Representations of two-mode squeezing transformations

Fan Hong-yi
*China Center of Advanced Science and Technology, (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China and Department of Material Science and Engineering, China University of Science and Technology, Hefei, Anhui 230026, People's Republic of China**

Fan Yue
Department of Material Science and Engineering, China University of Science and Technology, Hefei, Anhui 230026, People's Republic of China

(Received 16 January 1996)

We derive natural representations for two-mode squeezing operators. By natural we mean that the representations are composed by the eigenvectors of either one of the two two-mode quadrature operators for squeezing. The technique of integration within an ordered product of operators provides us with a convenient method of derivation. [S1050-2947(96)08207-8]

PACS number(s): 03.65.Ca, 03.65.Bz, 42.50.Dv

I. INTRODUCTION

In Ref. [1] the explicit form of the common eigenvectors of the relative position Q_1-Q_2 and the total momentum P_1+P_2, of two particles which were considered by Einstein, Podolsky, and Rosen in their argument that the quantum-mechanical state vector is not complete [2], is constructed. It is

$$|\eta\rangle=\exp(-\tfrac{1}{2}|\eta|^2+\eta a^\dagger-\eta^* b^\dagger+a^\dagger b^\dagger)|00\rangle, \tag{1}$$

where $\eta=\eta_1+i\eta_2$ is a complex number $a^\dagger,b^\dagger$ are two-mode creation operators, $|00\rangle$ is the two-mode vacuum state. On the other hand, the common eigenvector of Q_1+Q_2 and P_1-P_2 is

$$|\xi\rangle=\exp[(-\tfrac{1}{2}|\xi|^2+\xi a^\dagger+\xi^* b^\dagger-a^\dagger b^\dagger)|00\rangle, \quad \xi=\xi_1+i\xi_2. \tag{2}$$

A question thus naturally arises: what is the squeezing effect of $|\xi\rangle$ under the two-mode squeezing transformation? Note that the $|\xi\rangle$ state is an orthonormal and complete one [1],

$$\langle\xi'|\xi\rangle=\pi\delta(\xi_1-\xi_1')\delta(\xi_2-\xi_2'), \qquad \int\frac{d^2\xi}{\pi}|\xi\rangle\langle\xi|=1. \tag{3}$$

In this way $|\xi\rangle$ is in essence different from the two-mode squeezed state; the latter is in essence a generalized coherent state which is nonorthogonal and overcomplete [3]. Thus it seems necessary to answer the question. In Secs. II and III we shall investigate how $|\xi\rangle$ and $|\eta\rangle$ transform under the two-mode squeezing operator's action, respectively. The result will show that the $\langle\xi|$ (or $\langle\eta|$) representation is the natural language for describing the two-mode squeezing operators. In Sec. IV we explain this result and further point out the relationship between the $|\xi\rangle$ state and the eigenvector of a photocurrent operator $\hat{Z}$ of a heterodyne detector recently studied in Ref. [4]. Through our discussions we shall make full use of the newly developed technique of integration within an ordered product (IWOP) of operators [5].

*Mailing and permanent address.

II. NEW REPRESENTATION OF TWO-MODE SQUEEZING OPERATOR IN $\langle\xi|$ VECTOR SPACE

By two-mode squeezing operator we mean

$$S=\exp[\lambda(a^\dagger b^\dagger-ab)]. \tag{4}$$

We want to show that in $\langle\xi|$ representation S can be expressed as an integration projection operator, i.e.,

$$\mu\int\frac{d^2\xi}{\pi}|\mu\xi\rangle\langle\xi|\equiv U, \quad \mu>0, \tag{5}$$

where $\mu=e^\lambda$ is a squeezing parameter. Let us first show that U defined by (5) is unitary,

$$UU^\dagger=\mu^2\int\frac{d^2\xi}{\pi}|\mu\xi\rangle\langle\xi|\int\frac{d^2\xi'}{\pi}|\xi'\rangle\langle\mu\xi'|=1=U^\dagger U, \tag{6}$$

where the orthogonal property of $|\xi\rangle$ (see Ref. [1]) is used. Then by virtue of the IWOP technique we perform the integration in (5) [note that the normal ordering of $|00\rangle\langle 00|$ is $:\exp(-a^\dagger a-b^\dagger b):$],

$$U=\mu\int\frac{d^2\xi}{\pi}:\exp[-\tfrac{1}{2}|\xi|^2(1+\mu^2)+\xi(\mu a^\dagger+b)+\xi^*(\mu b^\dagger+a)-a^\dagger b^\dagger-ab-a^\dagger a-b^\dagger b]:$$
$$=2\mu(1+\mu^2)^{-1}:\exp\{(1+\mu^2)^{-1}2(\mu a^\dagger+b)(\mu b^\dagger+a)-(a^\dagger+b)(b^\dagger+a)\}:$$
$$=\text{sech}\lambda\exp(a^\dagger b^\dagger\tanh\lambda):\exp[(\text{sech}\lambda-1)(a^\dagger a+b^\dagger b)]:\exp(-ab\tanh\lambda), \tag{7}$$

which is just the normally ordered form of the two-mode squeezing operator S. Thus we can identify

$$S=U=\mu\int\frac{d^2\xi}{\pi}|\mu\xi\rangle\langle\xi|,\quad \mu=e^{\lambda}. \tag{8}$$

This is a new representation of S. With this representation one can immediately know the squeezing effect caused by the two-mode squeezing transformation, i.e.,

$$S|\xi\rangle=\mu|\mu\xi\rangle=\mu\exp[-\tfrac{1}{2}\mu^2|\xi|^2+\mu(\xi a^\dagger+\xi^*b^\dagger)-a^\dagger b^\dagger]|00\rangle,\quad \mu^2=\frac{1+\tanh\lambda}{1-\tanh\lambda}. \tag{9}$$

III. THE $\langle\eta|$ REPRESENTATION OF S

Using (8) we operate S on the state vector $|\eta\rangle$,

$$S|\eta\rangle=\mu\int\frac{d^2\xi}{\pi}|\mu\xi\rangle\langle\xi|\eta\rangle. \tag{10}$$

As a result of the overlap between $\langle\xi|$ and $|\eta\rangle$,

$$\langle\xi|\eta\rangle=\tfrac{1}{2}\exp[\tfrac{1}{2}(\xi^*\eta-\xi\eta^*)], \tag{11}$$

we have

$$S|\eta\rangle=\mu\int\frac{d^2\xi}{2\pi}|\mu\xi\rangle e^{(1/2)(\xi^*\eta-\xi\eta^*)}=\mu\int\frac{d^2\xi}{2\pi}\exp\left(-\frac{\mu^2}{2}|\xi|^2+\mu(\xi a^\dagger+\xi^*b^\dagger)+\tfrac{1}{2}(\xi^*\eta-\xi\eta^*)-a^\dagger b^\dagger\right)|00\rangle=\frac{1}{\mu}|\eta/\mu\rangle. \tag{12}$$

which tells us that in $\langle\eta|$ representation the squeezing operator S can be expressed as

$$S=\frac{1}{\mu}\int\frac{d^2\eta}{\pi}|\eta/\mu\rangle\langle\eta|; \tag{13}$$

this is consistent with the fact that $|\eta\rangle$ is a Fourier transformation of $|\xi\rangle$ (Ref. [6]), e.g.,

$$\int\frac{d^2\xi}{2\pi}|\xi\rangle e^{(1/2)(\xi^*\eta-\xi\eta^*)}=|\eta\rangle. \tag{14}$$

IV. PHYSICAL EXPLANATION FOR S'S TWO NEW REPRESENTATIONS

We now give a physical explanation for the results of Secs. II and III. As one knows, the two-mode squeezing's criterion lies in the quadratures' fluctuation

$$\langle\hat{X}_1^2\rangle-\langle\hat{X}_1\rangle^2=(\Delta\hat{X}_1)^2=\tfrac{1}{4}e^{2\lambda},\quad(\Delta\hat{X}_2)^2=\tfrac{1}{4}e^{-2\lambda}, \tag{15}$$

where $\hat{X}_j$ $(j=1,2)$ are two quadratures of optical field (3)

$$\hat{X}_1=\tfrac{1}{2}(Q_1+Q_2),\quad Q_1=\frac{1}{\sqrt{2}}(a+a^\dagger),\quad Q_2=\frac{1}{\sqrt{2}}(b+b^\dagger),$$
$$\hat{X}_2=\tfrac{1}{2}(P_1+P_2),\quad P_1=\frac{1}{\sqrt{2}i}(a-a^\dagger),\quad P_2=\frac{1}{\sqrt{2}i}(b-b^\dagger), \tag{16}$$

with $[\hat{X}_1,\hat{X}_2]=i/2$ and $\langle\ \rangle$ representing the expectation value of operators in the two-mode squeezed vacuum state. From (15) and (16) we notice that two-mode squeezing may get naturally characterized or described in the space spanned by the eigenstates of $\hat{X}_1$. Remember that $[Q_1+Q_2,P_1-P_2]=0$, $\hat{X}_1$ can share common eigenstates with $\hat{Y}_1=\tfrac{1}{2}(P_1-P_2)$. This is why the two-mode squeezing operator S can have natural representation in the $|\xi\rangle$ state space. On the other hand, $\hat{X}_2$ and its conjugate partner $\hat{Y}_2=\tfrac{1}{2}(Q_1-Q_2)$ can also be chosen as a quadrature phase, and X_2 shares a set of common eigenstates $|\eta\rangle$ with $\hat{Y}_2$; thus it is not strange that S can also have a natural representation by virtue of the $|\eta\rangle$ state. Based on [1]

$$\hat{X}_1|\xi\rangle=\frac{1}{\sqrt{2}}\xi_1|\xi\rangle,\quad\hat{X}_2|\eta\rangle=\frac{1}{\sqrt{2}}\eta_2|\eta\rangle \tag{17}$$

and (8) as well as (13) we deduce

$$S\hat{X}_1S^{-1}=\mu^2\int\frac{d^2\xi}{\pi}|\mu\xi\rangle\langle\xi|\hat{X}_1\int\frac{d^2\xi'}{\pi}|\xi'\rangle\langle\mu\xi'|$$

$$=\mu^2\int\frac{d^2\xi}{\pi}|\mu\xi\rangle\langle\mu\xi|\frac{1}{\sqrt{2}}\xi_1=\frac{1}{\mu}\hat{X}_1, \tag{18}$$

$$S\hat{X}_2S^{-1}=\frac{1}{\mu^2}\int\frac{d^2\eta}{\pi}|\eta/\mu\rangle\langle\eta/\mu|\frac{1}{\sqrt{2}}\eta_2=\mu\hat{X}_2. \tag{19}$$

Similarly we have $S\hat{Y}_1S^{-1}=\mu^{-1}\hat{Y}_1$, $S\hat{Y}_2S^{-1}=\mu\hat{Y}_2$.

The new representation of S can simplify many calculations about the effect of squeezing. As the first application of (8) we calculate the action of S on the two-mode coherent state

$$S|Z_1Z_2\rangle=\mu\int\frac{d^2\xi}{\pi}|\mu\xi\rangle\langle\xi|Z_1Z_2\rangle$$

$$=\text{sech}\lambda\exp[-\tfrac{1}{2}(|Z_1|^2+|Z_2|^2)+\text{sech}\lambda(Z_1a^\dagger+Z_2b^\dagger)+\tanh\lambda(a^\dagger b^\dagger-Z_1Z_2)]|00\rangle. \tag{20}$$

Then using (8) and

$$\langle\xi|mn\rangle=e^{-(1/2)|\xi|^2}\sum_{l=0}(-1)^l(\xi^*)^{m-l}\xi^{n-l}\times\frac{\sqrt{n!m!}}{l!(n-l)!(m-l)!}, \tag{21}$$

we can easily obtain the overlap between number state and squeezed number state:

$$\langle m'n'|S|mn\rangle=\mu\langle m'n'|\int\frac{d^2\xi}{\pi}|\mu\xi\rangle\langle\xi|mn\rangle$$

$$=\sum_{l,k}(-1)^{l+k}\delta_{n+m',m+n'}(\text{sech}\lambda)^{n+m'-l-k+1}\mu^{l-k+m'-m}\frac{(n+m'-l-k)![n!m!n'!m'!]^{1/2}}{l!(n-l)!(m-l)!k!(n'-k)!(m'-k)!}. \tag{22}$$

In summary, we have presented a natural representation for two-mode squeezing. This is consistent with the fact that the quadratures for judging two-mode squeezing are given by Eq. (16). The new representation is mathematically concise and physically appealing because the two-mode squeezing manifestly corresponds to the transformation from $|\xi\rangle$ to $\mu|\mu\xi\rangle$. Besides, it is interesting to point out that the state $|\xi\rangle$ can be realized in some quantum optical experiments, for example, in Ref. [4] the heterodyne eigenstate, which is the eigenvector of the photocurrent $\hat{Z}=a+b^\dagger$ has the same structure with the state $|\xi\rangle$ in our work. As one can easily see when one rewrites $|\xi\rangle$ in Eq. (2) as

$$|\xi\rangle=e^{\xi a^\dagger-\xi^*a}|0\rangle\rangle,\quad |0\rangle\rangle\equiv e^{-a^\dagger b^\dagger}|00\rangle. \tag{23}$$

This form is just Eq. (10) of Ref. [4], up to a phase factor.

[1] Fan Hong-yi and J. R. Klauder, Phys. Rev. A **49**, 704 (1994).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[3] See, e.g., R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).
[4] G. M. D'Arino and M. F. Sacchi, Phys. Rev. A **52**, R4309 (1995).
[5] Fan Hong-yi, H. R. Zaidi, and J. R. Klauder, Phys. Rev. D **35**, 1831 (1987); Fan Hong-yi and J. VanderLinde, Phys. Rev. A **39**, 2987 (1989).
[6] Fan Hong-yi and Chen Bo-zhan, Phys. Rev. A **53**, 2948 (1966).