Quantum-state homodyne measurement with vacuum ports

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We show that the quantum state of an N-mode light field can be reconstructed from the difference statistics recorded in multiport homodyning with N unused input ports. The theory is applied to balanced homodyne six-port detection, which is shown to be suited for measuring the Q function of a single-mode radiation field. [S1050-2947(96)01207-3]

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I. INTRODUCTION

The reconstruction of the quantum state of radiation from the data recorded in appropriate homodyne detection has been a subject of increasing interest. Any knowable information on the quantum state can be obtained provided that the density matrix of the radiation field be determined from the data measured. Since the density matrix can be given in various representations, a number of alternative approaches may be used to solve the problem of quantum-state reconstruction. Density matrix representations in orthonormal Hilbert space bases and others based on phase space quasidistributions, such as the Wigner function or the Q function, have been considered.

The first experimental reconstruction of the quantum state of a (single-mode) radiation field was performed by Smithey *et al.* [1], using balanced homodyne (four-port) detection. The signal-field quantum state is obtained from the difference-count distributions measured for a sufficiently large number of closely packed values of the (classical) phase difference between the interfering signal and local-oscillator fields within a π interval. The method, also called optical homodyne tomography, is based on the fact that the Wigner function of the signal field can be obtained from the measured data by means of inverse Radon transformation corresponding to a three-fold Fourier transformation [2]. The Fourier transform of the Wigner function then yields the density matrix (in a field-strength basis). Tomographic methods can also be applied to the determination of (vibrational) quantum states of molecules [3] and trapped ions [4]. Eventually, the method has been modified in order to measure the photon statistics on a short-time scale [5].

Experimental progress has stimulated a number of theoretical studies of the problem. In this context, it has been shown by Kühn *et al.* [6,7] that there is a more direct relation between the measured data and the density matrix in a field-strength basis which only requires a twofold Fourier transformation to be performed. Later D'Ariano *et al.* [8] had derived explicit relations between the count distributions measured and the density matrix in the photon-number basis.

On the other hand, it is well known that other than four-port schemes may also be used to measure the quantum state of light. A typical example is the combination of two four-port schemes to an eight-port scheme [9–11]. It was shown by Walker [10] that balanced eight-port homodyning is suited to realize all probability operator measures (POMs) of the signal mode. In particular, when two input modes are in the vacuum state the POM specifies (for perfect detection) the Q function [12–14] so that the measured joint difference-count distribution directly yields the Q function of the signal mode. Although the input phases need not be varied, the detection apparatus is more complex than in optical homodyne tomography and the data obtained are more noisy because of the additional vacuum fluctuations that are fed in by the (from the point of view of classical optics) unused input ports in the eight-port scheme.

Since the detection efficiencies are, in general, less than unity, the distributions recorded in optical homodyne tomography are typically convolutions of the desired ones with Gaussian noise distributions whose widths increase with decreasing quantum efficiencies [15]. Reconstruction of the (true) quantum state from the data measured therefore necessitates performing a deconvolution, which can formally be included in the reconstruction procedure for the density matrix, as has been shown for both the field-strength [6,7] and photon-number bases [16]. Application of the results requires particular care because the deconvolution may lead to an amplification of the errors associated with the inaccuracies in the measured data. In balanced homodyne eight-port detection the situation is quite similar. Here, the POM specifies a convolution of the Q function with a Gaussian noise distribution, which implies measurement of an s-parametrized quasidistribution, the value of s being less than minus one [7,17]. It should be noted that the only scheme known so far which does not require a deconvolution of the data measured is the recently proposed method of reconstruction of the (vibrational) quantum state of a trapped ion [4].

In order to reconstruct the quantum state of multimode light, methods of multiport homodyning may be applied. The quantum theory of multiport homodyning was pioneered by Walker [10] and general relations between the measurements and POMs were studied. The problem of reconstruction of the quantum state of a correlated two-mode radiation field was considered by Raymer *et al.* [18]. An analysis of the

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problem of reconstruction of the quantum state of an arbitrary N-mode radiation field by means of homodyne multiport detection has been given by Kühn *et al.* [19,20], with special emphasis on the relations between the characteristic functions of the count distributions measurable (in nonperfect detection) and the quantum state of the correlated N-mode signal field to be detected.

In the present paper we show that the quantum state of an arbitrary N-mode radiation field can be uniquely obtained from the data measurable in balanced homodyne multiport detection when at least N input ports are unused (or N other than vacuum reference quantum states are used). In such a scheme any kind of input phase variation, such as the variation of N phases in extended tomographic reconstruction of the quantum state of a correlated N-mode field [18–20], becomes superfluous. To give an example, we apply the theory to balanced homodyne six-port detection, which represents the simplest scheme that enables one to directly measure the quantum state of a single-mode field, without phase variation. In particular, we show that for perfect detection the Q function of the field may simply be given by the difference-count distribution in nonrectangular coordinates.

The paper is organized as follows. In Sec. II the theory of reconstruction of the quantum state of a correlated N-mode radiation field by means of multiport optical homodyning is briefly reviewed. Section III is devoted to the reconstruction of an N-mode signal field from the data measured in multiport homodyning in case of unused input ports. In Sec. IV the theory is applied to balanced homodyne six-port detection. A summary and some concluding remarks are given in Sec. V.

II. MULTIPORT HOMODYNING

Let us consider an N-mode radiation field whose quantum state is given by the density operator $\hat{\varrho}$ or, equivalently, by its characteristic function [21]

$$\Phi(\{\alpha_j\}) = \mathrm{Tr}\{\hat{\varrho}\hat{D}(\{\alpha_j\})\}. \tag{1}$$

Here, $\hat{D}(\{\alpha_j\})$ is the N-mode coherent displacement operator

$$\hat{D}(\{\alpha_j\}) = \exp\left[\sum_{j=1}^{N} (\alpha_j \hat{a}_j^\dagger - \alpha_j^* \hat{a}_j)\right], \tag{2}$$

where $\hat{a}_j^\dagger$ and $\hat{a}_j$, respectively, are the photon creation and destruction operators associated with the modes. The knowledge of $\Phi(\{\alpha_j\})$ enables one to calculate the density operator in any representation. Since the characteristic functions of the joint count distributions measured in multiport homodyning are closely related to the characteristic functions of the signal fields [10,19], multiport homodyne detection can be used to measure the quantum state of an N-mode radiation field.

Let us consider a multiport device and assume that N signal modes (channels $1, \ldots, N$) and a strong local oscillator (channel $N+1$) of equal frequencies are mixed to obtain $(N+1)$ output modes falling onto photodetectors whose counts m_k ($k=1, \ldots, N+1$) are recorded. Restricting attention to lossless devices that respond linearly to the input modes, the input and output photon operators $\hat{a}_k$ and $\hat{b}_k$, respectively, are related to each other by unitary transformations $U_{kk'} = |U_{kk'}|\exp(i\varphi_{kk'})$,

$$\hat{b}_k = \sum_{k'=1}^{N+1} U_{kk'}\hat{a}_{k'}. \tag{3}$$

Note that any discrete finite-dimensional unitary matrix can be constructed in the laboratory using devices, such as beam splitters, phase shifters, and mirrors [22].

In this way, $(N+1)$ input modes are transformed into $(N+1)$ output modes. Their simultaneous detection yields an $(N+1)$-fold joint count distribution $p(\{m_k\})$ whose characteristic function $\Omega(\{x_k\})$ can be obtained by Fourier transformation,

$$\Omega(\{x_k\}) = \sum_{\{m_k\}} p(\{m_k\}) \exp\left(i\sum_{k=1}^{N+1} x_k m_k\right). \tag{4}$$

From standard four-port detection it is already known that difference-count measurements have advantages in reducing the local-oscillator noise. In the multiport scheme under study it may therefore be advantageous to consider difference counts by choosing a reference output channel:

$$D_{lr} = \frac{m_l}{\eta_l|\alpha|} - \frac{|U_{lN+1}|^2}{|U_{rN+1}|^2}\frac{m_r}{\eta_r|\alpha|} \tag{5}$$

($l=1, \ldots, r-1, r+1, \ldots, N+1$, r denoting the reference channel). Here and in the following scaled counts $m_k/(\eta_k|\alpha|)$ are used, where η_k and $\alpha = |\alpha|\exp(i\varphi_\alpha)$, respectively, are the detector efficiencies and the complex amplitude of the local oscillator prepared in a coherent state $|\alpha\rangle$. The characteristic function $\Omega_{\mathrm{sdc}}(\{x_l\})$ of the N-fold joint scaled difference-count distribution $p_{\mathrm{sdc}}(\{D_{lr}\})$ is related to $\Omega(\{x_k\})$, Eq. (4), as

$$\Omega_{\mathrm{sdc}}(\{x_l\}) = \Omega\left(\left\{\frac{x_k}{\eta_k|\alpha|}\right\}\right) \times \exp\left\{-i\sum_{k=1}^{N+1} |U_{kN+1}|^2|\alpha|x_k\right\}, \tag{6}$$

where $x_k = x_l$ if $k \neq r$, and

$$x_r = -\sum_{l=1\,(\neq r)}^{N+1} \frac{|U_{lN+1}|^2}{|U_{rN+1}|^2} x_l. \tag{7}$$

The characteristic function of the joint count distribution in Eq. (6), $\Omega(\{x_k/(\eta_k|\alpha|)\})$, can be calculated using the standard theory of photoelectric detection of light. It can be shown that when the local oscillator is sufficiently strong, then Eq. (6) may be written as [10,19]

$$\Omega_{\mathrm{sdc}}(\{x_l\}) = \Phi(\{\beta_n\})\exp\left[-\frac{1-\eta}{2\eta}\sum_{n=1}^{N} |\beta_n|^2\right], \tag{8}$$

where, for simplicity, equal detection efficiencies have been assumed, and

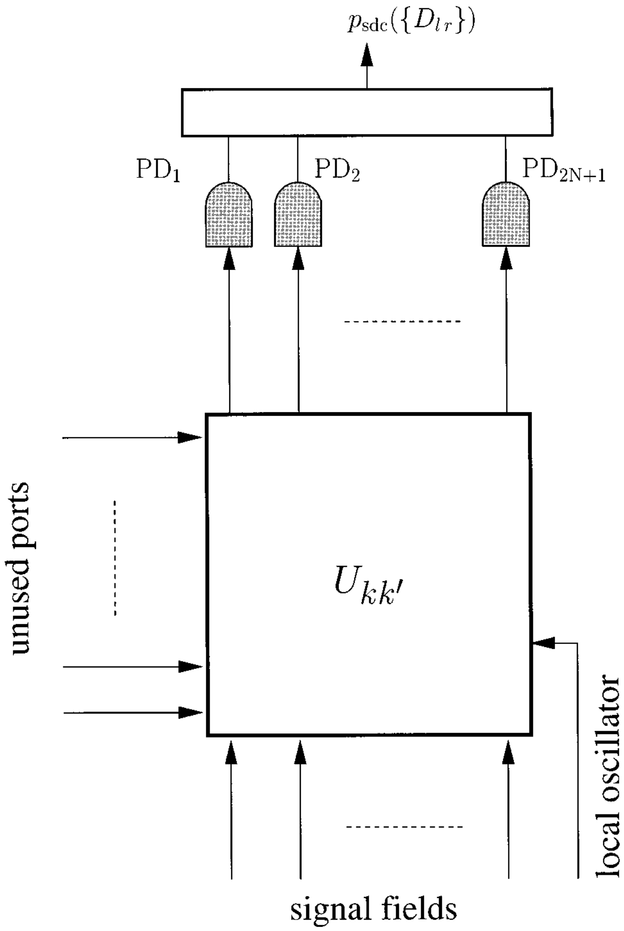

FIG. 1. Scheme of balanced multiport homodyning with unused input ports. The photodetectors PD_i $(i=1,\ldots,2N+1)$ are used to record the joint-difference count statistics $D_{\rm sdc}(\{D_{lr}\})$ in the output channels $(l=1,\ldots,r-1,r+1,\ldots,N+1$, r denoting the reference channel).

$$\beta_n = ie^{i\varphi_\alpha}\sum_{k=1}^{N+1} U_{kN+1}(U_{kn})^* x_k . \tag{9}$$

In the more general case, when the N modes belong to R groups that may differ in frequency, R subdevices of the type described above can be combined to transform the $(N+R)$ input modes into $(N+R)$ output modes. Their simultaneous detection then yields an $(N+R)$-fold joint count distribution. The generalization of the theory to this case is straightforward [19].

Equation (8) reveals that in the case of perfect detection the characteristic function of the joint scaled difference-count distribution represents, for appropriately chosen arguments, nothing else than the characteristic function of the quantum state of the N-mode signal field. To obtain $\Phi(\{\beta_n\})$ for arbitrary arguments, the $\beta_n=|\beta_n|\exp(i\varphi_{\beta_n})$ must be allowed to attain arbitrary complex values (note that the x_l are real quantities). This may be achieved by appropriately varying N phases in Eq. (9), which implies (successive) measurement of a set of joint difference-count distributions [19]. The method may be regarded as an extension of optical homodyne tomography to multimode fields.

III. QUANTUM-STATE RECONSTRUCTION IN THE CASE OF UNUSED INPUT PORTS

Equation (8) also offers the possibility to obtain the characteristic function of an N-mode signal field from the joint difference-count distribution measured in one experiment, provided that additional modes in reference quantum states (also called the quantum-ruler states) are introduced. In the simplest case unused input channels can be introduced, so that the reference modes are in the vacuum state. Let us suppose that the $2(N+1)$-port device is extended to a $2(2N+1)$-port device, N input channels being unused (see Fig. 1). Since the vacuum inputs are not correlated to each other and to the signal input, the characteristic function of the quantum state of the $(2N)$-mode radiation field can be factored according to

$$\Phi(\{\beta_n\}) = \Phi(\{\beta_{n_{\rm s}}\})\prod_{n_{\rm v}=N+1}^{2N}\Phi_{\rm v}(\beta_{n_{\rm v}}) \tag{10}$$

($n_{\rm s}=1,\ldots,N$, signal channels; $n_{\rm v}=N+1,\ldots,2N$, vacuum channels; $\{\beta_n\}=\{\beta_{n_{\rm s}},\beta_{n_{\rm v}}\}$), where $\Phi(\{\beta_{n_{\rm s}}\})$ is the desired characteristic function of the quantum state of the N-mode signal field, and

$$\Phi_{\rm v}(\beta_{n_{\rm v}}) = \exp\left[-\frac{1}{2}|\beta_{n_{\rm v}}|^2\right] \tag{11}$$

is a vacuum-input characteristic function.

Applying Eqs. (8) and (9), with $2N$ in place of N, and using Eq. (10) together with Eq. (11), we obtain the characteristic function of the quantum state of the N-mode signal field from the characteristic function of the $(2N)$-fold joint scaled difference-count distribution as

$$\Phi(\{\beta_{n_{\rm s}}\}) = \exp\left[\frac{1-\eta}{2\eta}\sum_{n=1}^{2N}|\beta_n|^2\right] \times\exp\left[\frac{1}{2}\sum_{n_{\rm v}=N+1}^{2N}|\beta_{n_{\rm v}}|^2\right]\Omega_{\rm sdc}(\{x_l\}), \tag{12}$$

where

$$\beta_n = ie^{i\varphi_\alpha}\sum_{k=1}^{2N+1} U_{kN+1}(U_{kn})^* x_k \tag{13}$$

$(l=1,\ldots,r-1,r+1,\ldots,2N+1)$. Inverting Eq. (13) and recalling that the x_l are real, we obtain

$$x_l = \sum_{n=1}^{2N}\left|\frac{U_{ln}\beta_n}{U_{l\,2N+1}}\right|\sin(\varphi_{ln}-\varphi_{l\,2N+1}+\varphi_{\beta_n}-\varphi_\alpha), \tag{14}$$

$$\sum_{n_{\rm s}=1}^{N}|U_{ln_{\rm s}}\beta_{n_{\rm s}}|\cos(\varphi_{ln_{\rm s}}-\varphi_{l\,2N+1}+\varphi_{\beta_{n_{\rm s}}}-\varphi_\alpha) = -\sum_{n_{\rm v}=N+1}^{2N}|U_{ln_{\rm v}}\beta_{n_{\rm v}}|\cos(\varphi_{ln_{\rm v}}-\varphi_{l\,2N+1}+\varphi_{\beta_{n_{\rm v}}}-\varphi_\alpha). \tag{15}$$

We see that in Eq. (14) the values of the $|\beta_{n_s}|$ and $\varphi_{\beta_{n_s}}$ can be varied freely. For chosen values of $|\beta_{n_s}|$ and $\varphi_{\beta_{n_s}}$ the conditions (15) that ensure that the x_l are real can always be satisfied, because they are simply the conditional equations for the values of $|\beta_{n_v}|$ and $\varphi_{\beta_{v_s}}$.

In other words, there is a one-to-one correspondence between the $(2N)$-fold joint difference-count distribution and the quantum state of an N-mode signal field in a detection scheme with N unused input channels. The result reveals that at least one local oscillator per group of (equal-frequency) signal modes and one unused input port per signal mode are required to reconstruct the quantum state of a multimode signal field from the joint difference-count distribution measured in the experiment. Clearly, both the number of local oscillators and the number of the unused input ports may be increased and the quantum state of the signal field may of course be reconstructed from an appropriately chosen joint difference-count distribution recorded in such an extended measurement scheme.

A typical example is the measurement of the Q function. When a single-mode signal field and a vacuum field are superimposed by a (1/2:1/2) beam splitter, the joint probability distribution of the field strengths in the two output modes is known to be related to the Q function of the signal field. For this reason, balanced homodyne eight-port detection has commonly been regarded as the method that is most adequate to direct measurement of the Q function [12–14]. On the other hand, from the classical theory a 3×3 coupler is already known to be the minimum required in order to determine the complex amplitude of a single-mode signal field [10]. Similarly, from the quantum theory a six-port scheme is expected to also be the minimum required for the detection of the single-mode quantum state in terms of the Q function.

IV. BALANCED SIX-PORT HOMODYNE DETECTION

Let us consider a homodyne six-port detection scheme of the type shown in Fig. 2 . We suppose that the beam-splitter coupling ratios and optical path lengths are chosen in such a way that the output and input photon operators are related to each other through the unitary transformation matrix [10]

$$\mathbf{U}=\frac{1}{\sqrt{3}}\begin{pmatrix}\phi & \phi^* & 1\\ \phi^* & \phi & 1\\ 1 & 1 & 1\end{pmatrix},\quad \phi=\exp\left(-i\frac{2\pi}{3}\right). \tag{16}$$

A practical realization of this transformation is considered in the Appendix [23]. We further assume that in balanced detection the difference counts $m_{13}=m_1-m_3$ and $m_{23}=m_2-m_3$ are recorded, so that the joint difference-count distribution $p_{\mathrm{dc}}(m_{13},m_{23})$ is obtained. Introducing the scaled difference counts

$$D_{13}=\frac{m_{13}}{\eta|\alpha|},\quad D_{23}=\frac{m_{23}}{\eta|\alpha|} \tag{17}$$

[cf. Eq. (5)], the characteristic function of the joint scaled difference-count distribution $p_{\mathrm{sdc}}(D_{13},D_{23})$ is defined by

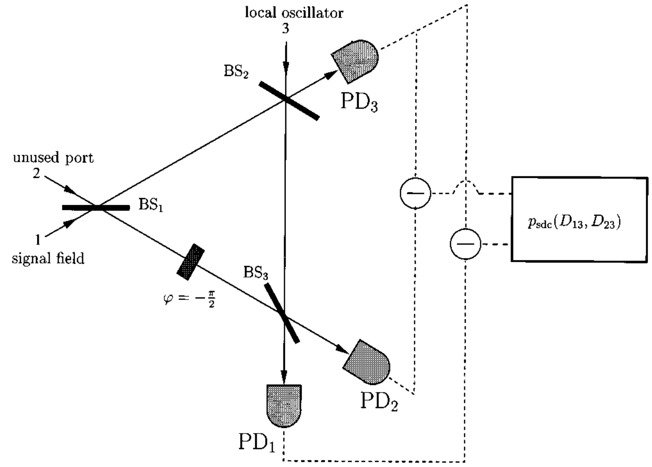

FIG. 2. Scheme of balanced six-port homodyning for the detection of the Q function. Three input fields (signal, local oscillator, vacuum) are combined by three symmetric beam splitters BS$_i$ ($i=1,2,3$) and a $(-\pi/2)$ phase shifter, where two (1/2:1/2) beam splitters (BS$_1$ and BS$_3$) and one (2/3:1/3) beam splitter (BS$_2$) are used. The joint difference statistics is recorded by the detectors PD$_i$ in the output channels.

$$\begin{aligned}\Omega_{\mathrm{sdc}}(x_1,x_2)&=\sum_{D_{13}}\sum_{D_{23}}p_{\mathrm{sdc}}(D_{13},D_{23})e^{iD_{13}x_1+iD_{23}x_2}\\&=\sum_{m_{13}}\sum_{m_{23}}p_{\mathrm{dc}}(m_{13},m_{23})\\&\quad\times\exp\left(i\frac{m_{13}}{\eta|\alpha|}x_1+i\frac{m_{23}}{\eta|\alpha|}x_2\right),\end{aligned} \tag{18}$$

where, for simplicity, $\eta_1=\eta_2=\eta_3=\eta$ has been assumed.

Applying Eq. (12) together with Eq. (13) (note that $N=1$) and recalling the unitary matrix (16), we may relate the characteristic function of the quantum state of the signal field, $\Phi(\beta_1)$, to the characteristic function of the joint scaled difference-count distribution, $\Omega_{\mathrm{sdc}}(x_1,x_2)$, as follows:

$$\begin{aligned}\Phi(\beta_1)&=\exp\left[\frac{1-\eta}{2\eta}(|\beta_1|^2+|\beta_2|^2)\right]\\&\quad\times\exp\left(\frac{1}{2}|\beta_2|^2\right)\Omega_{\mathrm{sdc}}(x_1,x_2),\end{aligned} \tag{19}$$

where

$$\beta_1=\frac{1}{3}ie^{i\varphi_\alpha}[(\phi^*-1)x_1+(\phi-1)x_2], \tag{20}$$

$$\beta_2=\frac{1}{3}ie^{i\varphi_\alpha}[(\phi-1)x_1+(\phi^*-1)x_2]. \tag{21}$$

Comparing Eqs. (20) and (21), we find that $|\beta_2|=|\beta_1|$, and hence Eq. (19) simplifies to

$$\Phi(\beta_1)=\exp\left(\frac{1}{2}|\beta_1|^2\right)\exp\left(\frac{1-\eta}{\eta}|\beta_1|^2\right)\Omega_{\mathrm{sdc}}(x_1,x_2). \tag{22}$$

From inspection of Eqs. (20) and (21) we see that variation of x_1 and x_2 along the real axis corresponds to variation of β_1 over the whole complex plane. As expected, the quantum state of the signal field is uniquely related to the joint difference-count distribution measured in the six-port junction.

The measured joint difference-count distributions can be related to s-parametrized quasidistributions of the signal field, $P(\alpha_1;s)$, with $s\leqslant -1$. For this purpose we express the quasidistributions in terms of their characteristic functions $\Phi(\beta_1;s)$,

$$P(\alpha_1;s)=\frac{1}{\pi^2}\int d^2\beta_1\Phi(\beta_1;s)\exp(\alpha_1\beta_1^*-\alpha_1^*\beta_1), \tag{23}$$

and recall that $\Phi(\beta_1;s)$ and $\Phi(\beta_1)\equiv\Phi(\beta_1;0)$ are related to each other as [24]

$$\Phi(\beta_1;s)=\exp\left(\frac{1}{2}s|\beta_1|^2\right)\Phi(\beta_1). \tag{24}$$

Combining Eqs. (22) and (24) yields

$$\Phi(\beta_1;s)=\exp\left[\frac{1}{2}|\beta_1|^2\left(s+\frac{2-\eta}{\eta}\right)\right]\Omega_{\mathrm{sdc}}(x_1,x_2), \tag{25}$$

which reveals that

$$\Phi\left(\beta_1;s=-\frac{2-\eta}{\eta}\right)=\Omega_{\mathrm{sdc}}(x_1,x_2). \tag{26}$$

In other words, the characteristic function of the joint difference-count distribution is (for appropriately transformed and scaled variables) the characteristic function of the signal-field quasidistribution $P(\alpha_1;s)$, with $s=-(2-\eta)/\eta$. In particular, when perfect detectors are used ($\eta=1$) the characteristic function of the Q function ($s=-1$) is observed. Inverting Eq. (18), we may write

$$\begin{aligned}p_{\mathrm{sdc}}(D_{13},D_{23})&=\eta^2|\alpha|^2p_{\mathrm{dc}}(m_{13},m_{23})\\&=\frac{1}{4\pi^2}\int dx_1\int dx_2\\&\quad\times\exp\left(-i\frac{m_{13}}{\eta|\alpha|}x_1-i\frac{m_{23}}{\eta|\alpha|}x_2\right)\Omega_{\mathrm{sdc}}(x_1,x_2).\end{aligned} \tag{27}$$

Note that in Eq. (18) D_{13} and D_{23} are effectively continuous variables, because of the strong local oscillator, which implies that the sums may be regarded as integrals. We now substitute in Eq. (27) for $\Omega_{\mathrm{sdc}}(x_1,x_2)$ the function $\Phi[\beta_1;s=-(2-\eta)/\eta]$ [Eq. (26)] and invert Eq. (20),

$$x_1=\frac{i}{|\alpha|}(\sqrt{\phi^*}\alpha^*\beta_1-\sqrt{\phi}\alpha\beta_1^*), \tag{28}$$

$$x_2=\frac{i}{|\alpha|}(\sqrt{\phi}\alpha^*\beta_1-\sqrt{\phi^*}\alpha\beta_1^*), \tag{29}$$

in order to change the variables of integration ($x_1,x_2\rightarrow\mathrm{Re}\{\beta_1\},\mathrm{Im}\{\beta_1\}$). Noting that $dx_1dx_2=4(\sqrt{3}/2)d^2\beta_1$ and recalling Eq. (23), we easily find that Eq. (27) can be rewritten to obtain the result that

$$\begin{aligned}p_{\mathrm{dc}}(m_{13},m_{23})&=\frac{\sqrt{3}}{2\eta^2|\alpha|^2}\\&\quad\times P\left(\alpha_1=-\frac{m_{13}}{\eta\alpha^*}\sqrt{\phi}-\frac{m_{23}}{\eta\alpha^*}\sqrt{\phi^*};s=-\frac{2-\eta}{\eta}\right).\end{aligned} \tag{30}$$

Equation (30) reveals that the measurement of the joint difference-count distribution in balanced six-port homodyne detection is equivalent to the measurement of the quasidistribution $P[\alpha_1;s=-(2-\eta)/\eta]$ of the signal field. The joint difference statistics as a function of the difference events m_{13},m_{23} directly yields a representation of the corresponding quasidistribution in terms of nonorthogonal coordinates in phase space. This quasidistribution approaches the Q function as the detection efficiency tends to unity $\{P[\alpha_1;s=-(2-\eta)/\eta]\rightarrow P(\alpha_1;-1)=Q(\alpha_1)$ for $\eta\rightarrow 1\}$. The distribution $P[\alpha_1;s=-(2-\eta)/\eta]$ can be expressed in terms of the Q function as

$$\begin{aligned}P&\left(\alpha_1;s=-\frac{2-\eta}{\eta}\right)\\&=\frac{\eta}{\pi(1-\eta)}\int d^2\alpha_1'Q(\alpha_1')\exp\left(-\frac{\eta|\alpha_1-\alpha_1'|^2}{1-\eta}\right).\end{aligned} \tag{31}$$

It is worth noting that $P[\alpha_1;s=-(2-\eta)/\eta]$ is the same quasidistribution as in the more complex eight-port scheme, where the signal field, the local-oscillator, and two vacuum fields (arising from two unused input channels) are superimposed and two difference signals of the four output fields are recorded [7,17]. Consequently, our result shows that two channels in such a detection scheme are superfluous.

V. SUMMARY AND CONCLUSIONS

Based on the quantum theory of multiport homodyne detection we have studied the quantum state measurement of N-mode signal fields by recording $(2N)$-fold joint difference-count distributions, in place of tomographic sets of N-fold joint difference-count distributions. Considering a signal field consisting of N modes of equal frequencies, we have shown that the entire information required to obtain the quantum state of the signal field can be extracted from a $(2N)$-fold joint difference-count distribution measured in $2(2N+1)$-port homodyne detection, where the input consists of N signal modes, one local oscillator mode, and N modes in appropriately chosen reference quantum states. This scheme has the minimum numbers of reference and local-oscillator inputs that are required in order to solve the problem.

We have applied the method to balanced homodyne six-port detection. Combining a single-mode signal field, a (strong) local oscillator, and a vacuum input by a linear lossless apparatus, we have shown that the joint difference-count measurement yields the quasidistribution $P(\alpha_1;s)$, $s=-(2-\eta)/\eta$, of the signal field, which for perfect detection ($\eta=1$) is the Q function. It is worth noting that this measurement scheme is fully equivalent to balanced homodyne eight-port detection. This shows that one input channel (and hence one output channel) in the eight-port scheme is superfluous for measuring the quantum state of a single-mode field.

Applications of the method to reconstruction of the quantum state of multimode light through unused input ports may closely follow the line shown for single-mode light. In particular, the theory reveals that a ten-port scheme already exhibits the minimum number of ports required to measure the quantum state of two correlated modes of equal frequency in terms of the two-mode Q function.

ACKNOWLEDGMENT

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APPENDIX: TRANSFORMATION MATRIX OF THE SIX-PORT SCHEME

Let us consider the unitary transformation corresponding to the six-port scheme in Fig. 2. Each of the symmetric beam splitters combines two input modes (operators $\hat{a}_1$ and $\hat{a}_2$) to give two output modes (operators $\hat{b}_1$ and $\hat{b}_2$) as

$$\hat{b}_1=R\hat{a}_1+T\hat{a}_2,$$
$$\hat{b}_2=T\hat{a}_1+R\hat{a}_2, \tag{A1}$$

where T and R, respectively, are the (complex) transmittance and reflectance of the beam splitter,

$$T=|T|e^{i\varphi_T}, \quad R=|R|e^{i\varphi_R}, \tag{A2}$$

which satisfy the conditions

$$|T|^2+|R|^2=1, \tag{A3}$$

$$\varphi_T-\varphi_R=\pm\frac{1}{2}\pi. \tag{A4}$$

Applying the beam splitter relations (A1) step by step to the scheme under study and taking into account the $(-\pi/2)$ phase shifter (see Fig. 2), the transformation matrix $\mathbf{U}$ defined in Eq. (3) is derived to be

$$\mathbf{U}=\begin{pmatrix} R_2T_1T_3-iR_1R_3 & R_1R_2T_3-iR_3T_1 & T_2T_3 \\ R_2R_3T_1-iR_1T_3 & R_1R_2R_3-iT_1T_3 & R_3T_2 \\ T_1T_2 & R_1T_2 & R_2 \end{pmatrix}, \tag{A5}$$

where T_i and R_i, respectively, are the transmittance and reflectance of the ith beam splitter BS_i ($i=1,2,3$). Using two (1/2:1/2) beam splitters (BS_1 and BS_3) and a (2/3:1/3) beam splitter (BS_2), that is to say, $|T_1|^2/|R_1|^2=|T_3|^2/|R_3|^2=1$ and $|T_2|^2/|R_2|^2=2$, and assuming that $\varphi_{R_i}=0$ and $\varphi_{T_i}=\pi/2$, Eq. (A5) reduces to

$$\mathbf{U}=\frac{1}{\sqrt{3}}\begin{pmatrix} \phi & -i\phi^* & -1 \\ -i\phi^* & -\phi & i \\ -1 & i & 1 \end{pmatrix}, \quad \phi=\exp\left(-i\frac{2\pi}{3}\right). \tag{A6}$$

This choice of the beam splitter parameters ensures that in balanced detection all the input channels have equal weight. Introducing the matrix

$$\mathbf{U}'=\mathbf{PUP}, \tag{A7}$$

where

$$\mathbf{P}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{A8}$$

we easily see that $\mathbf{U}'$ takes the form (16) (note that $\mathbf{U}$, $\mathbf{P}$, and $\mathbf{U}'$ are unitary matrices).

With regard to the measurement under consideration, the matrices $\mathbf{U}$ and $\mathbf{U}'$ are, of course, equivalent to each other. Multiplications by phase factors of the rows and columns of $\mathbf{U}$ introduce phase shifts of the output and input fields, respectively. The phase shifts of the output fields are irrelevant since they are not recorded by the photodetectors. The multiplications by phase factors of the second and third columns of $\mathbf{U}$ introduce an irrelevant phase shift of the vacuum input and a phase shift of the local oscillator. The latter simply gives rise to a redefinition of the (absolute) phase φ_α of the complex number α in Eq. (30).

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