Schrödinger-cat states in Paul traps

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We study the evolution of coherent and Schrödinger cat states in radio-frequency traps, which were used to bound Hg^+ ions. The quantum confinement is determined by evaluating the expectation values and dispersions of the position and momentum operators, together with the probability densities in coordinate and momenta representations. Also the Wigner functions associated to this nonstationary quantum system, a parametric oscillator, are constructed. [S1050-2947(97)07801-3]

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I. INTRODUCTION

Macroscopic quantum superposition of states has been achieved for unbound electrons since the middle 1950s [1]. In the late 1980s and the 1990s this has been also done for atom beam splitters [2]. Superposition of macroscopically distinguishable states called even and odd coherent states was suggested in [3]. Matos Filho and Vogel [4] have shown that this superposition is naturally realized as a steady state for a trapped ion in a bichromatic laser irradiation. More recently, Nieto [5] has studied the even and odd coherent and squeezed states of an ion in a Paul trap regarding possibilities of experimental realization of such states [6]. The measurements of nonclassical states of trapped ions, including even and odd coherent states, were discussed using the endoscopy method [7] and symplectic tomography [8]. There have also been proposals to build macroscopic superpositions of bound states [9] and to create Schrödinger-cat states in a single-mode electromagnetic field [10]. As pointed out by Spiridonov [11], the superposition states considered in Ref. [10] are a particular case of the generalized coherent states introduced by Titulaer and Glauber [12] and Bialynicka-Birula [13], and these states have the same Poisson statistics as the coherent states. This is the important difference between the states discussed in [10] and the ones discussed in [3]. The even and odd Schrödinger cats are more nonclassical states because both quadrature and number operator statistics are different from those in the coherent states. In generalized coherent states [12,13], including the particular Schrödinger cats in [10], only quadrature statistics are changed in comparison with the coherent states, meanwhile the number operator statistics is preserved. Using a Kerr nonlinear medium, the observation of nonclassical states of an atom moving in a Paul trap [14] and the subsequent construction of Schrödinger cats from such an atom have been reported [15]. These results have opened the possibility of testing the theory of quantum measurements in a quite clean way because the dispersion of the wave packet can be manipulated to a high degree. The experimental approach is based on a highly controlled harmonic motion of the trapped atom by exciting the motion from initial zero-point wave packets to coherent state wave packets of a well-defined amplitude and phase. This is acomplished by the entanglement of electronic and motional states of the atoms using a sequence of laser pulses.

In this contribution we are going to talk about the quantum effects in the dynamic stabilization of ions in three-dimensional radio-frequency quadrupole fields [16].

The quantum dynamics of a charged particle in a conventional Paul trap is separable into two independent motions, one of them in the plane X-Y and the other in the Z direction. Both motions are described by a parametric oscillator, that is, a harmonic oscillator with a time-dependent frequency. The study of time-dependent quantum systems is in general complicated, however for quadratic systems in position and momentum, the formalism of the linear time-dependent invariants developed and summarized in [17] can be used. It is also possible to apply algebraic methods to express the evolution operator as an element of the $\mathrm{Sp}(2,\mathbb{R})$ group with time-dependent parameters [18]. This algebraic procedure has been very successful in describing atomic and molecular collisions [19].

In this work we show for general one-dimensional quadratic systems that these two procedures are indeed equivalent. This result is also true for d-dimensional systems, at most quadratic, and in this case we have $2d$ independent linear constants of the motion to construct the Green function. The evolution operator is an element of the semidirect product group of the Weyl and symplectic groups in d dimensions, $W(d)\wedge \mathrm{Sp}(2d,\mathbb{R})$.

We study the quantum behavior of ions that at time $t=0$ are either in a generalized correlated [20], or even and odd Schrödinger-cat states, and their subsequent evolution in a Paul trap. We give general analytic expressions of several observables: the expectation values of the position and momentum operators, the dispersions in the position σ_{qq} and the momentum σ_{pp}, plus a measure of the correlation σ_{pq} between both variables. Also the probability densities in the coordinates and momenta representations, together with the structure of the associated quasiprobabilistic Wigner function [21] are found. These analytic expressions are written in terms of the two independent classical solutions of the parametric oscillator, the Mathieu functions.

Recently a single Hg^+ ion harmonically bound in a radio-frequency trap has been cooled to an extent where it spends most of the time in the ground state [22]. For this reason we

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consider the physical features of the traps used in that study to see the evolution of correlated and Schrödinger-cat states.

The plan of the present paper is as follows. In Sec. II we establish the equivalence between the linear time-dependent constants of motion formalism and the evolution operator procedure to get the solution of the nonstationary Schrödinger equation for general quadratic systems with time-dependent coefficients or strengths. In the Sec. III we determine the analytic expressions for the expectation values and dispersions of the position and momentum operators. The probability densities to find the ion in the position and momentum spaces are calculated, together with the Wigner function associated to the general homogeneous quadratic Hamiltonian. In Sec. IV, the analytic expressions developed in the previous section are used to study the quantum behavior of coherent and Schrödinger-cat states describing Hg^+ ions moving in radio-frequency traps. Finally a summary of the main quantum effects appearing in the evolution of the confined ions is given, emphasizing the differences in the features when the ions are represented by a Gaussian packet with minimum Heisenberg uncertainty with those of even or odd Schrödinger-cat states.

II. EVOLUTION OPERATOR

In this section we apply the time-dependent linear invariants theory to solve the nonstationary Schrödinger equation for the Hamiltonian

$$H=\frac{a(t)}{2}\hat{p}^2+\frac{b(t)}{2}(\hat{p}\hat{q}+\hat{q}\hat{p})+\frac{c(t)}{2}\hat{q}^2, \tag{2.1}$$

where here and in the next section we use the natural units of the system that is $m=\hbar=c(0)=1$.

First of all we construct the constants of motion of the system through the Hamiltonian formulation of Noether's theorem [23,24]. The invariants can be written in the form

$$\hat{p}_0(t)=h_1(t)\hat{p}-\frac{1}{a(t)}[\dot{h}_1(t)-h_1(t)b(t)]\hat{q}, \tag{2.2a}$$

$$\hat{q}_0(t)=h_2(t)\hat{p}-\frac{1}{a(t)}[\dot{h}_2(t)-h_2(t)b(t)]\hat{q}, \tag{2.2b}$$

where $h_k(t)$ with $k=1,2$, denote the independent solutions of the classical equations of motion

$$\ddot{h}_k(t)-\dot{h}_k(t)\left[\frac{\dot{a}(t)}{a(t)}\right]+h_k(t)\left[-\dot{b}(t)+a(t)c(t)-b^2(t)+b(t)\frac{\dot{a}(t)}{a(t)}\right]=0. \tag{2.3}$$

If the constants of motion are chosen to be the position and momentum operators at $t=0$, the solutions of (2.3) are completely determined by the initial conditions

$$h_1(t=0)=1,\dot{h}_1(t=0)=b(0), \tag{2.4a}$$

$$h_2(t=0)=0,\dot{h}_2(t=0)=-a(0). \tag{2.4b}$$

The equations (2.2) can be rewritten in matrix form, and the matrix that connects the position and momentum operators with the constants of the motion is a two-dimensional symplectic matrix. This means that expressions (2.2) are denoting a canonical transformation and therefore the commutation relations of the constants of the motion are identical to those of the position and momentum operators. Another convenient form to express the constants of the motion is in terms of creation and annihilation operators:

$$\begin{pmatrix}\hat{A}(t)\\ \hat{A}^\dagger(t)\end{pmatrix}=\begin{pmatrix}M_1 & M_2\\ M_2^* & M_1^*\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{a}^\dagger\end{pmatrix}, \tag{2.5}$$

where we have defined

$$M_1=\frac{1}{2}\left\{\left(1+i\frac{b}{a}\right)(h_1-ih_2)-\frac{1}{a}(\dot{h}_2+i\dot{h}_1)\right\}, \tag{2.6a}$$

$$M_2=\frac{1}{2}\left\{\left(-1+i\frac{b}{a}\right)(h_1-ih_2)-\frac{1}{a}(\dot{h}_2+i\dot{h}_1)\right\}, \tag{2.6b}$$

and these expressions satisfy the condition $|M_1|^2-|M_2|^2=1$. This property can be proved directly by making the indicated products and using the Wronskian of the differential equations (2.3), $W(h_2,h_1)=h_2\dot{h}_1-\dot{h}_2h_1=a(t)$.

The invariants are very useful because they allow us to define generalized correlated states, which are solutions of the time-dependent Schrödinger equation [23]. This is carried out by solving the differential equation

$$A(t)\Phi_0(q,t)=0, \tag{2.7}$$

where $A(t)$ is written in terms of the position and momentum operators. The solution is given by

$$\Phi_0(q,t)=\frac{1}{(2\pi)^{1/4}\lambda_p^{1/2}}\exp\left\{-\frac{i\lambda_q}{2\lambda_p}q^2\right\} \tag{2.8}$$

with the definitions

$$\lambda_p=\frac{1}{\sqrt{2}}[h_2+ih_1], \tag{2.9a}$$

$$\lambda_q=\frac{1}{a(t)}[b(t)\lambda_p-\dot{\lambda}_p]. \tag{2.9b}$$

The factor in Eq. (2.8) is a function of time, which was fixed by asking that Φ_0 satisfies the time-dependent Schrödinger equation associated to the Hamiltonian (2.1).

The action of a constant of motion, $A^\dagger(t)$ onto $\Phi_0(q,t)$ is also a solution of the nonstationary Schrödinger equation because $\Phi_0(q,t)$ is a solution. Therefore the action of the unitary operator

$$D(\alpha)=\exp(\alpha\hat{A}^\dagger-\alpha^*A), \tag{2.10}$$

onto (2.8) will be a solution and gives rise to the generalized correlated states. It can be rewritten as

$$\psi_\alpha(q,t)=\frac{e^{-|\alpha|^2/2}}{(2\pi)^{1/4}\lambda_p^{1/2}}\exp\left\{-\frac{i}{2\lambda_p}(\lambda_q q^2-2\alpha q+i\alpha^2\lambda_p^*)\right\}. \tag{2.11}$$

In summary the solutions of the nonstationary Schrödinger equation associated to the Hamiltonian (2.1) are obtained explicitly if it is possible to find $(h_1(t),h_2(t))$, the independent solutions of the corresponding classical equations of motion.

According to Refs. [18,19] in the algebraic method the evolution operator of Hamiltonian (2.1) must be an element of the symplectic group in two dimensions. Therefore we propose

$$U(t)=\exp(c_0K_0)\exp(c_-K_+)\exp(c_+K_-), \tag{2.12}$$

where we have defined the operators

$$K_0=\tfrac{1}{2}(a^\dagger a+\tfrac{1}{2}), \tag{2.13a}$$

$$K_+=\tfrac{1}{2}\,a^{\dagger 2}, \tag{2.13b}$$

$$K_-=\tfrac{1}{2}a^2, \tag{2.13c}$$

which have the commutation relations of a symplectic algebra in two dimensions [25].

Substituting the expression (2.12) into the differential equation for the time evolution operator, we get the following set of first-order coupled differential equations:

$$i\left(\frac{dc_0}{dt}-2c_-\frac{dc_+}{dt}\right)=a(t)+c(t), \tag{2.14a}$$

$$i\left(\frac{dc_-}{dt}-2c_-^2\frac{dc_+}{dt}\right)\exp(c_0)=\tfrac{1}{2}[a(t)-2ib(t)-c(t)], \tag{2.14b}$$

$$i\left(\frac{dc_+}{dt}\right)\exp(-c_0)=\tfrac{1}{2}[a(t)+2ib(t)-c(t)], \tag{2.14c}$$

with the initial conditions

$$c_0(0)=c_+(0)=c_-(0)=0. \tag{2.15}$$

In general these equations can be solved numerically, and for the parametric oscillator they are simplified. Thus if we know c_0, c_-, and c_+, the evolution of generalized correlated or Schrödinger-cat states according to the Hamiltonian (2.1) can be obtained.

To relate these two procedures, we remember that the constants of motion satisfy the relation

$$I(t)=U(t)I(0)U^\dagger(t), \tag{2.16}$$

where $I(t)$ denotes the invariants $A(t)$ or $A^\dagger(t)$. Throughout the commutation relation properties of the generators of the two-dimensional symplectic group with the creation and annihilation operators, the expression (2.16) can be evaluated and then we get the time evolution of the constants of motion

$$\begin{pmatrix}\hat{A}(t)\\ \hat{A}^\dagger(t)\end{pmatrix}=\begin{pmatrix}e^{(-c_0/2)} & -e^{(c_0/2)}c_-\\ -e^{(-c_0/2)}c_+ & e^{(c_0/2)}(1-c_-c_+)\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{a}^\dagger\end{pmatrix}, \tag{2.17}$$

which is a canonical transformation. We note that the matrix appearing in the last expression can also be obtained by considering a faithful representation of the symplectic generators, substituting them into the temporal evolution operator (2.12) and calculating the inverse of the resultant matrix.

Comparing Eq. (2.17) with the corresponding one obtained with the linear time-dependent constants of the motion formalism, Eq. (2.5), we have four algebraic equations, which give the following relations between c_0, c_+, and c_-:

$$c_0=-2\ln(M_1), \tag{2.18a}$$

$$c_+=M_2^*/M_1, \tag{2.18b}$$

$$c_-=-M_2M_1. \tag{2.18c}$$

In the expressions (2.5) and (2.12) there is not a contradiction between the number of independent parameters because the complex parameters c_+ and c_- are not independent.

Because M_1 and M_2 are written in terms of two independent solutions of the classical equations of motion (2.6), using Eqs. (2.18) we get

$$h_1(t)=\mathrm{Re}(e^{-c_0/2}+c_-e^{c_0/2}), \tag{2.19a}$$

$$h_2(t)=-\mathrm{Im}(e^{-c_0/2}-c_-e^{c_0/2}). \tag{2.19b}$$

In summary, if we know the classical solutions h_1 and h_2 we can determine M_1 and M_2 and, through the relations (2.18), the complex parameters appearing in the evolution operator. We can proceed in the other direction: if the functions c_0, c_+, and c_- are obtained, the solutions of the classical equations can be found by means of (2.19).

III. GAUSSIAN PACKETS

In this section we study the evolution of Gaussian packets under the Hamiltonian (2.1). We start by evaluating the expectation values and dispersions of the position and momentum operators in the generalized correlated states (2.11), which are solutions of the Hamiltonian (2.1).

To get the expectation values, the position and momentum operators are written in terms of the integrals of motion $A^\dagger$ and A. Afterwards through the actions $A|\alpha,t\rangle=\alpha|\alpha,t\rangle$ and $\langle\alpha,t|A^\dagger=\alpha^*\langle\alpha,t|$ it is immediate that

$$\langle\hat{q}\rangle_{\alpha,t}=i(\lambda_p^*\alpha-\lambda_p\alpha^*), \tag{3.1a}$$

$$\langle\hat{p}\rangle_{\alpha,t}=-i(\lambda_q^*\alpha-\lambda_q\alpha^*). \tag{3.1b}$$

In a similar form the dispersions can be obtained and the results are

$$\sigma_{qq}=|\lambda_p|^2, \tag{3.2a}$$

$$\sigma_{pq}=-\tfrac{1}{2}(\lambda_q^*\lambda_p+\lambda_p^*\lambda_q), \tag{3.2b}$$

$$\sigma_{pp}=|\lambda_q|^2. \tag{3.2c}$$

The last expressions are independent of the parameter α of the generalized correlated state and σ_{pq} is a measure of the correlation between the position and momentum. Also these dispersions minimize the Schrödinger-Robertson uncertainty relation

$$\sigma_{qq}\sigma_{pp}-\sigma_{pq}^2=\tfrac{1}{4}. \tag{3.3}$$

From the wave functions (2.11) associated to the Hamiltonian (2.1), the probability density in the position space can be calculated:

$$P_\alpha(q,t)=\frac{1}{\sqrt{2\pi\sigma_{qq}}}\exp\left\{-\frac{(q-\langle q\rangle_{\alpha,t})^2}{2\sigma_{qq}}\right\}, \tag{3.4}$$

where the expressions $\lambda_p\lambda_q^*-\lambda_q\lambda_p^+=i$, (3.1a), and (3.2a) were used.

The wave function in the momentum space is constructed through the usual Fourier transform of Eq. (2.11). From this result, we can evaluate the probability density in the momentum space:

$$P_\alpha(p,t)=\frac{1}{\sqrt{2\pi\sigma_{pp}}}\exp\left\{-\frac{(p-\langle p\rangle_{\alpha,t})^2}{2\sigma_{pp}}\right\}. \tag{3.5}$$

Finally we give the analytic expression for the Wigner function associated to the Hamiltonian (2.1). This function is defined by

$$W(q,p)=\int_{-\infty}^{\infty}\langle q+u/2|\hat{\rho}|q-u/2\rangle e^{-ipu}du, \tag{3.6}$$

where $\hat{\rho}$ is the density operator and in this case $\hat{\rho}=|\alpha,t\rangle\langle\alpha,t|$, with $|\alpha,t\rangle$ denoting the generalized correlated states, which are solutions of the time-dependent Schrödinger equation. Substituting this density operator and using the Eq. (2.11), we have the Fourier transform of a Gaussian term, which can be easily evaluated. Making some algebraic manipulations the Wigner function can be written as [24]

$$W(q,p,t)=\frac{1}{[(1-\mathcal{R}^2)\sigma_{pp}\sigma_{qq}]^{1/2}}\exp\left\{-\frac{1}{2(1-\mathcal{R}^2)}\times\left[\frac{\overline{q}^2}{\sigma_{qq}}+\frac{\overline{p}^2}{\sigma_{pp}}-2\mathcal{R}\frac{\overline{q}}{\sqrt{\sigma_{qq}}}\frac{\overline{p}}{\sqrt{\sigma_{pp}}}\right]\right\}, \tag{3.7}$$

where we have defined the correlation coefficient

$$\mathcal{R}=\frac{|\sigma_{qp}|}{\sqrt{\sigma_{pp}\sigma_{qq}}}, \tag{3.8}$$

and the variables

$$\overline{q}=q-\langle\hat{q}\rangle_{\alpha,t}, \tag{3.9a}$$

$$\overline{p}=p-\langle\hat{p}\rangle_{\alpha,t}. \tag{3.9b}$$

Now we study the evolution of Schrödinger-cat states under the Hamiltonian (2.1). These states are also called even and odd coherent states because they are related to irreducible representations of the finite point group of two elements, the identity and the reflection, acting on the complex plane of the parameters labelling the coherent states [26,27]. They are given by the linear combination of coherent states

$$|\alpha\rangle_\pm=\mathcal{N}_\pm(|\alpha\rangle\pm|-\alpha\rangle), \tag{3.10}$$

with

$$\mathcal{N}_+=\frac{\exp|\alpha|^2/2}{2\sqrt{\cosh|\alpha|^2}}, \tag{3.11a}$$

$$\mathcal{N}_-=\frac{\exp|\alpha|^2/2}{2\sqrt{\sinh|\alpha|^2}}. \tag{3.11b}$$

It is immediate to show that they are eigenfunctions of the square of the annihilation operator

$$a^2|\alpha\rangle_\pm=\alpha^2|\alpha\rangle_\pm. \tag{3.12}$$

Therefore in what follows we will find the corresponding analytic expressions for dispersions, densities, and Wigner functions but now associated to the eigenfunctions of $A^2(t)$ [23,26].

The expectation values of the position and momentum operator are equal to zero because the Schrödinger-cat states have the properties

$$A|\alpha,t\rangle_\pm=\left(\frac{\mathcal{N}_+}{\mathcal{N}_-}\right)\alpha|\alpha,t\rangle_\mp, \tag{3.13a}$$

$$_\pm\langle\alpha,t|A^\dagger=\left(\frac{\mathcal{N}_+}{\mathcal{N}_-}\right)\alpha^*{}_\mp\langle\alpha,t|, \tag{3.13b}$$

and the even and odd generalized correlated states are orthonormal.

It is easy to get the quadrature dispersions for the generalized Schrödinger-cat states [23]. The results are

$$\sigma_{pp}^\pm(t)=\lambda_4^2\sigma_{pp}^\pm(0)+\lambda_2^2\sigma_{qq}^\pm(0)-2\lambda_4\lambda_2\sigma_{pq}^\pm(0), \tag{3.14a}$$

$$\sigma_{pq}^\pm(t)=-\lambda_4\lambda_3\sigma_{pp}^\pm(0)-\lambda_2\lambda_1\sigma_{qq}^\pm(0)+\lambda_4\lambda_1\sigma_{pq}^\pm(0)+\lambda_2\lambda_3\sigma_{qp}^\pm(0), \tag{3.14b}$$

$$\sigma_{qq}^\pm(t)=\lambda_3^2\sigma_{pp}^\pm(0)+\lambda_1^2\sigma_{qq}^\pm(0)-2\lambda_3\lambda_1\sigma_{pq}^\pm(0), \tag{3.14c}$$

where $\lambda_1=\sqrt{2}\ \mathrm{Im}\lambda_p$, $\lambda_2=\sqrt{2}\ \mathrm{Im}\lambda_q$, $\lambda_3=\sqrt{2}\ \mathrm{Re}\lambda_p$, and $\lambda_4=\sqrt{2}\ \mathrm{Re}\lambda_q$. For $t=0$, they correspond to the quadratures for a standard cat state:

$$\sigma_{pp}^\pm(0)=r^2(\Delta_\pm-\cos 2\theta)+\tfrac{1}{2}, \tag{3.15a}$$

$$\sigma_{pq}^\pm(0)=r^2\sin 2\theta, \tag{3.15b}$$

$$\sigma_{qq}^\pm(0)=r^2(\Delta_\pm+\cos 2\theta)+\tfrac{1}{2}, \tag{3.15c}$$

where we have defined $\alpha=r\exp(i\theta)$ and

$$\Delta_{\pm}=\begin{cases}\tanh r^2\\ \coth r^2.\end{cases} \tag{3.16}$$

By means of the wave function of the even and odd generalized correlated states in the coordinate or momentum representations, we construct the corresponding probability densities. For the position case we have

$$P_{\alpha\pm}(q,t)=|\mathcal{N}_{\pm}|^2\Bigg\{P_{\alpha}(q,t)+P_{-\alpha}(q,t)\pm P_{\alpha}(q,t)\times\left[\exp\left(-2i\frac{\alpha q}{\lambda_p}\right)+\exp\left(2i\frac{\alpha^* q}{\lambda_p^*}\right)\right]\Bigg\}. \tag{3.17}$$

For the momentum we have a similar expression; it is only necessary to make the replacements: $P_{\alpha}(q,t)\mapsto P_{\alpha}(p,t)$ and in the exponential terms $\lambda_p\mapsto-\lambda_q$ and $q\mapsto p$.

To get the Wigner function we follow the procedure indicated in Ref. [26] and we get

$$W_{\pm}(q,p,t)=|\mathcal{N}_{\pm}|^2\{W_{\alpha,\alpha}(q,p,t)\pm W_{\alpha,-\alpha}(q,p,t)\pm W_{-\alpha,\alpha}(q,p,t)+W_{-\alpha,-\alpha}(q,p,t)\}, \tag{3.18}$$

where we have defined

$$W_{\alpha,\beta}(q,p,t)=\int\int du\ \exp(-ipu)\psi_{\alpha}(q+u/2,t)\times\psi_{\beta}^*(q-u/2,t). \tag{3.19}$$

Substituting the wave functions (2.11) into the last expression, and evaluating the integral we obtain

$$W_{\alpha,\beta}(q,p,t)=2\ \exp\{-2|z_0|^2+2\alpha z_0^*+2\beta^* z_0\}\times\exp\{-\alpha\beta^*-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2\}, \tag{3.20}$$

with

$$z_0(t)=\frac{i}{\sqrt{2}}(M_1-M_2)p+\tfrac{1}{2}(M_1+M_2)q. \tag{3.21}$$

Replacing the corresponding expressions (3.20) into (3.18), and making algebraic simplifications we arrive to the result

$$W_{\pm}(q,p,t)=4|\mathcal{N}_{\pm}|^2\exp\{-2|z_0|^2-|\alpha|^2\}\{\exp(-|\alpha|^2)\cosh[4\ \mathrm{Re}(\alpha z_0^*)]\pm\exp(|\alpha|^2)\cos[4\ \mathrm{Im}(\alpha z_0^*)]\}. \tag{3.22}$$

In the next section, we apply the developed formalism to study the behavior of an ion moving in a Paul trap.

IV. PAUL TRAPS

The quantum motion of a charged particle in a quadrupole radio-frequency trap has been solved in terms of the classical trajectories. It was shown that the quantum stability regions are exactly given by the stability regions for the associated Mathieu functions [28].

Although the possibility of confining charged particles by means of alternating and static electric fields was discovered forty years ago [16,29], the capacity to trap a single atomic particle was not obtained until the beginning of the 1980s [30]. Since this achievement, the use of ion traps to measure spectroscopic properties of isolated ions has been increasing.

For a Paul trap the potential energy is a combination of static and alternating quadrupole fields and so the Hamiltonian of an ion moving in the trap is given by

$$H=-\frac{\hbar^2}{2m}\nabla'^2-\frac{e}{r_0^2}(U+V\cos\Omega t')\left(x_3'^2-\frac{x_1'^2+x_2'^2}{2}\right), \tag{4.1}$$

where U is the direct current (dc) and V the radio-frequency voltages. As will be seen later, it is convenient to introduce the dimensionless parameters in the last expression:

$$a_3=-\frac{8eU}{mr_0^2\Omega^2}=-2a_i, \tag{4.2a}$$

$$b_3=\frac{4eV}{mr_0^2\Omega^2}=-2b_i, \tag{4.2b}$$

with $i=1,2$. The corresponding time-dependent Schrödinger equation yields a separable system of three independent differential equations, which are given by

$$\left(\frac{p_k'^2}{2m}+\frac{1}{2}m\Omega_k^2(t')x_k'^2\right)\phi_k(x_k',t')=i\hbar\frac{\partial\phi_k(x_k',t')}{\partial t'}, \tag{4.3}$$

where $k=1,2,3$, and we have defined

$$\Omega_k^2(t')=\frac{\Omega^2}{4}(a_k-2b_k\cos\Omega t'). \tag{4.4}$$

Now we introduce into Eq. (4.3) dimensionless positions, momenta, and time variables through the relations

$$x_k=\sqrt{\frac{m\Omega_i(0)}{\hbar}}x_k',\quad p_k=\sqrt{\frac{1}{\hbar m\Omega_i(0)}}p_k',\quad t=\Omega_i(0)t'. \tag{4.5}$$

Thus the expression (4.3) can be rewritten as

$$\frac{1}{2}(p_i^2+w_i^2(t)x_i^2)\phi_i=i\frac{\partial\phi_i}{\partial t}, \tag{4.6}$$

with $w_k(t)=\Omega_k(t')/\Omega_i(0)$.

Comparing the left-hand side of the last expression with Eq. (2.1) we have that

$$a(t)=1, \quad b(t)=0, \quad c(t)=w_k^2(t). \tag{4.7}$$

In summary, the quantum motion of an ion in a radio-frequency trap can be described by two kinds of parametric oscillators, one in direction Z and the other in the plane X-Y. Following the formalism described in the last section, to construct the constants of the motion we have to solve the classical equations of motion for the parametric oscillator. Substituting Eq. (4.7) into the expression (2.3) we get

$$\ddot{h}_k(t)+w_k^2(t)h_k(t)=0, \tag{4.8}$$

which corresponds to Mathieu equations. Then by means of the Bogoliubov transformation Eq. (2.6) we obtain the constants of the motion in terms of the independent solutions of the Mathieu equations. The solution of the time-dependent Schrödinger equation is obtained through the generalized correlated states also constructed in terms of $h_k(t)$. Finally

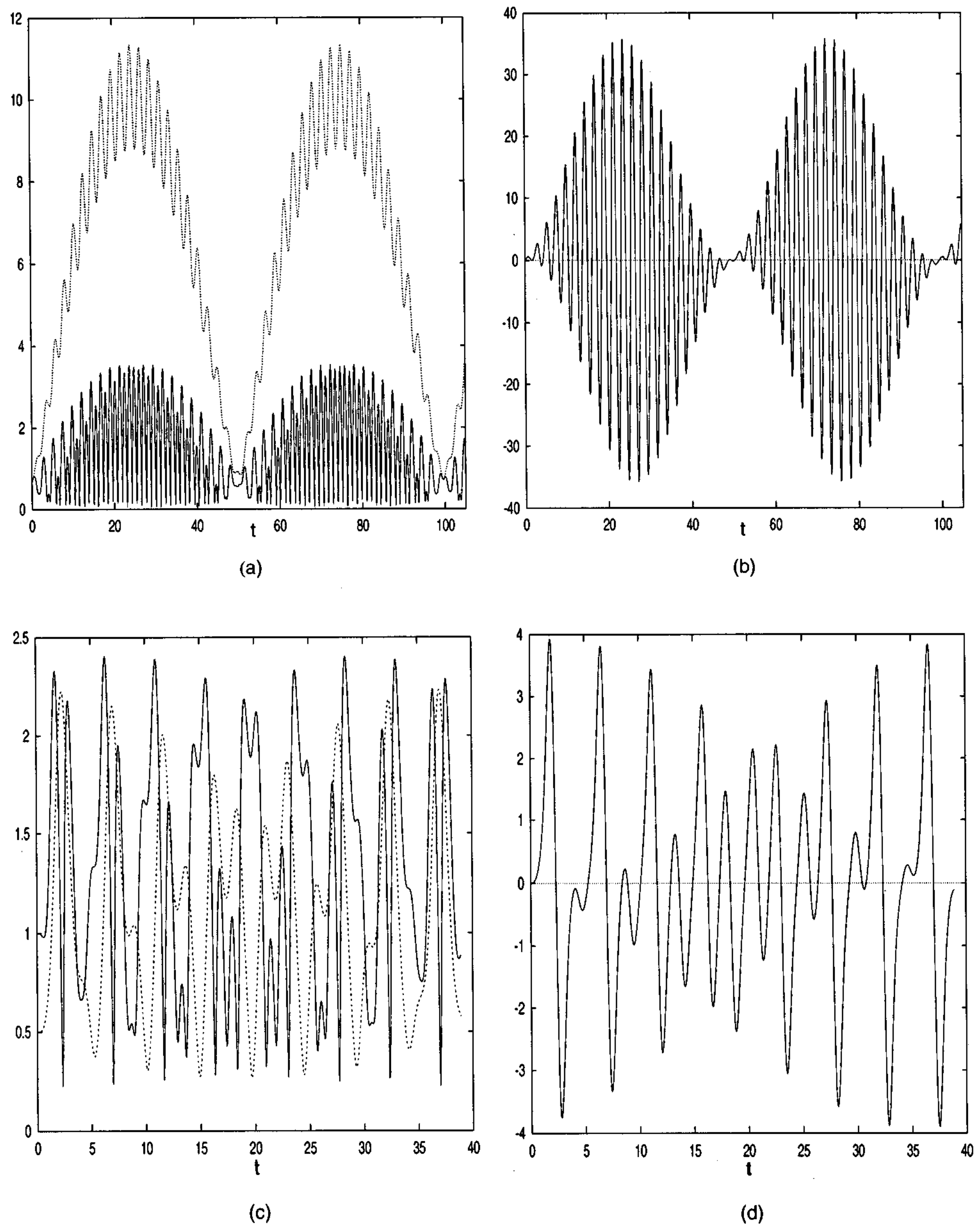

FIG. 1. Dispersions and correlations for a generalized correlated state are shown. In the left column, the dispersions in position Δ_q (dashed lines) and momenta Δ_p (full lines), are shown while in the right column the correlation factor σ_{pq} is shown. (a) and (b) correspond to the motion in the X-Y plane, whereas (c) and (d) to the motion along the Z direction. In the plots we are using adimensional units; the physical magnitudes are obtained by means of the expressions $[t]=1/\Omega_i(0)=1.87\times10^{-8}$ sec, $[q]=2.45\times10^{-7}$ cm, and $[p]=4.31\times10^{-21}$ g cm/sec.

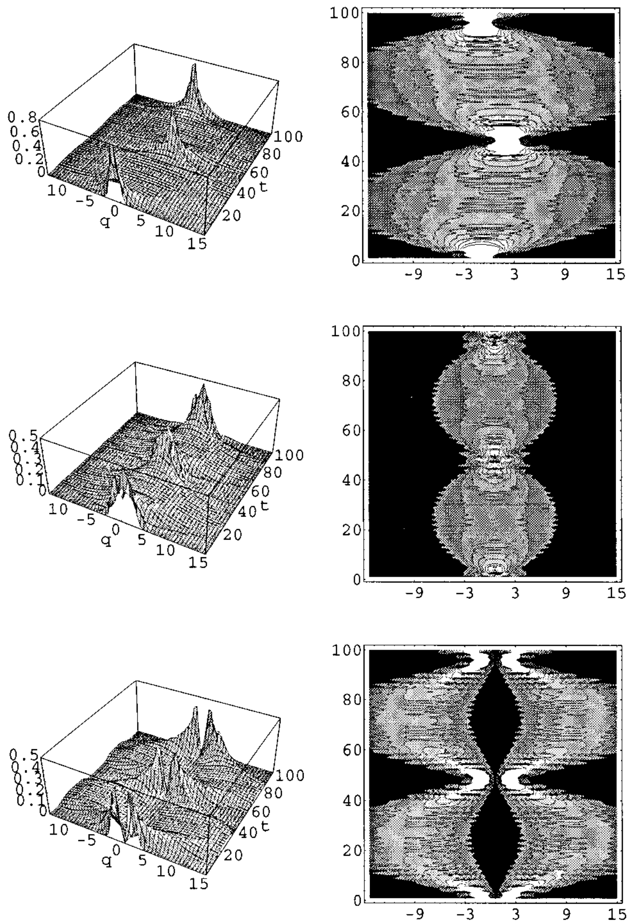

FIG. 2. The evolution of the probabilty densities in the configuration space, for the motion in the X-Y plane, are illustrated for the initial coherent state (top), even Schrödinger-cat state (middle), and odd Schrödinger-cat state (bottom), all for the amplitude $\alpha=(1,0)$. The corresponding contour plots are given in the right-hand side.

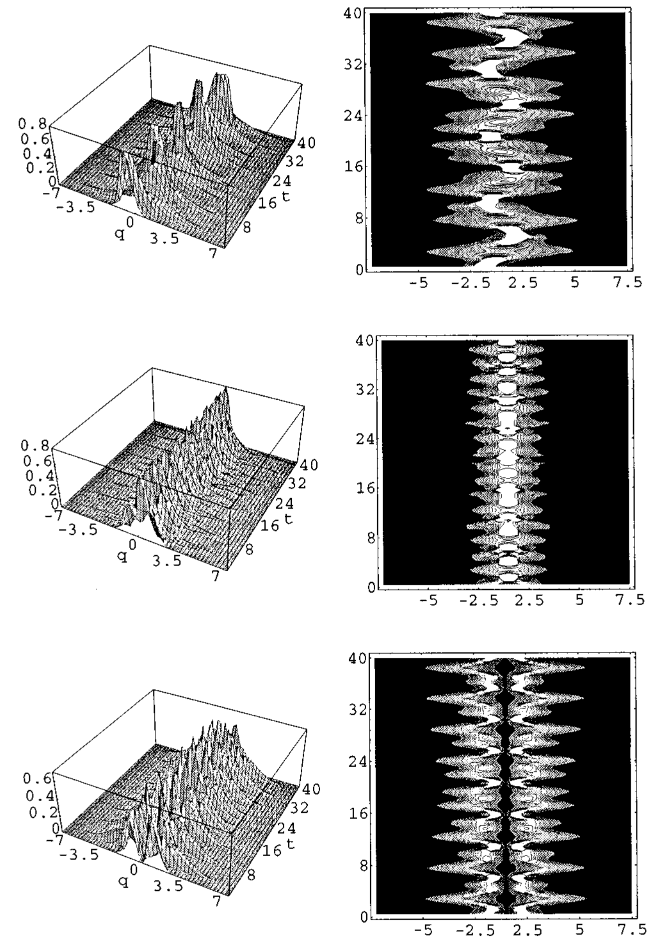

FIG. 3. The evolution of the probabilty densities in the configuration space, for the motion in the Z direction, are illustrated for the initial coherent state (top), even Schrödinger-cat state (middle), and odd Schrödinger-cat state (bottom), all for the amplitude $\alpha=(1,0)$. The corresponding contour plots are given in the right-hand side.

the coefficients of the evolution operator for this system (2.12) are given in terms of the Bogoliubov transformation matrix.

Next, we will describe the results obtained for the evolution of correlated and Schrödinger cat states in a Paul trap for the ion $^{198}\mathrm{Hg}^{+}$ of the following characteristics [22]:

$$V=1.2\ \text{kV},\quad U=71.4\ \text{V},\quad \frac{\Omega}{2\pi}=23.189\ \text{MHz},$$

$$r_0=466\ \mu\text{m}. \tag{4.9}$$

V. NUMERICAL RESULTS

In order to calculate the time evolution of correlated and Schrödinger-cat states in a Paul trap we can proceed in several forms. One of them is solving the Mathieu equations. Another one is through the solutions of the equations for the complex time-dependent coefficients $c_0(t)$, $c_+(t)$, and $c_-(t)$ [see Eq. (2.12]. The differential equations satisfied by these functions were solved numerically using the subroutine DE [31].

For the ion $^{198}\mathrm{Hg}^{+}$ trap the adimensional parameters of Eq. (4.2a) take the values

$$a_3=-0.0604,\quad b_3=0.508.$$

In the following results, we have used the units

$$[t]=1/\Omega_i(0)=1.87\times10^{-8}\ \text{sec},\quad [q]=2.45\times10^{-7}\ \text{cm},$$

$$[p]=4.31\times10^{-21}\,\text{g cm/sec}.$$

For the generalized correlated states, the expectation values of the position and momentum operators depend on the field amplitude α, Eq. (3.1). The classical trajectory coincides with the expectation value of the position if the initial conditions are given by $q_0=\sqrt{2}\,\mathrm{Re}\alpha$ and $p_0=\sqrt{2}\,\mathrm{Im}\,\alpha$. In fact, the corresponding behavior of the momentum expectation values are $\langle\hat{p}\rangle=d\langle\hat{q}\rangle/dt$. The trajectory of the ion is confined, the motion in the Z direction is more localized than the motion in the plane. The trajectory of the ion along the Z direction is similar to that shown in Ref. [32]. The expectation values of the position and momentum operators are zero with respect to the Schrödinger-cat states.

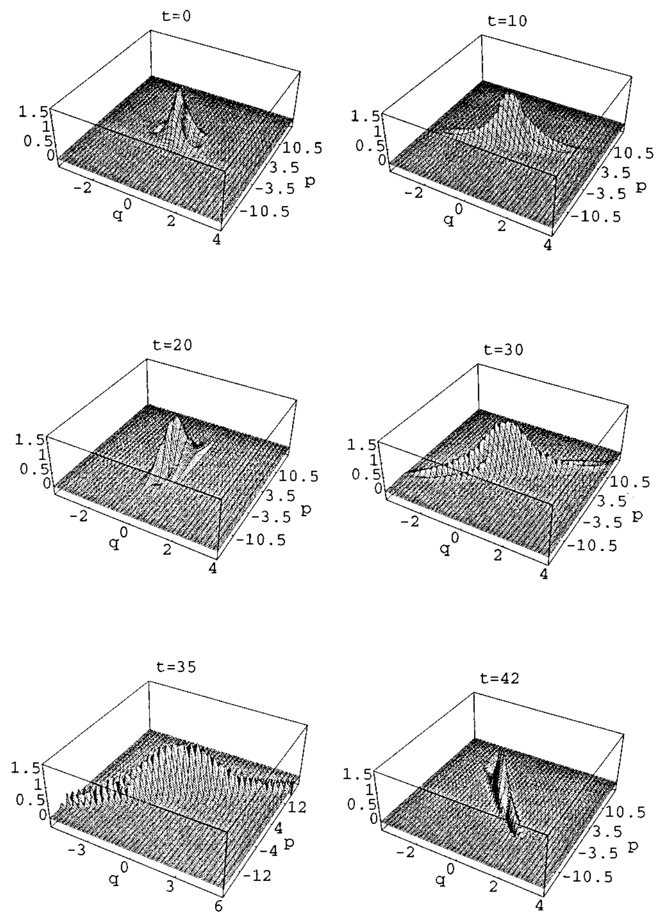

FIG. 4. Evolution of the Wigner function for the even Schrödinger-cat state in the Z direction, with $\alpha=(1,0)$.

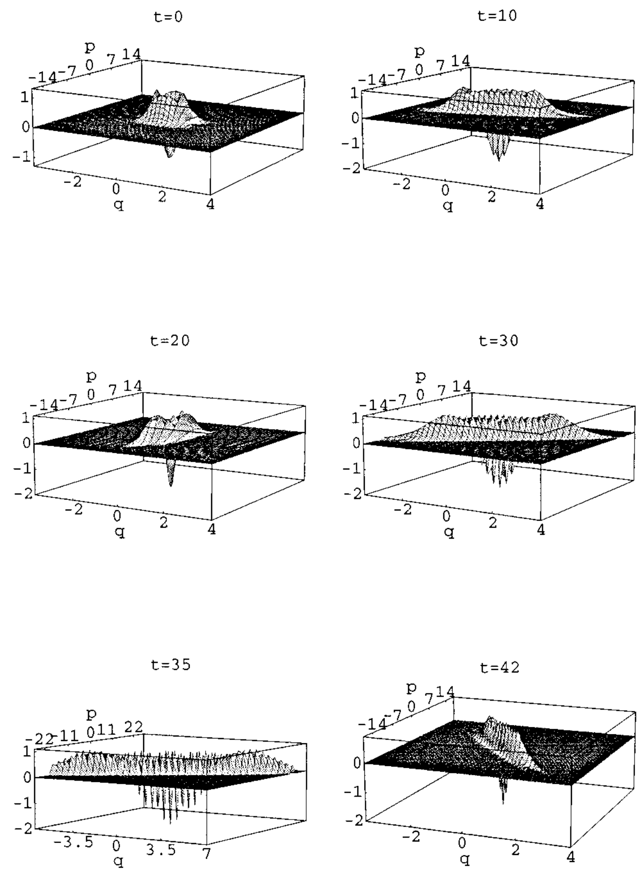

FIG. 5. Evolution of the Wigner function for the odd Schrödinger-cat state in the Z direction, with $\alpha=(1,0)$.

The dispersions for the generalized coherent states are independent of the α value, Eq. (3.2), and minimize the Schrödinger Robertson uncertainty relation, which also has been used to test the accuracy of the numerical calculations. In Fig. 1, we illustrate in the left column the dispersions in position $\Delta_q=\sqrt{\sigma_{qq}}$ (dashed lines) and momenta $\Delta_p=\sqrt{\sigma_{pp}}$ (full lines), while in the right column we illustrate the correlation factor σ_{pq}. Figures 1(a) and 1(b) correspond to the motion in the X-Y plane, whereas Figs. 1(c) and 1(d) to the motion along the Z axis. For the plane motion, a bigger dispersion in the position than in the momentum and a quasiperiodic behavior for the correlation are found. The squeezing phenomenon only occurs for the momentum variable. For the motion in the Z direction the dispersions are one order of magnitude less than for the plane motion. Also there is a quasiperiodic behavior for the correlation and the squeezing is present in position and momentum variables.

The quadratures for Schrödinger-cat states depend on the field amplitude α as can be seen from Eq. (3.14) and Eq. (3.15). For the Paul trap we have $\lambda_1=h_1$, $\lambda_2=-\dot{h}_1$, $\lambda_3=h_2$, and $\lambda_4=-\dot{h}_2$. To compare with the generalized correlated states, we select $\alpha=(1,0)$ and we get for the even and odd states a similar shape and structure for the position and momentum dispersions, together with the correlations σ_{pq}. The differences are in the sizes of the functions: for the even Schrödinger-cat state they are shrunk while for the odd Schrödinger-cat state they are stretched.

The behavior of the dispersions in the position and momentum variables can also be shown by means of the probability densities. To illustrate this, for the field amplitude $\alpha=(1,0)$, the probability densities in the configuration space in the X-Y plane (Fig. 2) and in the Z direction (Fig. 3) are plotted.

In these figures, we display, from the top to the bottom, the position probability densities of the generalized correlated state, the even Schrödinger-cat and odd Schrödinger-cat states, respectively. On the left hand side we show three-dimensional plots whereas on the right-hand side contour plots are given. In these, the quadrature values are better appreciated. At the top of Figs. 2 and 3, the maximum of the position probability density indicates the classical trajectory of the $^{198}Hg^{+}$ ion. The position probability densities of the even Schrödinger cat states have an absolute maximum at the origin with other maxima for larger values of the position variables. These maxima are easily seen through the white spots of the contour plots. They are clearer for the motion in the Z direction than in the plane. The position probability densities of the odd Schrödinger cat states have a minimum at the origin with two equal maxima for larger values of the position variable.

The position densities for all the studied cases are well localized packets, with a quasiperiodic behavior. To see the interference effects of these macroscopic superposition of states in the phase space it is convenient to display the corresponding Wigner distribution functions.

The evolution of the Wigner function of the ion, for the correlated vacuum state, exhibits a similar quasiperiodic behavior to the one shown in Ref. [21]. Thus, initially the Wigner function is a symmetric Gaussian in position and momentum, whereas it gets squeezed and rotated as a function of time.

In Figs. 4 and 5 we show the Wigner functions for the even and odd Schrödinger-cat states in the Z direction, with $\alpha=(1,0)$. In all these figures the squeezing phenomena can also be appreciated as well as the rotation of the Wigner functions. Finally, it is remarkable how the interference manifests itself for the odd Schrödinger-cat state, in particular at $t=35$.

VI. CONCLUSIONS

We have studied the evolution of generalized coherent and Schrödinger cat states moving in a radio-frequency $^{198}Hg^+$ trap. We found for the field amplitude $\alpha=(1,0)$ that the motion of the ion is confined. It is one order of magnitude bigger in the X-Y plane than in the Z direction, contrary to what is happening for the corresponding average kinetic energy.

The dispersions σ_{pp}, σ_{qq}, and σ_{pq} for these states are very similar. All of them present rapid oscillations around zero with a quasiperiodic amplitude. However, the order of magnitude of the dispersions is different for the motion in the X-Y plane from that in the Z direction.

The localization of the states is nicely seen in the contour plots of the probability densities in the configuration space for the coherent and Schrödinger-cat states. In the X-Y plane all of them are more localized around $t\simeq 0$, $t\simeq 50$, and $t\simeq 100$ whereas the opposite is true at $t\simeq 20$ and $t\simeq 80$. In the Z direction the oscillations are evident and the interference effects are more emphasized than in the X-Y plane motion. In both cases the even Schrödinger-cat state shows the most appropriate behavior to reach a pronounced localization of the ion.

The quasiprobabilistic Wigner function corroborates the presence of the squeezing phenomenon, which is present for the generalized coherent and Schrödinger-cat states. This is only illustrated for the Schrödinger cat states in the Z direction, which give information of the system in the phase space.

ACKNOWLEDGMENT

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