\Lambda_l. \quad (3.4)$$

Here γ_l and γ_u are the total rates of deexcitation of the populations of the sublevels $|l,m\rangle$ or $|u,m\rangle$. Λ_l and Λ_u are the pumping rates. For simplicity, we have removed the subscript v in the notations.

Equations for the optical and Zeeman coherences are

$$i\hbar\frac{d\rho_{lm,um\pm1}}{dt}=-\hbar\omega_0\rho_{lm,um\pm1}-\vec{E}\cdot\vec{\mu}_{lm,um\pm1}[\rho_{um\pm1,um\pm1}-\rho_{lm,lm}]$$
$$-\vec{E}[\vec{\mu}_{lm,um\mp1}\rho_{um\mp1,um\pm1}-\rho_{lm,lm\pm2}\vec{\mu}_{lm\pm2,um\pm1}]-i\hbar\gamma_{lu}\rho_{lm,um\pm1}, \quad (3.5)$$

$$i\hbar\frac{d\rho_{um,um\pm2}}{dt}=-\vec{E}\cdot\vec{\mu}_{um,lm\pm1}\rho_{lm\pm1,um\pm2}+\vec{E}\rho_{um,lm\pm1}\vec{\mu}_{lm\pm1,um\pm2}-i\hbar\gamma_{2u}\rho_{um,um\pm2}, \quad (3.6)$$

$$i\hbar\frac{d\rho_{lm,lm\pm2}}{dt}=-\vec{E}\cdot\vec{\mu}_{lm,um\pm1}\rho_{um\pm1,lm\pm2}+\vec{E}\rho_{lm,um\pm1}\vec{\mu}_{um\pm1,lm\pm2}-i\hbar\gamma_{2l}\rho_{lm,lm\pm2}. \quad (3.7)$$

γ_{lu}, γ_{2u}, and γ_{2l} are, respectively, the deexcitation rates of the optical coherence (homogeneous width) and of the Zeeman coherences inside the two levels. $\hbar\omega_0$ is the energy difference between both levels. Other coherences are neglected.

These equations will be used to obtain the populations of both levels to second order in field amplitude and the polarization to third order. The rotating-wave approximation is used, i.e., one writes the expansion for the forward and backward waves of optical coherences

$$\rho_{f(lm,um\pm1)}=\sum_q p_{fq(lm,um\pm1)}e^{i(\omega_q t-kz)}, \quad (3.8a)$$

$$\rho_{b(lm,um\pm1)}=\sum_q p_{bq(lm,um\pm1)}e^{i(\omega_q t+kz)}. \quad (3.8b)$$

We will use the relation

$$\vec{\mu}_{lm,um\pm1}=\sqrt{(j+1\pm m)(j+2\pm m)}\frac{\mu_{lu}}{\sqrt{2}}(\hat{\vec{x}}\pm i\hat{\vec{y}}), \quad (3.9)$$

which is valid for a $j\Rightarrow j+1$ transition (here $j=1$). μ_{lu} is the reduced matrix element. The field, being linearly polarized, has equal components along the two counterrotating circular vectors. The population distribution for atoms having a velocity v can be approximated at zero order [5]:

$${}^0\rho_{umum(v)}=\frac{\Lambda_u}{\gamma_u}=\frac{N_u}{\gamma_u}\frac{dv}{\sqrt{\pi}v_m}e^{-v^2/v_m^2}, \quad (3.10a)$$

$${}^0\rho_{lmlm(v)}=\frac{\Lambda_l}{\gamma_l}=\frac{N_l}{\gamma_l}\frac{dv}{\sqrt{\pi}v_m}e^{-v^2/v_m^2}. \quad (3.10b)$$

We make the approximation that the pumping does not depend on m. We will write

$$N_0(v)dv={}^0\rho_{umum(v)}-{}^0\rho_{lmlm(v)}=\Lambda_0\frac{dv}{\sqrt{\pi}v_m}e^{-v^2/v_m^2}, \quad (3.11)$$

where we have used the abreviation $\Lambda_0=(N_u/\gamma_u)-(N_l/\gamma_l)$. To first order, the optical coherences are, for the forward and backward q components:

$${}^1p_{fq(lm,um\pm1)}=-\frac{\mathcal{E}_q\mu_{lm,um\pm1}}{\hbar\sqrt{2}\,{}_q\Delta_f^*}N_0(v), \quad (3.12a)$$

$${}^1p_{bq(lm,um\pm1)}=-\frac{\mathcal{E}_q\mu_{lm,um\pm1}}{\hbar\sqrt{2}\,{}_q\Delta_b^*}N_0(v), \quad (3.12b)$$

with

$${}_q\Delta_f=\omega_0-(\omega_q-kv)-i\gamma_{lu} \quad (3.13a)$$

and

$${}_q\Delta_b=\omega_0-(\omega_q+kv)-i\gamma_{lu}. \quad (3.13b)$$

The second-order populations are

$${}^2\rho_{um,um}=i\frac{N_0(v)}{2\hbar^2\gamma_u}[|\mu_{um,lm+1}|^2+|\mu_{um,lm-1}|^2]\mathcal{I}, \quad (3.14a)$$

$${}^2\rho_{lm,lm}=-i\frac{N_0(v)}{2\hbar^2\gamma_l}[|\mu_{lm,um+1}|^2+|\mu_{lm,um-1}|^2]\mathcal{I}, \quad (3.14b)$$

with the abbreviation

$$\mathcal{I}=\sum_q I_q\left[\frac{1}{{}_q\Delta_f}+\frac{1}{{}_q\Delta_b}-\frac{1}{{}_q\Delta_f^*}-\frac{1}{{}_q\Delta_b^*}\right]. \quad (3.15a)$$

I_q is the intensity of the field at frequency ω_q. This expression will become important in saturation terms, i.e., when the linewidth is very narrow. It is thus justified to make the following approximation (see Appendix D):

$$\mathcal{I}=\left[\frac{1}{\Delta_f}+\frac{1}{\Delta_b}-\frac{1}{\Delta_f^*}-\frac{1}{\Delta_b^*}\right]\sum_q I_q, \quad (3.15b)$$

where the Δ's are evaluated at the central laser frequency (which will be denoted by ω_r). This frequency is generally different from the central resonance frequency ω_0 of the active medium.

Second-order Zeeman coherences are

$${}^2\rho_{um,um\pm2}=i\frac{N_0(v)dv}{2\hbar^2\gamma_{2u}}\mu_{um,lm\pm1}\mu_{lm\pm1,um\pm2}\mathcal{I}, \quad (3.16a)$$

$${}^2\rho_{lm,lm\pm2}=-i\frac{N_0(v)dv}{2\hbar^2\gamma_{2l}}\mu_{lm,um\pm1}\mu_{um\pm1,lm\pm2}\mathcal{I}. \quad (3.16b)$$

A sum of beating terms between frequency components appears in computing these expressions. However, in our case, the frequency differences are much more smaller than the deexcitation rates, and thus can be neglected in the Lorentzians. We are thus left with a sum over exponentials which is a δ function. This is why a field with a narrow spectrum can be modeled with a single-frequency function. Third-order n components of the velocity-dependent optical forward coherence are written

$$p_{fn(lm,um\pm1)}=-i\frac{1}{2\sqrt{2}\hbar^3}\frac{N_0(v)dv\mathcal{I}}{{}_n\Delta_f^*}\mu_{lm,um\pm1}\mathcal{E}_n\left\{\frac{|\mu_{um\pm1,lm\pm{}^2_0}|^2+|\mu_{um\pm1,lm\pm{}^0_2}|^2}{\gamma_u}+\frac{|\mu_{um,lm+1}|^2+|\mu_{um,lm-1}|^2}{\gamma_l}+\frac{|\mu_{um\mp1,lm}|^2}{\gamma_{2u}}+\frac{|\mu_{um\pm1,lm\pm2}|^2}{\gamma_{2l}}\right\}. \quad (3.17)$$

We have neglected the spatial hole burning effect, i.e., terms containing $e^{\pm 2ikz}$.

IV. CALCULATION OF THE SOURCE TERM

We want to know the source term which feeds the laser field, i.e., the term which corresponds to S in Eq. (2.3) or $S_0 - aI$ in Eq. (2.8). This source corresponds to photons which are spontaneously emitted in the mode with the correct polarization, at angular frequency ω_q in the interval $\delta\nu$ during the round-trip time $2l/c$. Another constraint is that the spontaneous field is emitted with a random phase Φ with a probability $d\Phi/2\pi$: the projection of the source field onto the laser field thus contains $\cos\Phi$; when integrated over Φ the intensity of the source will thus be multiplied by $\frac{1}{2}$ (the mean value of $\cos^2\Phi$). In order to compute S , we will make an energy balance between the number of spontaneously deexcited atoms and the number n_s of spontaneous photons. Amplification of these photons is neglected in this approach, but can be included later. Let γ_{urad} be the rate at which the upper state deexcites through the particular transitions $|u\rangle \Rightarrow |l\rangle$. This theory is limited to the cases where γ_{urad} is not modified by the cavity properties [21]. One has $\gamma_{\mathrm{urad}} < \gamma_u$.

The nonsaturated population term of the upper state for the velocity v is

$$\sum_m {}^0\rho_{umum}(v) = (2j_u+1)\frac{N_u}{\gamma_u}\frac{dv}{\sqrt{\pi}v_m}e^{-v^2/v_m^2} \simeq 3N_0(v)dv \tag{4.1}$$

(with $j_u = 1$), when pumping terms onto the lower level are neglected. The saturation term can be written

$$\sum_m {}^2\rho_{um,um} = 20i\frac{N_0(v)dv|\mu_{lu}|^2}{2\hbar^2\gamma_u}\mathcal{I}. \tag{4.2}$$

Using the Wigner-Eckart theorem [Eq. (3.9)] indicated above, one finds:

$$S_0 = \sum_m |\mu_{lm,um+1}|^2 = \sum_m |\mu_{lm,um-1}|^2 = 10|\mu_{lu}|^2. \tag{4.3}$$

The saturated population term for a class of atoms having a velocity v can be written explicitly,

$$\rho_{uu}(v) = \frac{N_u}{\gamma_u}\frac{dv}{\sqrt{\pi}v_m}e^{-v^2/v_m^2}\left\{3 + 20i\frac{|\mu_{lu}|^2}{2\hbar^2\gamma_u}\sum_q I_q \times\left[\frac{1}{{}_q\Delta_f} + \frac{1}{{}_q\Delta_b} - \frac{1}{{}_q\Delta_f^*} - \frac{1}{{}_q\Delta_b^*}\right]\right\}. \tag{4.4}$$

Now, the number $\mathcal{N}(v)$ of atoms which deexcite spontaneously during the time δt and in the volume $\delta\mathcal{V}$, for this class of velocity, is

$$\mathcal{N}(v) = \gamma_{\mathrm{urad}}\delta t\rho_{uu}(v)\delta\mathcal{V}. \tag{4.5}$$

This number is also that of the spontaneous photons emitted by this category of atoms. They are randomly emitted in space and in frequency, and the field associated to them has a random phase Φ. Half of them are emitted with the linear polarization of interest. Moreover, we are interested here only in these photons which are emitted inside the solid angle $\Omega = 2\pi(1-\cos\theta)$ of the laser mode. Here $\theta = \arctan\lambda/\pi W_0$, where W_0 is the radius of the beam waist. The ratio $\Omega/4\pi$ is often called the spontaneous-emission coupling factor β in the literature. Now the probability P_{ω_n} for a photon to be emitted at frequency $\omega_n \pm kv$ in the laboratory reference frame is

$$P_{\omega_n} = \frac{2\gamma_{ab}d\nu}{[\omega_n \pm kv - \omega_0]^2 + \gamma_{ab}^2}, \tag{4.6}$$

with

$$\int_{-\infty}^{\infty}\frac{2\gamma_{ab}d\nu}{[\omega_n \pm kv - \omega_0]^2 + \gamma_{ab}^2} = 1. \tag{4.7}$$

Finally, the number of photons emitted at frequency ω_n in the solid angle Ω with the x polarization during time δt and volume $\mathcal{V}$ and integrated over Φ is

$$N = \frac{1}{2}\frac{1}{2}\frac{\Omega}{4\pi}\gamma_{\mathrm{urad}}\delta t\mathcal{V}\int_{\mathrm{velocities}}\rho_{uu}(v)\left[\frac{\gamma_{ab}d\nu}{(\omega_n + kv - \omega_0)^2 + \gamma_{ab}^2} + \frac{\gamma_{ab}d\nu}{(\omega_n - kv - \omega_0)^2 + \gamma_{ab}^2}\right]. \tag{4.8}$$

The first factor $\frac{1}{2}$ comes from integration over Φ as explained above, and the second from the polarization. After integration over velocities (see Appendix D), one obtains

$$N = \frac{1}{8}\frac{\Omega}{4\pi}\gamma_{\mathrm{urad}}\delta t\mathcal{V}\frac{N_u}{\gamma_u}d\nu\left\{\frac{12}{kv_m}Z_x^i - 20\frac{|\mu_{lu}|^2}{\hbar^2\gamma_u}\frac{1}{k^2v_m^2}Z_0^iV_x\sum_q I_q\right\}, \tag{4.9}$$

where V_x is the pure (positive) real function

$$V_x = \frac{2}{Z_0^i}\Re\left\{-\frac{Z_x - Z_r}{x - X_r} + \frac{Z_x + Z_r}{x + X_r + 2i\mathcal{Y}}\right\}. \tag{4.10}$$

Here Z is the plasma dispersion function defined at the central laser resonance frequency ω_r (Z_r) or at the test frequency x (Z_x). At line center ω_0, one has $Z_0 = iZ_0^i$, and Z_0^i is used as a normalization factor. The function is sometimes denoted Z_ξ, with $\xi = x + i\mathcal{Y}$. x is the reduced frequency normalized with respect to kv_m, half the Doppler width. $\mathcal{Y}$ is the ratio of the homogeneous to the inhomogeneous linewidths. The function V_x is always positive.

The intensity S of the source field which we are looking for is related to the power carried by these spontaneous photons through Poynting's theorem

$$\frac{\epsilon_0 c}{2}\pi W_0^2 S = \frac{Nh\nu}{\delta t}, \tag{4.11}$$

where W_0 is the beam radius. If we write $\delta t = 2l/c$ and introduce the emitting volume $\mathcal{V} \simeq \pi l W_0^2$, one obtains the familiar formula

$$S=\frac{Nh\nu}{\epsilon_0\mathcal{V}} \tag{4.12}$$

The final formula for the source field for the normalized frequency x is

$$S=\frac{h\nu}{\epsilon_0}\frac{\Omega}{2\pi}\gamma_{\rm urad}\delta t\frac{N_u}{\gamma_u}d\nu\left\{\frac{12}{kv_m}Z_x^i - 20\frac{|\mu_{lu}|^2}{\hbar^2\gamma_u}\frac{1}{k^2v_m^2}Z_0^iV_x0.43\sum_q I_q\right\}, \tag{4.13}$$

where we have introduced the geometrical factor 0.43 which reduces the saturation with respect to that given by a plane wave (see Appendix B). We will use this formula in a compressed form

$$S=dxC_1[z_x^i-C_2V_xI_T]. \tag{4.14}$$

z_x^i is the imaginary part of the plasma dispersion function normalized by Z_0^i. C_1 and C_2 are defined by

$$C_1=\frac{3h\nu}{\pi\epsilon_0}\frac{\Omega}{\pi}\gamma_{\rm urad}\delta t\frac{N_u}{\gamma_u}Z_0^i, \tag{4.15}$$

$$C_2=\frac{5}{3}\frac{|\mu_{lu}|^2}{\hbar^2\gamma_u}\frac{1}{kv_m}0.43. \tag{4.16}$$

The total intensity

$$I_T=\sum_q I_q \tag{4.17}$$

has been introduced, and dx refers to a variation of the normalized frequency,

$$dx=\frac{d(\omega-\omega_0)}{kv_m}. \tag{4.18}$$

V. CALCULATION OF THE AMPLIFICATION TERM

The polarization of the medium is obtained from the relation

$$\vec{P}_{({\rm real})fn}={\rm Tr}\{\rho\vec{\mu}\} \tag{5.1}$$

for a component at frequency ω_n. The complex component polarized along x for the forward wave can be written

$$P_{fn}=\frac{1}{\sqrt{2}}\sum_m [p_{fn(lm,um+1)}\mu_{um+1,lm} + p_{fn(lm,um-1)}\mu_{um-1,lm}]. \tag{5.2}$$

When using the matrix elements which have been previously calculated, one obtains

$$P_{fn}=-\mathcal{E}_n\frac{2S_0|\mu_{l,u}|^2}{\hbar}\int_{-\infty}^{\infty}\frac{N_0(v)dv}{{}_n\Delta_f^*}-i\mathcal{E}_n\frac{|\mu_{l,u}|^4}{4\hbar^3}\left[\frac{2(S_2+S_1)}{\gamma_u}+\frac{2(S_1+S_3)}{\gamma_l}+\frac{2S_3}{\gamma_{2u}}+\frac{2S_2}{\gamma_{2l}}\right]\int_{-\infty}^{\infty}\frac{N_0(v)dv}{{}_n\Delta_f^*}\mathcal{I}. \tag{5.3}$$

The sums S_0,S_1,S_2, and S_3 are again obtained from Eq. (3.9) applied to the transition $j\Rightarrow j+1$. One finds

$$\sum_m |\mu_{lmum+1}|^4=46|\mu_{lu}|^4, \tag{5.4}$$

$$\sum_m |\mu_{lm,um+1}|^2|\mu_{um+1,lm+2}|^2=|\mu_{lu}|^4, \tag{5.5}$$

$$\sum_m |\mu_{lmum-1}|^2|\mu_{lmum+1}|^2=21|\mu_{lu}|^4, \tag{5.6}$$

which give $S_1=46, S_2=1$, and $S_3=21$. It remains to integrate over the velocity which appears in $N_0(v)dv$ and in the denominators of $\mathcal{I}$. This is easily done and the result appears as a combination of plasma dispersion functions as in the preceding paragraph. One has

$$\int_{-\infty}^{\infty}\frac{N_0(v)dv}{{}_n\Delta_f^*}=\frac{\Lambda_0}{(kv_m)^2}Z_x^*, \tag{5.7}$$

$$\int_{-\infty}^{\infty}\frac{N_0(v)dv}{{}_n\Delta_f^*}\mathcal{I}=\frac{\Lambda_0}{(kv_m)^2}Z_0^iW_x\sum_q I_q, \tag{5.8}$$

where we have introduced the complex function

$$W_x=\frac{1}{Z_0^i}\left\{\frac{Z_x^*-Z_r}{\xi_x^*-\xi_r}-\frac{Z_x^*+Z_r}{\xi_x^*+\xi_r}-\frac{Z_x^*-Z_r^*}{\xi_x^*-\xi_r^*}+\frac{Z_x^*+Z_r^*}{\xi_x^*+\xi_r^*}\right\}. \tag{5.9}$$

Here Z_r is the plasma dispersion function taken at the central laser frequency; ξ_r is also the value of ξ_x for this frequency. Finally, for the saturated polarizability which relates the field and the polarization $(P_x=\alpha E_x)$ at frequency x, one writes

$$\alpha=-\frac{20|\mu_{l,u}|^2}{\hbar}\frac{\Lambda_0}{kv_m}Z_x^*-i\frac{|\mu_{l,u}|^4}{4\hbar^3}\left[\frac{94}{\gamma_u}+\frac{134}{\gamma_l}+\frac{42}{\gamma_{2u}}+\frac{2}{\gamma_{2l}}\right]\times\frac{\Lambda_0}{(kv_m)^2}Z_0^iW_x0.43I_T. \tag{5.10}$$

This can be related to the real and imaginary parts of the polarizability, which will appear in the Airy function

$$\alpha^i=\frac{\mu_{l,u}^2}{\hbar}\frac{\Lambda_0}{kv_m}Z_x^i, \tag{5.11a}$$

$$\alpha^r=-\frac{\mu_{l,u}^2}{\hbar}\frac{\Lambda_0}{kv_m}Z_x^r, \tag{5.11b}$$

$$\beta^i=\frac{|\mu_{l,u}|^4}{4\hbar^3}\left[\frac{94}{\gamma_u}+\frac{134}{\gamma_l}+\frac{42}{\gamma_{2u}}+\frac{2}{\gamma_{2l}}\right]\frac{\Lambda_0}{(kv_m)^2}Z_0^iW_x^r0.43, \tag{5.12a}$$

$$\beta^r=-\frac{|\mu_{l,u}|^4}{4\hbar^3}\left[\frac{94}{\gamma_u}+\frac{134}{\gamma_l}+\frac{42}{\gamma_{2u}}+\frac{2}{\gamma_{2l}}\right]\frac{\Lambda_0}{(kv_m)^2}Z_0^iW_x^i0.43. \tag{5.12b}$$

VI. EXPRESSION OF THE LASER AIRY FUNCTION AND NORMALIZATION

Using the expressions for the source term and for the polarizability, one obtains the Airy function for the laser,

$$I_x dx=\frac{dx C_1[z_x^i-C_2V_xI_T]}{[1-e^{-L}e^{\omega l({}^0\alpha^i-\beta^iI_T)/\epsilon_0c}]^2+4e^{-L+\omega l({}^0\alpha^i-\beta^iI_T)/\epsilon_0c}\sin^2\{\omega l[1+({}^0\alpha^r-\beta^rI_T)/2\epsilon_0]/c\}},\tag{6.1}$$

where we have written $I_x dx$ instead of I as in Eq. (2.8). $I(x)$ corresponds to the spectral density. I_T can thus be understood as a sum over $I(x)$,

$$I_T=\int I(x)dx.\tag{6.2}$$

We have now obtained everything needed to compute the line shape. However, we will write the formulas in a normalized form, more suitable for this computation. The gain can be written

$$\frac{\omega l({}^0\alpha^i-\beta^iI_T)}{\epsilon_0c}=G\left[z_x^i-W_x^r\frac{I_T}{I_n}\right],\tag{6.3}$$

with

$$G=\frac{20\omega l\Lambda_0Z_0^i}{\epsilon_0c\hbar kv_m}|\mu_{l,u}|^2\tag{6.4}$$

and

$$\frac{1}{I_n}=0.0215\frac{|\mu_{l,u}|^2}{4\hbar^2kv_m}\left[\frac{94}{\gamma_u}+\frac{134}{\gamma_l}+\frac{42}{\gamma_{2u}}+\frac{2}{\gamma_{2l}}\right].\tag{6.5}$$

G has no dimension, and I_n is a normalization factor for the intensity. We thus define

$$y=\frac{I_x}{I_n}\tag{6.6}$$

and

$$Y=\frac{I_T}{I_n}=\int_{-\infty}^{\infty}y\,dx.\tag{6.7}$$

Now the gain term appearing in the exponential in Eq. (6.1) can be written in the following form:

$$-L+G[z_n^i-W_n^rY]=L[r(z_n^i-W_n^rY)-1],\tag{6.8}$$

where we have introduced the ratio

$$r=\frac{N_u}{N_{\rm uth}}=\frac{G}{G_t},\tag{6.9}$$

where $N_{\rm uth}$ is the value of N_u at threshold. As usual, the threshold is defined when the gain G compensates exactly for losses at line center when there is no saturation. At this point G becomes $G_t=L$. r is the relative gain, as traditionally defined. One can thus compute $N_{\rm uth}$ from Eq. (6.4) for $G=G_t$,

$$N_{\rm uth}=\frac{L\epsilon_0c\hbar kv_m\gamma_u}{20\omega lZ_0^i|\mu_{l,u}|^2}.\tag{6.10}$$

$N_{\rm uth}$ represents the pumping term, i.e., the number of atoms arriving in state $|u\rangle$ per unit volume and unit of time at threshold.

Now the phase ϕ at frequency ω in Eq. (6.1) can be written

$$\phi=-\frac{\omega}{c}l[1+({}^0\alpha_\omega^r-\beta_\omega^rI_T)/2\epsilon_0]=-\frac{\omega l}{c}+rL(z_\omega^r-W_\omega^iY).\tag{6.11}$$

The resonance is attained for the angular frequency ω_r such that $2\phi(\omega_r)=N2\pi$. Thus one can write the phase

$$\phi\equiv-(\omega-\omega_r)\frac{l}{c}+rL[z^r(\omega)-z^r(\omega_r)-(W^i(\omega)-W^i(\omega_r))Y].\tag{6.12}$$

One sees that ϕ includes the intensity in a natural way. The phase amplitude coupling factor belongs thus to the theory: this effect is very small in the case of gas lasers. Around and above threshold, the line is very narrow, as we will see in Sec. VII, and ω is very close to ω_r, which allows us to expand $z^r(\omega)$ and $W^i(\omega)$ to first order around ω_r. We thus obtain

$$\phi\simeq-(\omega-\omega_r)\left\{\frac{l}{c}-\frac{rL}{kv_m}[z'^r(\omega_r)-W'^i(\omega_r)Y]\right\},\tag{6.13}$$

where the prime indicates the derivative with respect to ω/kv_m. One obtains

$$\phi\simeq-(x-x_r)\mathcal{A},\tag{6.14}$$

where $x:=(\omega-\omega_0)/kv_m$ is the normalized frequency detuning as defined above. x_r corresponds to ω_r. $\mathcal{A}$ is such that

$$\mathcal{A}:=kv_m\frac{l}{c}-rL[z'^r(\omega_r)-W'^i(\omega_r)Y].\tag{6.15}$$

When the influence of the second term inside the brackets is neglected, $\mathcal{A}$ reduces to the constant $U:=kv_ml/c$ which is the ratio of the Doppler width to the cavity free spectral range.

Equation (61) in normalized form thus becomes

$$y=\frac{rD_1[z_x^i-D_2V_xY]}{[1-e^{L[r(z_x^i-W_x^rY)-1]}]^2+4e^{L[r(z_x^i-W_x^rY)-1]}\sin^2((x-x_r)\mathcal{A})},\tag{6.16}$$

with

$$D_1=\frac{C_1}{I_n}=\frac{0.00645}{4\pi^2}\Omega\,\gamma_{\rm urad}L\left[\frac{94}{\gamma_u}+\frac{134}{\gamma_l}+\frac{42}{\gamma_{2u}}+\frac{2}{\gamma_{2l}}\right] \tag{6.17}$$

and

$$D_2=\frac{400}{3}\,\frac{1}{[94+134\gamma_u/\gamma_l+42\gamma_u/\gamma_{2u}+2\,\gamma_u/\gamma_{2l}]}. \tag{6.18}$$

One sees that Eq. (6.16) contains in a synthetic way the three basic effects mentioned in Sec. I: when the losses L are very large, the effect of the cavity disappears, which is expressed in Eq. (6.16) by the exponential which tends toward zero. There remains the numerator which represents the line shape of the source. When the medium is not saturated $(Y\sim 0)$, we recover the usual Airy function. Finally one can see that the physics of optical stability is also included in Eq. (6.16).

Depending on the value of r, Eq. (6.16) can or cannot be simplified. Below threshold where Y is negligible, Eq. (6.16) gives the usual Airy function. Above threshold, the variation of y is essentially given by the denominator. Around and above threshold, the variation of $\phi=\mathcal{A}(x-x_r)$ is very small inside the laser line profile. Thus the expansion $\sin\phi\simeq(x_r-x)\mathcal{A}$ is allowed, which shows that the Airy function becomes a Lorentzian-like function

$$y=\frac{rD_1[z_x^i-D_2V_xY]}{4e^{L[r(z_x^i-W_x^rY)-1]}\mathcal{A}^2}\,\frac{1}{\Gamma^2+(x-x_r)^2}. \tag{6.19}$$

The first fraction does not depend sensibly upon x, the normalized frequency: the spectral dependence of z^r, z^i, V, W^r, and W^i can be safely removed because of the narrowness of the laser line. These quantities are thus evaluated at the central laser frequency (this is labeled by the index r). Γ can thus be understood as the normalized half-width at half maximum,

$$\Gamma=\frac{1-e^{L[r(z_r^i-W_r^rY)-1]}}{2e^{L/2[r(z_r^i-W_r^rY)-1]}\mathcal{A}}. \tag{6.20}$$

Integrating Eq. (6.16) over x [always with $\sin\phi\simeq\mathcal{A}(x_r-x)$] gives an equation for the total normalized intensity Y:

$$Y=\frac{\pi rD_1[z_r^i-D_2V_rY]}{2e^{L/2[r(z_r^i-W_r^rY)-1]}\mathcal{A}[1-e^{L[r(z_r^i-W_r^rY)-1]}]}. \tag{6.21}$$

Since the gain does not exceed losses apreciably, even around threshold, the exponential can be expanded to first order and one obtains the third order equation

$$2Y\{-L[r(z_r^i-W_r^rY)-1]\}\{U-rL(z'^r_r-W'^i_rY)\}$$
$$-\pi rD_1[z_r^i-D_2V_rY]=0. \tag{6.22}$$

This equation is different from Eq. (2.10) in two respects: the dispersion does not appear in Eq. (2.10), and the net gain is not squared in Eq. (6.22). In order to obtain an approximate expression for Γ which can be compared to standard expressions for the linewidth, one can make the same approximations as those made to obtain Eq. (2.13), i.e., one develops

$$Y=Y_L+\delta Y \tag{6.23}$$

around Y_L defined by Eq. (2.11), or

$$Y_L=\frac{rz_r^i-1}{rW_r^r}. \tag{6.24}$$

This approximation can be introduced in Eq. (6.22), which gives

$$\delta Y=\frac{\pi rD_1[z_r^i-D_2V_rY_L]}{\{U-rL(z'^r_r-W'^i_rY_L)\}2Y_LLrW_r^r}, \tag{6.25}$$

where we have replaced Y by Y_L in the source term and in the dispersion term, and neglected δY as compared to Y_L when necessary. One can now write an approximate expression for the linewidth from Eq. (6.20),

$$\Gamma=\pi rD_1[z_r^i-D_2V_rY_L]\frac{1}{4[U-rL(z'^r_r-W'^i_rY_L)]^2}\,\frac{1}{Y_L}. \tag{6.26}$$

The first factor contains the effect of the amplitude and the saturated line shape of the source, while the second describes a mixed effect of the medium and cavity field (including the phase-amplitude coupling), and the third is the well-known reducing factor inversely proportional to the intensity Y_L. Note that this expression has been obtained with the simplifying assumption that the pumping of the lower level is negligible as compared to that of the upper level $(N_u/\gamma_u\gg N_l/\gamma_l)$. If this is not the case, one obtains more complicated expressions for the symbols D_1 and D_2, and one recovers a factor close to what is usually called $n_{\rm sp}$, the spontaneous-emission factor [7]. This is developed in Appendix E.

Let us now compare Eq. (6.26) with the expression given by Schawlow and Townes [1] for the laser linewidth,

$$\Delta\nu=\frac{h\nu}{4\pi}\Gamma_0^2\frac{1}{P_{\rm out}}, \tag{6.27}$$

where $\Gamma_0=(c/2d)\ln(r_1r_2)$ is the empty cavity loss rate at resonance, and $P_{\rm out}$ the total output power through both mirrors. One sees that both Eqs. (6.26) and (6.27) have a similar structure, with the difference that the source and the phase terms are expressed more precisely in Eq. (6.26).

The laser line shape obeys formula Eq. (6.1), which is very close to the Lorentzian expressed by Eq. (6.16) even below threshold. We have numerically computed the intensity and the linewidth using these formulas, and the data are given in Table I for the case of the Ne line at 3.39 μm. The laser central frequency ω_r is taken at line center ω_0. The dipole moment is obtained from Fermi's golden rule,

$$\mu_{ul}^2=\tfrac{1}{10}\epsilon_0\hbar\lambda^3\gamma_{\rm urad}/8\pi^2. \tag{6.28}$$

Figure 1 displays the variation of intensity as computed from Eq. (6.21), when the gain is increased from below [Fig. 1(a)],

TABLE I. Data used in the numerical calculations.

Mirror reflectances:	$r_1 = r_2 = 0.8$
Cavity length:	0.3 m
Wavelength:	$3.3913 \mu m$
kv_m:	$2\pi 15 \times 10^7$ rad/s
Homogeneous linewidth:	$\gamma_{ul} = 75 \times 10^6$ Hz
Radiative deexcitation rate of the upper level for the 3.39-μm radiation:	$\gamma_{\text{urad}} = 2.87 \times 10^6$ Hz
Deexcitation rate of the upper level:	$\gamma_u = 18.04 \times 10^6$ Hz
Deexcitation rate of the lower level:	$\gamma_l = 10^8$ Hz
Deexcitation rate of Zeeman coherences:	$\gamma_{2u} = \gamma_{2l} = 3 \times 10^7$ Hz
Solid angle of the laser mode:	$\Omega/\pi = 10^{-6}$

to around [Fig. 1(b)] and above [Fig. 1(c)] threshold. Figure 2 shows the variation of the linewidth as a function of the gain as calculated from Eq. (6.20). The laser transition is clearly seen on both figures: the linewidth, which is 9.7 MHz for $r = 0.5$, decreases toward 34 kHz at $r = 1$ and down to 192 Hz for $r = 1.2$. The same calculation, made with $r_1 = r_2 = 0.9$, gives $\Gamma' s = 3.8$ MHz, 11.2 KHz, and 44 Hz instead for the same three values of r. It is interesting to note that simple equations like Eq. (6.20) or (6.21) are able to describe variations of intensity or linewidth on several orders of magnitude, which corresponds effectively to what happens in a laser.

Up to this point, and while the obtained numbers are of the correct order of magnitude [7,22,23], we have not yet tried to make any precise quantitative comparison with experimental results, because we have focused on a description of the laser transition while measurements have mainly been directed toward finding the ultimate width limit at high intensity. It is clear that such a comparison would need another calculation in which the saturation term is not of perturbative nature; i.e., we need an extension to the high-intensity limit. However, as the present theory does not contain any nonmeasurable fitting parameter, we hope to be able to experimentally verify Eq. (6.20) quantitatively.

VII. CONCLUSION

In this study we have essentially adapted the Airy function of the passive Fabry-Pérot cavity to the laser with an explicit numerical calculation performed on the single-mode He-Ne laser at 3.39 μm in the context of the weak, mean-field approximation. This function is usually obtained using a monochromatic external source, and performing a calculation with the concept of multiple path interferometry. The latter method is not mandatory: one has to abandon it sometimes, for instance in the study of multiple thin films. In the

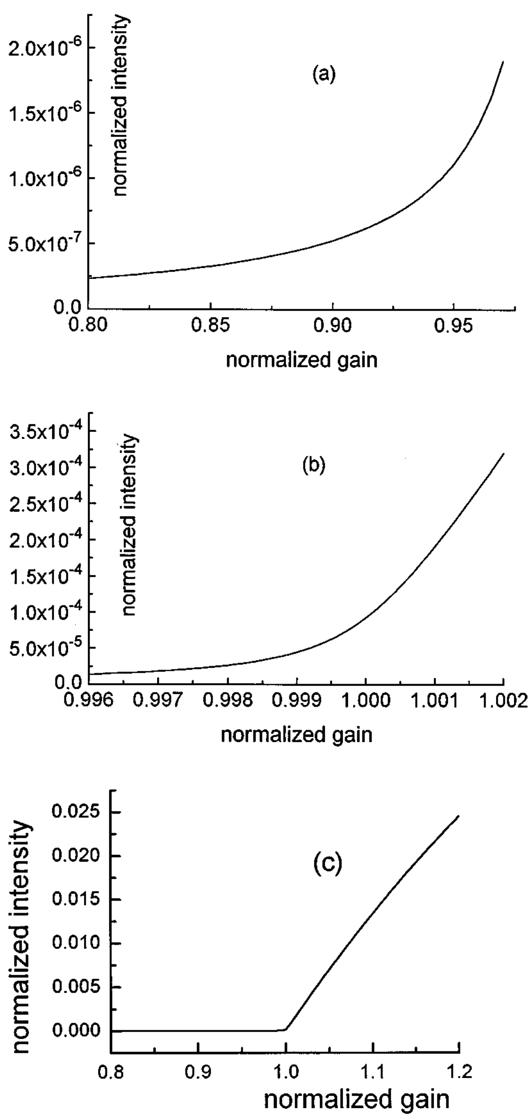

FIG. 1. Laser characteristic curve as computed from Eq. (6.21) and data in Table I (a) below, (b) around, and (c) above threshold.

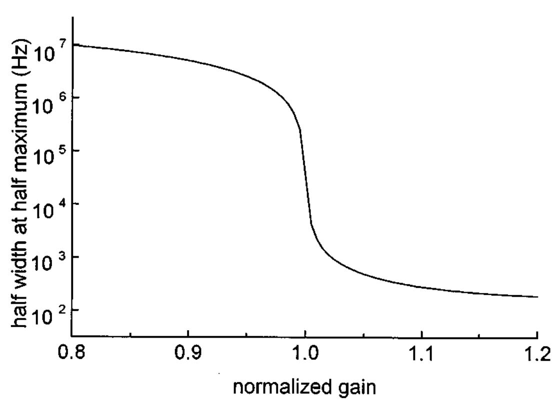

FIG. 2. Evolution of linewidth in Hz as a function of gain normalized at threshold. The scale is logarithmic.

case of a laser, the hypothesis of a monochromatic source has to be relaxed as well. However, as soon as we have the concept of a spontaneous source whose intensity is small and spectrum wide, one is naturally led to adopt the generalized Airy function in the frequency domain. Then it is important to make a precise energy balance between the number of atoms which deexcite spontaneously and the number of spontaneous photons. The projection of the associated spontaneous field onto the laser mode gives the source of the laser field. Thus, in our theory, the spontaneous emission is not simply a secondary phenomenon which is only responsible for the laser linewidth; it is a fundamental effect without which the laser would not exist.

The generalized Airy function gives a simple and powerful description of the laser. It shows in a continuous way how a wide, weak spectrum transforms itself into a narrow, intense line. This transformation begins well below threshold. The synthetic formulation of the laser static behavior through this function includes the spontaneous source, the stimulated emission, and the resonant cavity. The same formula [i.e., Eq. (6.16)] simultaneously describes the line shape of the source (with or without inversion), which is a Voigt function here, its amplification (or absorption) properties and the effect of the resonant cavity. The methods given here can be extended to describe the transformations of statistical properties of the source into those of the laser light. Many quantitative results can still be obtained from the present theory, which is the first of a series in which the laser is considered as an active Fabry-Pérot cavity. For instance, we have given a method [24] to study coupled lasers which is based on this interpretation. The link between this study and quantum treatments can be made through an equation for the field operators which includes the properties of the resonant cavity as described here and a detailed calculation of the Langevin forces (in the frequency domain) which usually represent the spontaneous emission phenomenologically. Finally, we stress again that, together with the description of the laser transition, the formula given here offers a complementary (and more precise) interpretation of the laser linewidth, based on a characteristic transfer function of the Fabry-Pérot cavity in the frequency domain.

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APPENDIX A: ROUND-TRIP METHOD

In this method, Maxwell equations for the field are solved at a given point of the laser. This allows us to write the forward-propagating field $\mathcal{E}_{f2}$ on one mirror as a function of the field $\mathcal{E}_{f1}$ on the other mirror. For instance,

$$\mathcal{E}_{f2}(d)=\mathcal{E}_{f1}(0)\exp\left[-\int_0^d k(z)dz\right]. \tag{A1}$$

When the cavity is empty, $k(z)$ obeys the usual dispersion relation of vacuum, i.e., $k=\omega/c$. In the case of a saturated medium, $k(z)$ is a complicated nonlinear function which depends upon the local intensities of both counterpropagating fields. We have simplified this problem by taking the mean value of the integral in Eq. (A1). This is the essence of the mean-field approximation. Then the two counterpropagating fields are related to each other from the boundary conditions on the mirrors. For instance, the backward and the forward fields are related by

$$\mathcal{E}_{f1}(0)=r_1\mathcal{E}_{b1}(0) \tag{A2}$$

on mirror number 1 having a reflectivity r_1 . This can be used to obtain an equation of evolution for the field.

Let us call $\mathcal{E}_A(t)$ the slowly varying, complex amplitude of the field centered at the angular frequency ω, at point A inside the cavity, and at time t. Let Δt be the duration of a round trip: $\Delta t=2l/c$. The field $\mathcal{E}_A(t+\Delta t)$ at point A and time $t+\Delta t$ results from the superposition of (i) $\mathcal{E}_A(t)$ after a round trip and (ii) the source field $\mathcal{S}$:

$$\mathcal{E}_A(t+\Delta t)=\mathcal{E}_A(t)r_1r_2e^{i\phi}+\mathcal{S}, \tag{A3}$$

with $\phi=-\omega/c2l$. The difference equation follows:

$$\frac{1}{\Delta t}[\mathcal{E}_A(t+\Delta t)-\mathcal{E}_A(t)]=-\frac{c}{2l}[1-r_1r_2e^{i\phi}]\mathcal{E}_A(t)+\frac{c}{2l}\mathcal{S}. \tag{A4}$$

Now, if one considers only phenomena which vary on *time scales much larger* than $2l/c$, this equation can be transformed into the differential equation

$$\frac{d\mathcal{E}}{dt}=-\frac{c}{2l}[1-e^{-L}e^{i\phi}]\mathcal{E}+\frac{c}{2l}\mathcal{S}, \tag{A5}$$

where we have introduced the losses L in an exponential form for later convenience: $r_1r_2=e^{-L}$. This equation gives the stationary regime

$$\mathcal{E}=\frac{\mathcal{S}}{1-r_1r_2e^{i\phi}}. \tag{A6}$$

APPENDIX B: GEOMETRICAL SATURATION FACTOR FOR A GAUSSIAN BEAM

The laser field is never a plane wave. Its intensity is in general better represented by a bell-shaped function centered along the laser axis z at $r=0$, r being the transverse coordinate. The saturation is thus lower than that of the plane wave having the same intensity at $r=0$. Let us recall some properties of the Gaussian beam in an amplifying medium [17,18]. The complex field is represented by the expression:

$$E_f(t,r,z)=\mathcal{E}_0e^{i(\omega t-kz)}e^{-iP_f}e^{-ikr^2/2q_f}, \tag{B1a}$$

$$E_b(t,r,z)=\mathcal{E}_0e^{i(\omega t+kz)}e^{-iP_b}e^{ikr^2/2q_b}, \tag{B1b}$$

respectively, for the forward and the backward beams. Here: $k=\omega/c(1+\alpha/2\epsilon_0)$. The complex functions P_f, P_b, q_f, and q_b are those of the cavity [19] modified by the inhomogeneities of the medium. They can be written

$$\frac{1}{q_f}=\frac{1}{{}^0q_f}+\epsilon_{qf}, \tag{B2}$$

$$P_f={}^0P_f+\epsilon_{pf}, \tag{B3}$$

$$\frac{1}{q_b}=\frac{1}{{}^0q_b}+\epsilon_{qb}, \tag{B4}$$

$$P_b={}^0P_b+\epsilon_{pb}. \tag{B5}$$

ϵ_{qf}, ϵ_{qb}, ϵ_{pf}, and ϵ_{pb} are perturbations with respect to the quantities ${}^0P_f$, ${}^0P_b$, ${}^0q_f$, and ${}^0q_b$ which characterize the cavity with a linear medium. These perturbations are proportional to the intensities of the fields, and bring a relative correction of the order of 10^{-2} to the beam parameters of a gas laser, i.e., to its radius of curvature and its diameter. They explain asymmetries [18] in the gas laser line shape vs frequency. However, these small transverse effects will be neglected in the following. For our purpose here, the main result of this theory concerns the saturation term: when the transverse Gaussian distribution of the beam is approximated by a parabola, the saturating intensity $I=\mathcal{E}_0\mathcal{E}_0^*$ which appears in the polarizability, is reduced by a factor of 0.43 with respect to the plane-wave case (i.e., I is replaced by $0.43I$). We will later include this factor 0.43 in the saturation term.

APPENDIX C: MIXED TIME-FREQUENCY DOMAIN

In a complex form, the field $E(t)$ is connected to his frequency components by

$$E_c(t)=\int_0^\infty E(\nu)e^{2i\pi\nu t}d\nu. \tag{C1}$$

In the usual case of a single-mode laser, one writes the expansion in the useful interval $\Delta\nu$ around the central frequency ν_0,

$$E_c(t)=e^{2i\pi\nu_0 t}\int_{\Delta\nu}E(\nu)e^{2i\pi(\nu-\nu_0)t}d\nu=\mathcal{E}(t)e^{2i\pi\nu_0 t}, \tag{C2}$$

where the slowly varying amplitude $\mathcal{E}(t)$ is introduced:

$$\mathcal{E}(t)=\int_{\Delta\nu}E(\nu)e^{2i\pi(\nu-\nu_0)t}d\nu. \tag{C3}$$

Here we do not care about supplementary resonances such as population resonances observable in class-B lasers. In usual laser line-shape theories $\mathcal{E}(t)$ is the central quantity which is studied. In our case, we are interested in applying the properties of the Fabry-Pérot cavity in the same way they are usually applied in the case of an empty cavity, i.e., in a mixed time-frequency domain where a frequency component of the field is well identified. Thus we will divide the spectral line into intervals, the width of which are $\delta\nu$ and centered around frequencies ν_q:

$$E_c(t)=\sum_q e^{2i\pi\nu_q t}\int_{\nu_q-\delta\nu/2}^{\nu_q+\delta\nu/2}E(\nu)e^{2i\pi(\nu-\nu_q)t}d\nu$$
$$=\sum_q \mathcal{E}_q(t)e^{2i\pi\nu_q t}, \tag{C4}$$

with the amplitude for each component,

$$\mathcal{E}_q(t)=\int_{\nu_q-\delta\nu/2}^{\nu_q+\delta\nu/2}E(\nu)e^{2i\pi(\nu-\nu_q)t}d\nu. \tag{C5}$$

Again, we insist on this aspect of the physics: such an amplitude $\mathcal{E}_q(t)$ varies very slowly in time. Let us take as an example the case of a line whose width is 1 Hz. When it is divided into 100 intervals, the difference is $\nu-\nu_q\leqslant 5.10^{-3}$ Hz, which, to be measured, will need at least 200 s. The mathematical limit to Eq. (C4) is Eq. (C1), in which $E(\nu)$ does not depend on time.

APPENDIX D: VELOCITY INTEGRALS

Integrals over velocities are related to the plasma dispersion functions

$$Z(\xi)=\frac{1}{\sqrt{\pi k v_m}}\int_{-\infty}^{+\infty}du e^{-u^2}\frac{1}{u-\xi}, \tag{D1}$$

with the notations

$$u=\frac{v}{v_m}, \tag{D2}$$

and

$$\xi:=\frac{1}{kv_m}[\omega-\omega_0+i\gamma_{ul}]=x+i\mathcal{Y}, \tag{D3}$$

where x is the detuning expressed in units of half the Doppler width, and $\mathcal{Y}$ the ratio of the homogeneous to the inhomogeneous linewidths.

The integrals appear in the expression of the population as well as in those of the polarization. Let us consider first the following expression in Eq. (4.8):

$$\int_{\text{velocities}}\rho_{uu}(v)\left[\frac{\gamma_{ab}d\nu}{(\omega_n+kv-\omega_0)^2+\gamma_{ab}^2}+\frac{\gamma_{ab}d\nu}{(\omega_n-kv-\omega_0)^2+\gamma_{ab}^2}\right]=-\frac{i}{2}\frac{N_u}{\gamma_u}d\nu\int_{-\infty}^{\infty}\frac{dv}{\sqrt{\pi}v_m}e^{-v^2/v_m^2}$$
$$\times\left[\frac{1}{{}_n\Delta_f}+\frac{1}{{}_n\Delta_b}-\frac{1}{{}_n\Delta_f^*}-\frac{1}{{}_n\Delta_b^*}\right]\left\{3+20i\frac{|\mu_{lu}|^2}{2\hbar^2\gamma_u}\mathcal{I}\right\}. \tag{D4}$$

The first part gives

$$\int_{-\infty}^{\infty}\frac{dv}{\sqrt{\pi}v_m}e^{-v^2/v_m^2}\left[\frac{1}{{}_n\Delta_f}+\frac{1}{{}_n\Delta_b}-\frac{1}{{}_n\Delta_f^*}-\frac{1}{{}_n\Delta_b^*}\right]=\frac{4i}{kv_m}Z^i(\xi). \tag{D5}$$

The second part is a little bit more complicated:

$$\int_{-\infty}^{\infty}\frac{dv}{\sqrt{\pi}v_m}e^{-v^2/v_m^2}\left[\frac{1}{{}_n\Delta_f}+\frac{1}{{}_n\Delta_b}-\frac{1}{{}_n\Delta_f^*}-\frac{1}{{}_n\Delta_b^*}\right]\sum_q I_q\left[\frac{1}{{}_q\Delta_f}+\frac{1}{{}_q\Delta_b}-\frac{1}{{}_q\Delta_f^*}-\frac{1}{{}_q\Delta_b^*}\right]$$
$$=\frac{2}{k^2v_m^2}\sum_q I_q\left[\frac{Z_n-Z_q}{x_n-x_q}+\frac{Z_n^*-Z_q^*}{x_n-x_q}-\frac{Z_n+Z_q}{x_n+x_q+2i\mathcal{Y}}-\frac{Z_n^*+Z_q^*}{x_n+x_q-2i\mathcal{Y}}\right]. \tag{D6}$$

This term becomes important especially above threshold, where the linewidth becomes very small. Thus x_q does not vary very much, and a good approximation is to take x_q invariant and equal to the laser central frequency; i.e., we take $x_q=x_r$. In this case,

$$\int_{\text{velocities}}\rho_{uu}(v)\left[\frac{\gamma_{ab}d\nu}{(\omega_n+kv-\omega_0)^2+\gamma_{ab}^2}+\frac{\gamma_{ab}d\nu}{(\omega_n-kv-\omega_0)^2+\gamma_{ab}^2}\right]=-\frac{i}{2}\frac{N_u}{\gamma_u}d\nu\left\{\frac{12i}{kv_m}Z_x^i-20i\frac{|\mu_{l,u}|^2}{2\hbar^2\gamma_u}\frac{2}{k^2v_m^2}V_x\sum_q I_q\right\}, \tag{D7}$$

with the abbreviation for the real function V_x,

$$V_x=\frac{Z_x-Z_r}{x-x_r}+\frac{Z_x^*-Z_r^*}{x-x_r}-\frac{Z_x+Z_r}{x+x_r+2i\mathcal{Y}}-\frac{Z_x^*+Z_r^*}{x+x_r-2i\mathcal{Y}}. \tag{D8}$$

APPENDIX E: SPONTANEOUS-EMISSION FACTOR

The spontaneous-emission factor [7] N_{sp} is traditionally defined as the ratio

$$N_{\text{sp}}=\frac{N_2}{N_2-N_1}, \tag{E1}$$

where N_2 and N_1 are, respectively, the populations of the upper and lower levels. In the context of this work, one can also define a closely related factor

$$n_{sp}=\frac{N_u}{\gamma_u}\frac{1}{\Lambda_0}, \tag{E2}$$

with

$$\Lambda_0=\frac{N_u}{\gamma_u}-\frac{N_l}{\gamma_l}. \tag{E3}$$

When $(N_l)/\gamma_1$ is negligible as compared to $(N_u)/\gamma_u$, $n_{sp}=1$, which corresponds to the formulas given above. If this is not the case, one finds, instead of Eq. (4.13), another expression for the source term:

$$S=\frac{h\nu}{\epsilon_0}\frac{\Omega}{2\pi}\gamma_{\text{urad}}\delta t\frac{N_u}{\gamma_u}d\nu\left\{\frac{12}{kv_m}Z_x^i-\frac{20}{n_{sp}}\frac{|\mu_{lu}|^2}{\hbar^2\gamma_u}\times\frac{1}{k^2v_m^2}Z_0^iV_x0.43\sum_q I_q\right\}. \tag{E4}$$

Formula (4.16) is replaced by

$$C_2=\frac{5}{3n_{sp}}\frac{|\mu_{lu}|^2}{\hbar^2\gamma_u}\frac{1}{kv_m}0.43. \tag{E5}$$

Instead of Eq. (6.9), one has $r=\Lambda_0/\Lambda_{0\text{th}}$ where $\Lambda_{0\text{th}}$ is the value of Λ_0 at threshold, which is now defined by

$$\Lambda_{0\text{th}}=\frac{L\epsilon_0c\hbar kv_m\gamma_u}{20\omega lZ_0^i|\mu_{l,u}|^2}. \tag{E6}$$

Equation (6.16) thus becomes

$$y=\frac{r_uD_1[z_x^i-D_2V_xY/n_{\text{sp}}]}{[1-e^{L[r(z_x^i-W_x^rY)-1]}]^2+4e^{L[r(z_x^i-W_x^rY)-1]}\sin^2((x-x_r)\mathcal{A})} \tag{E7}$$

with the normalized pumping term for the upper level:

$$r_u:=\frac{N_u}{\gamma_u}\frac{1}{\Lambda_{0\text{th}}}. \tag{E8}$$

From Eq. (6.26), the expression for the linewidth is

$$\Gamma=\pi r_uD_1\left[z_r^i-\frac{D_2}{n_{sp}}V_rY_L\right]\frac{1}{4[U-rL(z_r'^r-W_r'^iY_L)]^2}\frac{1}{Y_L}. \tag{E9}$$

as $r<r_u$, this value leads to a larger linewidth than Eq. (6.26).

[1] A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1940 (1958).
[2] J. T. Verdeyen, in *Laser Electronics*, 3rd ed. (Prentice Hall, Englewood, Cliffs, NJ, 1995), Secs. 8.8 and 9.3.
[3] R. D. Hempstead and M. Lax, Phys. Rev. **161**, 350 (1967).
[4] M. I. Kolobov, L. Davidovitch, E. Giacobino, and C. Fabre, Phys. Rev. A **47**, 1431 (1993).
[5] M. Sargent, M. O. Scully, and W. Lamb, Jr., in *Laser Physics* (Addison-Wesley, Reading, MA, 1974), Vol. 17-3.
[6] C.H. Henry, IEEE J. Quantum Electron. **QE18**, 259 (1982); **QE19**, 1391 (1983).
[7] For a comprehensive recent study of the linewidth in gas lasers, see S. Kuppens, Ph.D. thesis, Leiden University, Huygens Lab., The Netherlands, 1995.
[8] H. Gerhardt, H. Welling, and A. Guttner, Z. Phys. **253**, 113 (1972).
[9] A. Güttner, H. Welling, K. H. Gericke, and W. Seifert, Phys. Rev. **18**, 1157 (1978).
[10] H. Risken and H. D. Volmer, Z. Phys. **201**, 323 (1967); **204**, 240 (1967).
[11] H. Risken and R. Seybold, Phys. Lett. **38A**, 63 (1972).
[12] R. Seybold and H. Risken, Z. Phys. **267**, 323 (1974).
[13] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, New York, 1995).
[14] P. Goldberg, P. W. Milonni, and B. Sundaram, Phys. Rev. **44** 1969 (1991); **44**, 4556 (1991).
[15] K. Ujihara, IEEE J. Quantum Electron. **QE20**, 814 (1984).
[16] S. Prasad, Phys. Rev. **46**, 1540 (1992).
[17] G. Stéphan and H. Taleb, J. Phys. (Paris) **42**, 1623 (1981).
[18] G. Stéphan and M. Trümper, Phys. Rev. A **28**, 2344 (1983); **28**, 3450 (1983); **30**, 1925 (1984).
[19] H. Kogelnik, Appl. Opt. **5**, 1562 (1965).
[20] G. Stéphan, J. Nonlinear Opt. Phys. Mater. **5**, 551 (1996).
[21] P. Goy, J. M. Raimond, M. Gross, and S. Haroche, Phys. Rev. Lett. **50**, 1903 (1983).
[22] S. J. M. Kuppens, M. P. Van Exter, and J. P. Woerdman, Phys. Rev. Lett. **72**, 3815 (1994).
[23] S. J. M. Kuppens, M. P. Van Exter, M. Van Duin, and J. P. Woerdman, IEEE J. Quantum Electron. **31**, 1237 (1995).
[24] P. Even, K. Aït-Ameur, and G. Stéphan, Phys. Rev. A (to be published).