Quantum state reconstruction and detection of quantum coherences on different observation levels

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(Received 16 May 1995; revised manuscript received 4 January 1996)

We apply the Jaynes principle of maximum entropy [Phys. Rev. **106**, 620 (1957); **108**, 171 (1957)] for a reconstruction of Wigner functions of quantum-mechanical states of light on different observation levels. We study how quantum interference between components of superpositions of coherent states, which is responsible for the appearance of nonclassical effects, can be detected on different observation levels. We analyze in detail the reconstruction of Wigner functions of squeezed states on different observation levels in the case of nonunit detection efficiency modeled as a decay of the state under consideration into a zero-temperature reservoir. [S1050-2947(96)08706-9]

PACS number(s): 42.50.Dv, 03.65.Bz, 05.30.Ch

I. INTRODUCTION

The Wigner function [1] of the quantum-mechanical state which is described by the density operator $\hat{\rho}$ can, in principle, be reconstructed either via a set of single-observable measurements (the so-called optical homodyne tomography [2,3]) or via a simultaneous measurement of two noncommuting observables (see, for instance, the concept of propensities as discussed by Wódkiewicz [4] and others [5,6]). The completely reconstructed Wigner function, or equivalently the reconstructed density operator, contains information about *all* independent moments of the system operators, i.e., in the case of the quantum harmonic oscillator the knowledge of the Wigner function is equivalent to the knowledge of all moments $\langle(\hat{a}^\dagger)^m\hat{a}^n\rangle$ of the creation $(\hat{a}^\dagger)$ and annihilation $(\hat{a})$ operators.

In many cases it turns out that the state under consideration is characterized by an *infinite* number of independent moments $\langle(\hat{a}^\dagger)^m\hat{a}^n\rangle$ (for all m and n). To perform a *complete* measurement of these moments can take an infinite time. This means that even though the Wigner function can in principle be reconstructed the collection of experimental data takes an infinite time. In addition, the data processing and numerical reconstruction of the Wigner function are time consuming as well. Therefore an experimental realization of the reconstruction of the Wigner function can be questionable.

In practice, it is possible to perform a measurement of just a finite number of independent moments of the system operators. The aim of this paper is to analyze how the Wigner function of a quantum state of a single-mode light field can be (partially) reconstructed from not necessarily complete data obtained from the measurement of the system (i.e., from a finite number of moments of system operators). Simultaneously we address the question of how to quantify the precision with which the Wigner function is reconstructed. To accomplish this task we utilize the concept of observation levels [7], where each observation level is specified by a set of linearly independent operators $\hat{G}_\nu$ $(\nu=1,2,\ldots,n)$ for which expectation values G_ν are given (measured). With the help of the Jaynes principle of maximum entropy [8] (see also [7,9]) we will show how to reconstruct in the most reliable way the Wigner function of the measured state within a given observation level. In addition, we analyze how quantum coherences can be detected on different observation levels. In other words, we address the problem: *Which is the most incomplete observation level which still allows us to distinguish between a pure state and the corresponding statistical mixture?* To model nonunit efficiency measurements we analyze the ‘‘decay’’ of quantum-mechanical states into a zero-temperature reservoir (heat bath). The paper is organized as follows. In Sec. II we briefly review basic elements of the phase-space formalism used in quantum optics. In Sec. III we introduce the concept of observation levels applied to quantum optics. In Sec. IV we show how with the help of the maximum entropy principle Wigner functions on given observation levels can be reconstructed. In Sec. V we analyze Wigner functions of a squeezed vacuum state of light on different observation levels. Section VI is devoted to a discussion of detection of quantum coherences on different observation levels and description of the decay of superposition states of light. We finish our paper with conclusions.

II. PHASE-SPACE DESCRIPTION OF STATES OF A SINGLE-MODE FIELD

Utilizing a close analogy between the operator for the electric component $\hat{E}(r,t)$ of a monochromatic light field and the quantum-mechanical harmonic oscillator, we will

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consider a dynamical system which is described by a pair of canonically conjugated Hermitian observables $\hat{q}$ and $\hat{p}$, with $[\hat{q},\hat{p}]=i\hbar$. Eigenvalues q and p of these operators range continuously from $-\infty$ to $+\infty$. The annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ can be expressed as a complex linear combination of $\hat{q}$ and $\hat{p}$:

$$\hat{a}=\frac{1}{\sqrt{2\hbar}}(\lambda\hat{q}+i\lambda^{-1}\hat{p}),\quad \hat{a}^\dagger=\frac{1}{\sqrt{2\hbar}}(\lambda\hat{q}-i\lambda^{-1}\hat{p}), \tag{2.1}$$

where λ is an arbitrary real parameter. The operators $\hat{a}$ and $\hat{a}^\dagger$ obey the Weyl-Heisenberg commutation relation $[\hat{a},\hat{a}^\dagger]=1$.

A particularly useful set of states is the overcomplete set of coherent states $|\alpha\rangle$ which are the eigenstates of the annihilation operator $\hat{a}$, i.e., $\hat{a}|\alpha\rangle=\alpha|\alpha\rangle$. These coherent states can be generated from the vacuum state $|0\rangle$ [defined as $\hat{a}|0\rangle=0$] by the action of the unitary displacement operator $\hat{D}(\alpha)$ [6],

$$\hat{D}(\alpha)\equiv\exp[\alpha\hat{a}^\dagger-\alpha^*\hat{a}],\quad |\alpha\rangle=\hat{D}(\alpha)|0\rangle. \tag{2.2}$$

The parametric space of eigenvalues, i.e., the *phase space* for our dynamical system, is the *infinite* plane of eigenvalues (q,p) of the Hermitian operators $\hat{q}$ and $\hat{p}$. An equivalent phase space is the complex plane of eigenvalues

$$\alpha=\frac{1}{\sqrt{2\hbar}}(\lambda q+i\lambda^{-1}p) \tag{2.3}$$

of the annihilation operator $\hat{a}$. The parameters q and p in Eq. (2.3) can be interpreted as the expectation values of the operators $\hat{q}$ and $\hat{p}$ in the state $|\alpha\rangle$.

The phase-space description of the quantum-mechanical oscillator which is in the state described by the density operator $\hat{\rho}=|\Psi\rangle\langle\Psi|$ (in what follows we will consider mainly pure states) is based on the definition of the Wigner function [1] $W_{|\Psi\rangle}(\xi)$. The Wigner function of the system described by the density operator $\hat{\rho}$ is defined as [10]

$$W_{|\Psi\rangle}(\xi)=\frac{1}{\pi}\int \mathrm{Tr}[\hat{\rho}\hat{D}(\eta)]\exp(\xi\eta^*-\xi^*\eta)d^2\eta, \tag{2.4}$$

where $\hat{D}(\eta)$ is given by Eq. (2.2).

The Wigner function can also be defined as a particular Fourier transform of the density operator expressed in the basis of the eigenvectors $|q\rangle$ of theposition operator $\hat{q}$:

$$W_{|\Psi\rangle}(q,p)\equiv\int_{-\infty}^{\infty}d\zeta\langle q-\zeta/2|\hat{\rho}|q+\zeta/2\rangle e^{ip\zeta/\hbar}. \tag{2.5}$$

Both definitions (2.4) and (2.5) of the Wigner function are identical (see Hillery *et al.* [1]), providing the parameters ξ and ξ^* are related to the coordinates q and p of the phase space as

$$\xi=\frac{1}{\sqrt{2\hbar}}(\lambda q+i\lambda^{-1}p),\quad \xi^*=\frac{1}{\sqrt{2\hbar}}(\lambda q-i\lambda^{-1}p). \tag{2.6}$$

The Wigner function can be interpreted as the quasiprobability density distribution (see below) through which a probability can be expressed to find a quantum-mechanical system (harmonic oscillator) around the ''point'' (q,p) of the phase space.

III. MAXIMUM ENTROPY PRINCIPLE AND OBSERVATION LEVELS

The state of a quantum system can always be described by a statistical density operator $\hat{\rho}$. Depending on the system preparation, the density operator represents either a pure quantum state (complete system preparation) or a statistical mixture of pure states (incomplete preparation). The degree of deviation of a statistical mixture from the pure state can be best described by the *uncertainty measure* $\eta[\hat{\rho}]$ (see [7,9]),

$$\eta[\hat{\rho}]=-k_B\,\mathrm{Tr}(\hat{\rho}\,\ln\hat{\rho}), \tag{3.1}$$

where k_B is the Boltzmann constant. The uncertainty measure $\eta[\hat{\rho}]$ is equal to zero for pure states and $\eta[\hat{\rho}]>0$ for statistical mixtures. For an isolated system the uncertainty measure is a constant of motion, i.e., $d\eta(t)/dt=0$.

A. Maximum entropy principle

There are situations when instead of the density operator $\hat{\rho}$, expectation values G_ν of a set $\mathcal{O}$ of operators $\hat{G}_\nu$ $(\nu=1,\ldots,n)$ are given. The set of linearly independent operators is referred to as the *observation level* $\mathcal{O}$ [7]. A large number of density operators $\hat{\rho}_{\{\hat{G}\}}$ which fulfill the conditions

$$\mathrm{Tr}\hat{\rho}_{\{\hat{G}\}}=1, \tag{3.2a}$$

$$\mathrm{Tr}(\hat{\rho}_{\{\hat{G}\}}\hat{G}_\nu)=G_\nu,\quad \nu=1,2,\ldots,n, \tag{3.2b}$$

can be found for a given set of expectation values $G_\nu=\langle\hat{G}_\nu\rangle$. Each of these density operators $\hat{\rho}_{\{\hat{G}\}}$ can possess a different value of the uncertainty measure $\eta[\hat{\rho}_{\{\hat{G}\}}]$. If we wish to use only the expectation values G_ν of the chosen observation level for determining the density operator, we must select a particular density operator $\hat{\rho}_{\{\hat{G}\}}=\hat{\sigma}_{\{\hat{G}\}}$ in an unbiased manner. According to the Jaynes principle of maximum entropy [8] this density operator $\hat{\sigma}_{\{\hat{G}\}}$ must be the one which has the largest uncertainty measure $\eta[\hat{\sigma}_{\{\hat{G}\}}]$ and simultaneously fulfills constraints (3.2). As a consequence,

$$\begin{aligned}\eta[\hat{\sigma}_{\{\hat{G}\}}]&=-k_B\mathrm{Tr}(\hat{\sigma}_{\{\hat{G}\}}\ln\hat{\sigma}_{\{\hat{G}\}})\geqslant\eta[\hat{\rho}_{\{\hat{G}\}}]\\&=-k_B\mathrm{Tr}(\hat{\rho}_{\{\hat{G}\}}\ln\hat{\rho}_{\{\hat{G}\}})\end{aligned} \tag{3.3}$$

for all possible $\hat{\rho}_{\{\hat{G}\}}$ which fulfill Eqs. (3.2). The variation determining the maximum of $\eta[\hat{\sigma}_{\{\hat{G}\}}]$ under the conditions (3.2) leads to a generalized canonical density operator [8,11]

$$\hat{\sigma}_{\{\hat{G}\}}=\frac{1}{Z_{\{\hat{G}\}}}\exp\left(-\sum_\nu\lambda_\nu\hat{G}_\nu\right), \tag{3.4}$$

$$Z_{\{\hat{G}\}}(\lambda_1,\ldots,\lambda_n)=\mathrm{Tr}\left[\exp\left(-\sum_\nu\lambda_\nu\hat{G}_\nu\right)\right], \tag{3.5}$$

where λ_n are the Lagrange multipliers and $Z_{\{\hat{G}\}}(\lambda_1,\ldots,\lambda_n)$ is the generalized partition function. The Lagrange multipliers can be expressed as functions of the expectation values $\lambda_\nu = \lambda_\nu(G_1, \ldots, G_n)$. The maximum uncertainty measure regarding an observation level $\mathcal{O}_{\{\hat{G}\}}$ will be referred to as the entropy $S_{\{\hat{G}\}} \equiv \eta[\hat{\sigma}_{\{\hat{G}\}}] = -k_B \mathrm{Tr}(\hat{\sigma}_{\{\hat{G}\}} \ln \hat{\sigma}_{\{\hat{G}\}})$. This entropy can be expressed exclusively in terms of the mean values G_ν, i.e., $S_{\{\hat{G}\}} = S(G_1, \ldots, G_n)$.

B. Extension and reduction of the observation level

If an observation level $\mathcal{O}_{\{\hat{G}\}} \equiv \{\hat{G}_1, \ldots, \hat{G}_n\}$ is extended by including further operators $\hat{M}_1, \ldots, \hat{M}_l$, then additional expectation values $M_1 = \langle \hat{M}_1 \rangle, \ldots, M_l = \langle \hat{M}_l \rangle$ can only increase the amount of available information about the state of the system. This procedure is called the *extension* of the observation level (from $\mathcal{O}_{\{\hat{G}\}}$ to $\mathcal{O}_{\{\hat{G},\hat{M}\}}$) and is associated with a decrease of the entropy. The generalized canonical density operator on the observation level $\mathcal{O}_{\{\hat{G},\hat{M}\}}$,

$$\hat{\sigma}_{\{\hat{G},\hat{M}\}} = \frac{1}{Z_{\{\hat{G},\hat{M}\}}} \exp\left(-\sum_{\nu=1}^{n} \lambda_\nu \hat{G}_\nu - \sum_{\mu=1}^{l} \kappa_\mu \hat{M}_\mu \right), \tag{3.6}$$

belongs to the set of density operators $\hat{\rho}_{\{\hat{G}\}}$ which fulfill Eq. (3.2). The entropy $S_{\{\hat{G},\hat{M}\}}$ of the extended observation level $\mathcal{O}_{\{\hat{G},\hat{M}\}}$ can only be smaller than or equal to the entropy $S_{\{\hat{G}\}}$ of the original observation level $\mathcal{O}_{\{\hat{G}\}}$, i.e., $S_{\{\hat{G},\hat{M}\}} \leqslant S_{\{\hat{G}\}}$ [a special case of Eq. (3.3)]. The Lagrange multipliers can be expressed as functions of the expectation values: $\lambda_\nu = \lambda_\nu(G_1, \ldots, G_n, M_1, \ldots, M_l)$ and $\kappa_\mu = \kappa_\mu(G_1, \ldots, G_n, M_1, \ldots, M_l)$. In the special case $\kappa_\mu = 0$ the expectation values M_μ are functions of the expectation values G_ν. This means that the measurement of observables $\hat{M}_\mu$ does not increase information about the system. Consequently, $\hat{\rho}_{\{\hat{G},\hat{M}\}} = \hat{\rho}_{\{\hat{G}\}}$ and $S_{\{\hat{G},\hat{M}\}} = S_{\{\hat{G}\}}$.

We can also consider a *reduction of the observation level* if we decrease the number of independent observables which are measured, e.g., $\mathcal{O}_{\{\hat{G},\hat{M}\}} \to \mathcal{O}_{\{\hat{G}\}}$ (here $\hat{G}_\nu$ and $\hat{M}_\mu$ are independent). This reduction is accompanied by an increase of the entropy due to the decrease of the information available about the state of the system.

C. Time-dependent entropy of an observation level

If the dynamical evolution of the system is governed by the evolution superoperator $\hat{U}(t,t_0)$, such that $\hat{\rho}(t) = \hat{U}(t,t_0)\hat{\rho}(t_0)$, then the expectation values of the operators $\hat{G}_\nu$ on the given observation level at time t are given by $G_\nu(t) = \mathrm{Tr}[\hat{G}_\nu \hat{U}(t,t_0)\hat{\rho}(t_0)]$. By using these time-dependent expectation values as constraints for maximizing the uncertainty measure $\eta[\hat{\rho}_{\{\hat{G}\}}(t)]$, we get the generalized canonical density operator $\hat{\sigma}_{\{\hat{G}\}}$ [see Eq. (3.4)] with the time-dependent Lagrange multipliers $\lambda_\nu(t) = \lambda_\nu(G_1(t), \ldots, G_n(t))$ and the time-dependent entropy $S_{\{\hat{G}\}}(t)$ which is associated with the given observation level. This generalized canonical density operator is not governed by the von Neumann equation but it satisfies an integro-differential equation derived by Robertson [11] (see also [12]). The time-dependent entropy $S_{\{\hat{G}\}}(t)$ is defined for any system that is arbitrarily far from equilibrium. In the case of an isolated system the entropy can increase or decrease during the time evolution (see, for example, the book by Hobson, Ref. [9], Sec. 5.6).

D. Wigner functions on different observation levels

With the help of a generalized canonical density operator $\hat{\sigma}_{\{\hat{G}\}}$ which is associated with our *actual* knowledge about the state of the physical system, we define the Wigner function in the ξ phase space at the corresponding observation level as

$$W_{\{\hat{G}\}}(\xi) = \frac{1}{\pi} \int d^2\eta \, \mathrm{Tr}[\hat{D}(\eta)\hat{\sigma}_{\{\hat{G}\}}] \exp(\xi\eta^* - \xi^*\eta). \tag{3.7}$$

An analogous expression can be found for the Wigner function in the (q,p) phase space [see Eq. (2.5)].

E. Maximum entropy principle and laws of physics

It has been pointed out by Jaynes [8] that there is a strong formal resemblance between the maximum entropy formalism and the rules of calculations in statistical mechanics and thermodynamics. Simultaneously, he has emphasized that the maximum entropy principle ''has nothing to do with the laws of physics.'' In fact, this is the reason why the maximum entropy principle is applicable in so many fields of human activity, such as economy or sociology (for more details, see the book by Kapur and Kesavan [9]). To be more specific, it is worth citing a paragraph from Jaynes' Brandeis lectures (see p. 183 of these lectures [8]): ''Conventional quantum theory has provided an answer to the problem of setting up initial state descriptions only in the limiting case where measurements of a complete set of commuting observables have been made, the density matrix $\hat{\rho}(0)$ then reducing to the projection operator onto a pure state $\psi(0)$ which is the appropriate simultaneous eigenstate of all measured quantities. But there is almost no experimental situation in which we really have all this information, and before we have a theory able to treat actual experimental situations, existing quantum theory must be supplemented with some principle that tells us how to translate, or encode, the results of measurements into a definite state description $\hat{\rho}(0)$. Note that the problem is not to find $\hat{\rho}(0)$ which correctly describes true physical situation.' That is unknown, and always remains so, because of incomplete information. In order to have a usable theory we must ask the much more modest question: *What $\hat{\rho}(0)$ best describes our state of knowledge about the physical situation?*'' In other words, the maximum entropy principle is *the most conservative assignment in the sense that it does not permit one to draw any conclusions not warranted by the data*.

We can conclude that a measurement itself is a physical process and is governed by the laws of physics. On the other hand, formal procedures by means of which we reconstruct information about the system from the measured data are based on certain principles which cannot be directly expressed in terms of the physical laws.

IV. OBSERVATION LEVELS FOR A SINGLE-MODE FIELD

In our paper we will consider two different classes of observation levels; namely, we will consider the phase-sensitive and phase-insensitive observation levels. Phase-sensitive observation levels are related to operators which provide some information about off-diagonal matrix elements of the density operator in the Fock basis (i.e., these observation levels reveal some information about the phase of states under consideration). On the contrary, phase-insensitive observation levels are based exclusively on a measurement of diagonal matrix elements in the Fock basis. Before we proceed to a detailed description of the phase-sensitive and phase-insensitive observation levels, we introduce two exceptional observation levels, the complete and thermal observation levels.

Complete observation level $\mathcal{O}_0 \equiv \{(\hat{a}^\dagger)^k \hat{a}^l; \forall k,l\}$. The set of operators $|n\rangle\langle m|$ (for all n and m) is referred to as the *complete* observation level. The expectation values of the operators $|n\rangle\langle m|$ are the matrix elements of the density operator in the Fock basis,

$$\langle m|\hat{\rho}|n\rangle = \mathrm{Tr}[\hat{\rho}|n\rangle\langle m|], \quad \forall n,m, \tag{4.1}$$

and therefore the generalized canonical density operator is identical with the statistical density operator,

$$\hat{\sigma}_0 = \frac{1}{Z_0}\exp\left[-\sum_{m,n=0}^{\infty} \lambda_{m,n}|n\rangle\langle m|\right] = \hat{\rho}. \tag{4.2}$$

In this case the entropy S_0 is determined by the density operator $\hat{\rho}$ as

$$S_0 = -k_B \mathrm{Tr}[\hat{\sigma}_0 \ln \hat{\sigma}_0] = -k_B \mathrm{Tr}[\hat{\rho}\ln\hat{\rho}]. \tag{4.3}$$

This entropy is usually called the von Neumann entropy [13].

As a consequence of the relation (cf. Sec. 3.3 in [14])

$$|n\rangle\langle m| = \lim_{\varepsilon\to 1}\sum_{k=0}^{\infty}\frac{(-\varepsilon)^k}{k!\sqrt{n!m!}}(\hat{a}^\dagger)^{k+n}\hat{a}^{k+m}, \tag{4.4}$$

the complete observation level $\mathcal{O}_0$ can also be given by a set of operators $\{(\hat{a}^\dagger)^k \hat{a}^l; \forall k,l\}$ or $\{\hat{q}^k \hat{p}^l; \forall k,l\}$. The Wigner function on the complete information level is equal to the Wigner function of the state itself, i.e., $W^{(0)}_{|\Psi\rangle}(\xi) = W_{|\Psi\rangle}(\xi)$.

Thermal observation level $\mathcal{O}_{\mathrm{th}} \equiv \{\hat{a}^\dagger \hat{a}\}$. The total reduction of the complete observation level $\mathcal{O}_0$ results in a thermal observation level $\mathcal{O}_{\mathrm{th}}$ characterized just by one observable, the photon number operator $\hat{n}$, i.e., quantum-mechanical states of light on this observation level are characterized only by their mean photon number $\bar{n} \equiv \langle \hat{n}\rangle$. The generalized canonical density operator of this observation level is the well-known density operator of the harmonic oscillator in thermal equilibrium,

$$\hat{\sigma}_{\mathrm{th}} = \frac{1}{Z_{\mathrm{th}}}\exp[-\lambda_{\mathrm{th}}\hat{n}]. \tag{4.5}$$

To find an explicit expression for the Lagrange multiplier λ_{th} we have to solve the equation $\mathrm{Tr}[\sigma_{\mathrm{th}}\hat{n}] = \bar{n}$, from which we find that

$$\exp(-\lambda_{\mathrm{th}}) = \frac{\bar{n}}{\bar{n}+1}, \tag{4.6}$$

so that the partition function corresponding to the operator $\hat{\sigma}_{\mathrm{th}}$ reads $Z_{\mathrm{th}} = \bar{n}+1$. Now we can rewrite the generalized canonical density operator $\hat{\sigma}_{\mathrm{th}}$ in the Fock basis in the form

$$\hat{\sigma}_{\mathrm{th}} = \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}|n\rangle\langle n|. \tag{4.7}$$

For the entropy of the thermal observation level we find a familiar expression:

$$S_{\mathrm{th}} = k_B(\bar{n}+1)\ln(\bar{n}+1) - k_B \bar{n}\ln\bar{n}. \tag{4.8}$$

The fact that the entropy S_{th} is larger than zero for any $\bar{n} > 0$ reflects the fact that on the thermal observation level *all* states with the same mean photon number are indistinguishable. This is the reason why Wigner functions of different states on the thermal information level are identical. The Wigner function of the state $|\Psi\rangle$ on the thermal observation level is given by the relation

$$W^{(\mathrm{th})}_{|\Psi\rangle}(\xi) = \frac{2}{1+2\bar{n}}\exp\left[-\frac{2|\xi|^2}{1+2\bar{n}}\right]. \tag{4.9}$$

From Eq. (4.8) it also follows that the vacuum state can be completely reconstructed on $\mathcal{O}_{\mathrm{th}}$ because $S_{\mathrm{th}} = 0$ for $\bar{n} = 0$. Extending the thermal observation level we can obtain more ''complete'' Wigner functions, which in the limit of the complete observation level are equal to the Wigner function of the measured state itself, i.e., they are not biased by the lack of information (measured data) about the state.

A. Phase-sensitive observation levels

1. Observation level $\mathcal{O}_1 \equiv \{\hat{a}^\dagger\hat{a}, \hat{a}^\dagger, \hat{a}\}$

We can extend the thermal observation level if in addition to the observable $\hat{n}$ we consider also the measurement of mean values of the operators $\hat{a}$ and $\hat{a}^\dagger$ (that is, a measurement of the observables $\hat{q}$ and $\hat{p}$ is performed). If we denote the (measured) mean values of these operators as $\langle \hat{a}\rangle = \gamma$ and $\langle \hat{a}^\dagger\rangle = \gamma^*$, then the generalized canonical density operator $\hat{\sigma}_1$ can be written as

$$\hat{\sigma}_1 = \frac{1}{Z_1}\exp[-\lambda_1(\hat{a}^\dagger - \gamma^*)(\hat{a} - \gamma)], \tag{4.10}$$

with the partition function Z_1 given by the relation $Z_1 = (1 - e^{-\lambda_1})^{-1}$. To find the Lagrange multiplier λ_1 we have to solve the equation $\mathrm{Tr}[\hat{a}^\dagger \hat{a}\hat{\sigma}_1] = \bar{n}$, from which we find

$$e^{-\lambda_1} = \frac{\bar{n} - |\gamma|^2}{1 + \bar{n} - |\gamma|^2}. \tag{4.11}$$

The entropy S_1 on the observation level $\mathcal{O}_1$ can be expressed in a form very similar to S_{th} [see Eq. (4.8)]:

$$S_1 = k_B[\bar{n} - |\gamma|^2 + 1]\ln[\bar{n} - |\gamma|^2 + 1] - k_B[\bar{n} - |\gamma|^2]\ln[\bar{n} - |\gamma|^2]. \tag{4.12}$$

The Wigner function $W^{(1)}_{|\Psi\rangle}(\xi)$ corresponding to the generalized canonical density operator $\hat{\sigma}_1$ reads

$$W^{(1)}_{|\Psi\rangle}(\xi)=\frac{2}{1+2(\overline{n}-|\gamma|^2)}\exp\left[-\frac{2|\xi-\gamma|^2}{1+2(\overline{n}-|\gamma|^2)}\right]. \quad (4.13)$$

From the expression (4.12) it follows that $S_1=0$ for those states for which $\overline{n}=|\gamma|^2$. In fact, there is only one state with this property. It is a coherent state $|\alpha\rangle$ (2.2). In other words, because of the fact that $S_1=0$, the coherent state can be *completely* reconstructed on the observation level $\mathcal{O}_1$. In this case $W^{(1)}_{|\alpha\rangle}(\xi)=W^{(0)}_{|\alpha\rangle}(\xi)=2\exp[-2|\xi-\alpha|^2]$. For other states $S_1>0$ and therefore to improve our information about the state we have to perform further measurements, i.e., we have to extend the observation level $\mathcal{O}_1$.

2. Observation level $\mathcal{O}_2\equiv\{\hat{a}^\dagger\hat{a},(\hat{a}^\dagger)^2,\hat{a}^2,\hat{a}^\dagger,\hat{a}\}$

One of the possible extensions of the observation level $\mathcal{O}_1$ can be performed with the help of observables $\hat{q}^2$ and $\hat{p}^2$, i.e., when not only the mean photon number $\overline{n}$ and mean values of $\hat{q}$ and $\hat{p}$ are known, but also the variances $\langle(\Delta\hat{q})^2\rangle$, $\langle(\Delta\hat{p})^2\rangle$, and $\langle\{\Delta\hat{q}\Delta\hat{p}\}\rangle$ are measured. On the observation level $\mathcal{O}_2$ we can express the generalized canonical operator $\hat{\sigma}_2$ as

$$\hat{\sigma}_2=\frac{1}{Z_2}\exp\bigg[-\frac{\lambda_2}{2}(\hat{a}^\dagger-\gamma^*)^2 - \frac{\lambda_2^*}{2}(\hat{a}-\gamma)^2-\lambda_1(\hat{a}^\dagger-\gamma^*)(\hat{a}-\gamma)\bigg], \quad (4.14a)$$

where the Lagrange multiplier λ_1 is real while λ_2 can be complex: $\lambda_2=|\lambda_2|e^{-i\theta}$. We can rewrite $\hat{\sigma}_2$ in a form similar to the thermal density operator:

$$\hat{\sigma}_2=\frac{1}{\tilde{Z}_2}\hat{D}(\gamma)\hat{U}(\theta/2)\hat{S}(r) \times\exp[-(\lambda_1^2-|\lambda_2|^2)^{1/2}\hat{a}^\dagger\hat{a}]\hat{S}^\dagger(r)\hat{U}^\dagger(\theta/2)\hat{D}^\dagger(\gamma), \quad (4.14b)$$

where the displacement operator $\hat{D}(\gamma)$ is given by Eq. (2.2), while the operators $\hat{U}(\theta/2)$ and $\hat{S}(r)$ are given by the relations

$$\hat{U}(\theta)=\exp[-i\theta\hat{a}^\dagger\hat{a}],$$

$$\hat{S}(r)=\exp\left[-\frac{ir}{2\hbar}(\hat{q}\hat{p}+\hat{p}\hat{q})\right]=\exp\left[\frac{r}{2}(\hat{a}^{\dagger 2}-\hat{a}^2)\right], \quad (4.15)$$

where $\tanh 2r=-|\lambda_2|/\lambda_1$. The partition function $\tilde{Z}_2$ in Eq. (4.14b) can be evaluated in an explicit form:

$$\tilde{Z}_2^{-1}=1-\exp[-(\lambda_1^2-|\lambda_2|^2)^{1/2}]. \quad (4.16)$$

Instead of finding explicit expressions for the Lagrange multipliers λ_1 and λ_2 we can find solutions for the parameters $\tanh 2r$ and χ defined as

$$\chi=\{\exp[(\lambda_1^2-|\lambda_2|^2)^{1/2}]-1\}^{-1}. \quad (4.17)$$

We express these parameters as

$$\tanh 2r=\frac{|M|}{N+1/2}, \quad \chi=[(N+1/2)^2-|M|^2]^{1/2}-1/2, \quad (4.18)$$

where $N=\overline{n}-|\gamma|^2>0$ and $M=|M|e^{-i\theta}=\zeta-\gamma^2$.

We recall that physical requirements [15] lead to the following restrictions on the parameters N and M:

$$N\geqslant 0, \quad N(N+1)\geqslant|M|^2. \quad (4.19)$$

Once $\tanh 2r$ and χ are found we can reconstruct the Wigner function $W^{(2)}_{|\Psi\rangle}(\xi)$ on the observation level $\mathcal{O}_2$. This Wigner function reads [15]

$$W^{(2)}_{|\Psi\rangle}(\xi)=\frac{1}{[(N+1/2)^2-|M|^2]^{1/2}}\exp\left[-\frac{(N+1/2)|\xi-\gamma|^2-(M^*/2)(\xi-\gamma)^2-(M/2)(\xi^*-\gamma^*)^2}{[(N+1/2)^2-|M|^2]}\right]. \quad (4.20)$$

Analogously, we can find an expression for the entropy S_2:

$$S_2=k_B(\chi+1)\ln(\chi+1)-k_B\chi\ln\chi. \quad (4.21)$$

It has the form of the thermal entropy (4.8) with a mean thermal photon number equal to χ [see Eq. (4.18)].

Using the expression for the Wigner function (4.20) we can rewrite the variances of the position and momentum operators in terms of the parameters N and M as follows:

$$\langle(\Delta\hat{q})^2\rangle=\frac{\hbar}{2}[1+2N+2\mathrm{Re}M],$$

$$\langle(\Delta\hat{p})^2\rangle=\frac{\hbar}{2}[1+2N-2\mathrm{Re}M]. \quad (4.22)$$

The product of these variances reads

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle=\frac{\hbar^2}{4}[(1+2N)^2-4(\mathrm{Re}M)^2]. \quad (4.23a)$$

From the expression (4.21) for the entropy S_2 it is seen that those states for which $N(N+1)=|M|^2$ can be completely reconstructed of the observation level $\mathcal{O}_2$, because for these states $S_2=0$. In fact, it has been shown by Dodonov *et al.* [16] that the states for which $N(N+1)=|M|^2$ are the *only* pure states which have non-negative Wigner functions. For these states the product of variances (4.23a) reads

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle=\frac{\hbar^2}{4}[1+4(\mathrm{Im}M)^2], \tag{4.23b}$$

which means that if in addition $\mathrm{Im}M=0$ (see, for instance, a squeezed vacuum state with a real parameter of squeezing) then these states also belong to the class of minimum uncertainty states. From Eqs. (4.18) and (4.21) it follows that all pure Gaussian states for which $N(N+1)=|M|^2$ can be completely reconstructed on the observation level $\mathcal{O}_2$.

B. Phase-insensitive observation levels

1. Observation level $\mathcal{O}_A \equiv \{\hat{P}_n=|n\rangle\langle n|;\forall n\}$

The most general phase-insensitive observation level corresponds to the case when *all* diagonal elements $P_n=\langle n|\hat{\rho}|n\rangle$ of the density operator $\hat{\rho}$ describing the state under consideration are measured. The observation level $\mathcal{O}_A$ can be obtained via a reduction of the complete observation level $\mathcal{O}_0$ and it corresponds to the measurement of the photon number distribution P_n such that $\Sigma_n P_n=1$. Because of the relation (4.4) we can conclude that the observation level $\mathcal{O}_A$ corresponds to the measurement of all moments of the creation and annihilation operators of the form $(\hat{a}^\dagger)^k\hat{a}^k$ or, which is the same, it corresponds to a measurement of all moments of the photon number operator, i.e.,

$$\mathcal{O}_A\equiv\{\hat{P}_n=|n\rangle\langle n|;\forall n\}=\{(\hat{a}^\dagger)^k\hat{a}^k;\forall k\}=\{\hat{n}^k;\forall k\}. \tag{4.24}$$

The generalized canonical operator $\hat{\sigma}_A$ at the observation level $\mathcal{O}_A$ reads

$$\hat{\sigma}_A=\frac{1}{Z_A}\exp\left[-\sum_{n=0}^{\infty}\lambda_n|n\rangle\langle n|\right]=\sum_{n=0}^{\infty}P_n|n\rangle\langle n|. \tag{4.25}$$

The Lagrange multipliers λ_n have to be evaluated from an infinite set of equations $P_n=\mathrm{Tr}[\hat{\sigma}_A\hat{P}_n]=e^{-\lambda_n}/Z_A$ (for $\forall n$), from which we find $\lambda_n=-\ln[Z_AP_n]$. The entropy S_A at the observation level $\mathcal{O}_A$ is given by the expression first derived by Shannon [17]:

$$S_A=-k_B\sum_{n=0}^{\infty}P_n\ln P_n. \tag{4.26}$$

The Wigner function $W^{(\mathrm{A})}_{|\Psi\rangle}(\xi)$ of the state $|\Psi\rangle$ at the observation level $\mathcal{O}_A$ can be reconstructed in the form

$$W^{(\mathrm{A})}_{|\Psi\rangle}(\xi)=\sum_{n=0}^{\infty}P_nW_{|n\rangle}(\xi), \tag{4.27a}$$

where $W_{|n\rangle}(\xi)$ is the Wigner function of the Fock state $|n\rangle$,

$$W_{|n\rangle}(\xi)=2(-1)^n\exp(-2|\xi|^2)\mathcal{L}_n(4|\xi|^2), \tag{4.27b}$$

and $\mathcal{L}_n(x)$ is the Laguerre polynomial of order n.

The phase-insensitive observation level $\mathcal{O}_A$ can be further reduced if only a finite number of operators $\hat{P}_n$ [where $n\in\mathcal{M}$] is considered. In this case, in general, we have $\Sigma_{n\in\mathcal{M}}P_n<1$ and therefore it is essential that apart from the mean values P_n the mean photon number $\bar{n}$ is also known from the measurement.

2. Observation level $\mathcal{O}_B\equiv\{\hat{n},\hat{P}_N=|N\rangle\langle N|\}$

We can reduce observation levels $\mathcal{O}_A$ when we consider only a measurement of the mean photon number $\bar{n}$ and the probability P_N to find the system under consideration in the Fock state $|N\rangle$. The generalized density operator $\hat{\sigma}_B$ in this case reads

$$\hat{\sigma}_B=\frac{1}{Z_B}\exp[-\lambda\hat{n}-\lambda_N\hat{P}_N]=P_N|N\rangle\langle N|+\sum_{n\neq N}^{\infty}P_n|n\rangle\langle n|, \tag{4.28}$$

where $P_n=\exp(-\lambda n)/Z_B$ gives the photon number distribution on the subspace of the Fock space without the vector $|N\rangle$. If we introduce the notation $x=\exp(-\lambda)$, $y=\exp(-\lambda_N)$, then the Lagrange multipliers λ and λ_N can be found from the equations

$$P_N=\frac{(1-x)x^Ny}{1+x^N(y-1)(1-x)}, \tag{4.29a}$$

$$\bar{n}=\frac{x+Nx^N(1-x)^2(y-1)}{(1-x)[1+x^N(y-1)(1-x)]}. \tag{4.29b}$$

Generally, we cannot express the Lagrange multipliers λ and λ_N as functions of $\bar{n}$ and P_N in an analytical way for arbitrary N and Eqs. (4.29) have to be solved numerically. Nevertheless, there are two cases when these equations can be solved in a closed analytical form.

(1) If $N=0$ (we will denote this observation level as $\mathcal{O}_{B1}$), then we can find for Lagrange multipliers λ and λ_0 the following expressions:

$$e^{-\lambda}=1-\frac{1-P_0}{\bar{n}},\quad e^{-\lambda_0}=\frac{P_0}{(1-P_0)^2}[\bar{n}-(1-P_0)], \tag{4.30}$$

and after some straightforward algebra we can evaluate the parameters P_n as

$$P_n=\begin{cases}P_0 & \text{for } n=0,\\ \dfrac{(1-P_0)^2}{\bar{n}-(1-P_0)}\left[\dfrac{\bar{n}-(1-P_0)}{\bar{n}}\right]^n & \text{for } n>0.\end{cases} \tag{4.31}$$

From Eq. (4.31) which describes the photon number distribution of the generalized density operator $\hat{\sigma}_{B1}$, it follows that the reconstructed state on the observation level $\mathcal{O}_{B1}$ has a thermal-like character on the subspace formed of Fock states except the vacuum. Nevertheless, in this case the reconstructed Wigner function can be negative (unlike in the case of the thermal observation level). This can happen if P_0 is close to zero and $\bar{n}$ is small. Using explicit expressions for the parameters P_n given by Eq. (4.31), we can evaluate the entropy S_{B1} corresponding to the present observation level:

$$S_{B1}=-k_BP_0\ln P_0-k_B(\bar{n}-P)\ln(\bar{n}-P)-2k_BP\ln P+k_B\bar{n}\ln\bar{n}, \quad (4.32)$$

where we have used the notation $P=1-P_0$. In the limit $P_0\to(1+\bar{n})^{-1}$ expression (4.32) represents the entropy on the thermal observation level [see Eq. (4.8)]. In this limit $\mathcal{O}_{B1}$ reduces to the thermal observation level $\mathcal{O}_{\rm th}$. On the other hand, in the limit $P_0\to 0$, $\bar{n}\to 1$ the entropy $S_{B1}=0$ and $P_n=\delta_{n,1}$ which means that the Fock state $|1\rangle$ can be completely reconstructed on the observation level $\mathcal{O}_{B1}$.

(2) If the mean photon number is an integer, then in the case $N=\bar{n}$ (we will denote this observation level as $\mathcal{O}_{B2}$) we find for the Lagrange multipliers λ and $\lambda_{N=\bar{n}}\equiv\lambda_{\bar{n}}$ the expressions

$$e^{-\lambda}=\frac{\bar{n}}{1+\bar{n}}, \quad e^{-\lambda_{\bar{n}}}=\frac{(1+\bar{n})^{1+\bar{n}}-\bar{n}^{\bar{n}}}{(1-P_{\bar{n}})\bar{n}^{\bar{n}}}P_{\bar{n}}. \quad (4.33)$$

The reconstructed photon number distribution has again a thermal-like character:

$$P_n=\langle n|\hat{\sigma}_{B2}|n\rangle=\frac{e^{-n\lambda}}{Z_{B2}}[1+\delta_{n,\bar{n}}(e^{-\lambda_{\bar{n}}}-1)]. \quad (4.34)$$

The corresponding entropy can be evaluated in a closed analytical form:

$$S_{B2}=-k_BP_{\bar{n}}\ln P_{\bar{n}}-k_B(1-P_{\bar{n}})\ln(1-P_{\bar{n}})+k_B(1-P_{\bar{n}})\times\ln\left[\frac{(1+\bar{n})^{1+\bar{n}}}{\bar{n}^{\bar{n}}}-1\right]. \quad (4.35)$$

It is interesting to note that if $P_{\bar{n}}$ is given by its value in the thermal photon number distribution then the entropy (4.35) reduces to the entropy of the thermal observation level [see Eq. (4.8)]. In such a case the reconstructed density operator $\hat{\sigma}_{B2}=\hat{\sigma}_{\rm th}$ [see Eq. (4.7)] and so the reduction $\mathcal{O}_{B2}\to\mathcal{O}_{\rm th}$ takes place. On the other hand, if $P_{\bar{n}}=1$ then $S_{B2}=0$ and the Fock state $|\bar{n}\rangle$ can be completely reconstructed on the observation level $\mathcal{O}_{B2}$.

C. Relations between observation levels

The various observation levels considered in this section can be obtained as a result of a sequence of mutual reductions. Therefore we can order the observation levels under consideration. This ordering can be done separately for phase-sensitive and phase-insensitive observation levels. In particular, phase-sensitive observation levels are ordered as follows:

$$\mathcal{O}_0\supset\mathcal{O}_2\supset\mathcal{O}_1\supset\mathcal{O}_{\rm th}. \quad (4.36a)$$

The corresponding entropies are related as

$$S_0\leqslant S_2\leqslant S_1\leqslant S_{\rm th}. \quad (4.36b)$$

The ordering of phase-insensitive observation levels $\mathcal{O}_A$, $\mathcal{O}_{B1}$, and $\mathcal{O}_{B2}$ is more complex:

$$\mathcal{O}_0\supset\mathcal{O}_A\supset\left\{\begin{matrix}\mathcal{O}_{B1}\\ \mathcal{O}_{B2}\end{matrix}\right\}\supset\mathcal{O}_{\rm th}, \quad (4.37a)$$

which reflects the fact that observation levels $\mathcal{O}_{B1}$ and $\mathcal{O}_{B2}$ cannot be obtained as a result of mutual reduction or extension. The corresponding entropies are related as

$$S_0\leqslant S_A\leqslant\left\{\begin{matrix}S_{B1}\\ S_{B2}\end{matrix}\right\}\leqslant S_{\rm th}. \quad (4.37b)$$

For a particular quantum-mechanical state of light, the observation levels $\mathcal{O}_k$ can be ordered with respect to increasing values of the entropies S_k. From the above it also follows that if the entropy S_k on the observation level $\mathcal{O}_k$ is equal to zero, then the entropies on the extended observation levels are equal to zero as well. This means that the Wigner function of a pure state can be completely reconstructed on the observation level $\mathcal{O}_k$, i.e., the complete reconstruction can be performed via the measurement of a finite number of observables.

D. Choice of the observation level

We stress here that the entropies S_k associated with different observation levels do not reflect only the purity of the state itself but also the degree of our knowledge (data obtained from a measurement) about the state. In other words, the entropies S_k can be taken as a measure of the error of a reconstruction procedure on a given observation level. The higher (i.e., more complete) the observation level, the better is the reconstruction and the smaller is the value of S_k. This behavior is clearly seen from the chain of inequalities presented by Eqs. (4.36b) and (4.37b).

If *a priori* information that the states which are going to be reconstructed are *pure* states is available (i.e., the von Neumann entropy S_0 associated with the complete observation level is equal to zero) then the entropies S_k associated with $\mathcal{O}_k$ uniquely quantify the precision with which a particular reconstruction has been performed. To be more specific, if $S_k=0$ on $\mathcal{O}_k$ lower than $\mathcal{O}_0$ we can conclude with certainty that a complete reconstruction of a *pure* state has been performed on $\mathcal{O}_k$ (for instance, one can perform a complete reconstruction of a coherent state on $\mathcal{O}_1$ because the entropy S_1 is equal to zero). This means that there is no need to perform any further measurements (extending the observation level) because we already have complete information about the state.

In the case of statistical mixtures the von Neumann entropy S_0 associated with $\mathcal{O}_0$ is larger than zero. Therefore the quantification of the precision of the reconstruction with the help of entropies S_k associated with $\mathcal{O}_k$ is more difficult. We do not know whether we have performed complete reconstruction before a measurement on the complete observation level $\mathcal{O}_0$ has been performed. Only when we know *a priori* that S_0 has a given value, then if $S_k=S_0$ we can say that on $\mathcal{O}_k$ a complete (i.e., the best possible) reconstruction has been performed.

If there is no *a priori* information available about the state which is going to be reconstructed, there does not exist any universal prescription which would suggest to an experimentalist which observation level is the most suitable for the reconstruction of the given state. In any case, there is at least one general rule which has to be satisfied, i.e., any consistent observation level has to reveal information about the mean photon number (mean energy) of the state. This means that the thermal observation level can be taken as the initial step for any state reconstruction. In successive steps this observation level can be extended to more and more complete observation levels. A sequence of obtained entropies $\{S_k\}$ associated with the observation levels $\{\mathcal{O}_k\}$, or more precisely differences between neighboring entropies in this sequence, can give us some indication how close we are to the completely reconstructed state (this procedure is suitable also for statistical mixtures). We note that there are other measures which can also be utilized for this purpose. For example, the Hilbert-Schmidt norm (i.e., ''distance'') $\mathrm{dist}(\hat{\sigma}_k,\hat{\sigma}_l)$ between the density operators defined as

$$\mathrm{dist}(\hat{\sigma}_k,\hat{\sigma}_l)=||\hat{\sigma}_k-\hat{\sigma}_l||\equiv[\mathrm{Tr}(\hat{\sigma}_k-\hat{\sigma}_l)^2]^{1/2} \quad (4.38)$$

can serve as the measure of how close the two states described by density operators $\hat{\sigma}_k$ and $\hat{\sigma}_l$ are. Nevertheless, the distance $\mathrm{dist}(\hat{\sigma}_k,\hat{\sigma}_l)$ does not tell us which reconstruction (i.e., which density operator $\hat{\sigma}_k$ or $\hat{\sigma}_l$) is more complete. This can only be done with the help of the corresponding entropies S_k and S_l.

A completely different picture appears if one has some *a priori* information about the state which is going to be reconstructed. For instance, if from the preparation procedure some properties of the state are known, then this information can significantly improve our choice of ''the most efficient'' observation level which would yield, if not complete, then at least a very good reconstruction. As an example, we can briefly discuss the experiment by Raymer *et al.* [3] in which Wigner functions of the vacuum state and the squeezed vacuum state have been reconstructed via the optical homodyne technique. The preparation part of the setup in Raymer *et al.*'s experiment was designed to generate squeezed vacuum states, i.e., *pure* Gaussian states. If this is taken as *a priori* information then one can conclude that the measurement performed on the observation level $\mathcal{O}_2$ reveals the complete information about the state. Consequently, instead of performing a very sophisticated homodyne tomography (which in an ideal case corresponds to the measurement on $\mathcal{O}_0$) one can perform a simple homodyne measurement in which the variance of relevant quadratures can be measured and the Wigner function of the state can be reconstructed. On the other hand, for *non-Gaussian* states optical homodyne tomography can be considered as the most efficient way to gain information about the system.

We note that if the density operator $\hat{\rho}_0$ of the measured state is known *a priori*, then the Hilbert-Schmidt norm (4.38) can be used to measure how close the reconstructed state is to the original state.

In our previous discussion we have not analyzed the role of experimental errors in a reconstruction scheme based on the maximum entropy principle. That is, we have considered that all mean values of observables are measured precisely. We note that any inclusion of ''errors'' on a given observation level is implicitly associated with an extension of this observation level. For instance, an error related to the mean value of $\hat{n}$ is associated with a measurement of the mean value of the operator $\hat{n}^2$.

V. RECONSTRUCTION OF WIGNER FUNCTIONS

As an illustration we will analyze in this section a reconstruction of the Wigner function of the squeezed vacuum state on different observation levels. The squeezed vacuum state [18] can be expressed in the Fock basis as

$$|\eta\rangle=(1-\eta^2)^{1/4}\sum_{n=0}^{\infty}\frac{[(2n)!]^{1/2}}{2^n n!}\eta^n|2n\rangle, \quad (5.1)$$

where the squeezing parameter η (for simplicity we assume η to be real) ranges from -1 to $+1$. The squeezed vacuum state (5.1) can be obtained by the action of the squeezing operator $\hat{S}(r)$ given by Eq. (4.15) on the vacuum state $|0\rangle$, i.e., $|\eta\rangle=\hat{S}(r)|0\rangle$, where the squeezing parameter r is related to the parameter η as $\eta=\tanh r$. The mean photon number in the squeezed vacuum (5.1) is given by the relation $\bar{n}=\eta^2/(1-\eta^2)$. The variances of the position and momentum operators can be expressed in the form

$$\langle(\Delta\hat{q})^2\rangle=\hbar\sigma_q^2, \quad \langle(\Delta\hat{p})^2\rangle=\hbar\sigma_p^2, \quad (5.2a)$$

with the parameters σ_q and σ_p given by the relations

$$\sigma_q^2=\frac{1}{2}\left(\frac{1+\eta}{1-\eta}\right)=\frac{1}{2}+\frac{\sqrt{\bar{n}}}{\sqrt{1+\bar{n}}-\sqrt{\bar{n}}},$$

$$\sigma_p^2=\frac{1}{2}\left(\frac{1-\eta}{1+\eta}\right)=\frac{1}{2}-\frac{\sqrt{\bar{n}}}{\sqrt{1+\bar{n}}+\sqrt{\bar{n}}}. \quad (5.2b)$$

If we assume the squeezing parameter to be real and $\eta\in[0,-1]$ then from Eq. (5.2) it follows that fluctuations in the momentum are reduced below the vacuum state limit $\hbar/2$ at the expense of increased fluctuations in the position. Simultaneously, it is important to stress that the product of variances $\langle(\Delta\hat{q})^2\rangle$ and $\langle(\Delta\hat{p})^2\rangle$ is equal to $\hbar^2/4$, which means that the squeezed vacuum state belongs to the class of minimum uncertainty states.

The Wigner function of the squeezed vacuum state is of Gaussian form:

$$W_{|\eta\rangle}(q,p)=\frac{1}{\sigma_q\sigma_p}\exp\left[-\frac{1}{2\hbar}\frac{q^2}{\sigma_q^2}-\frac{1}{2\hbar}\frac{p^2}{\sigma_p^2}\right], \quad (5.3a)$$

with the parameters σ_q^2 and σ_p^2 given by Eq. (5.3). In the $(\mathrm{Re}\xi;\mathrm{Im}\xi)$ phase space the Wigner function of the squeezed vacuum reads

$$W_{|\eta\rangle}(\xi)=\frac{1}{\sigma_q\sigma_p}\exp\left[-\frac{(\mathrm{Re}\xi)^2}{\sigma_q^2}-\frac{(\mathrm{Im}\xi)^2}{\sigma_p^2}\right]. \quad (5.3b)$$

From Eq. (5.3) it follows that the mean values of the position and the momentum operators in the squeezed vacuum state

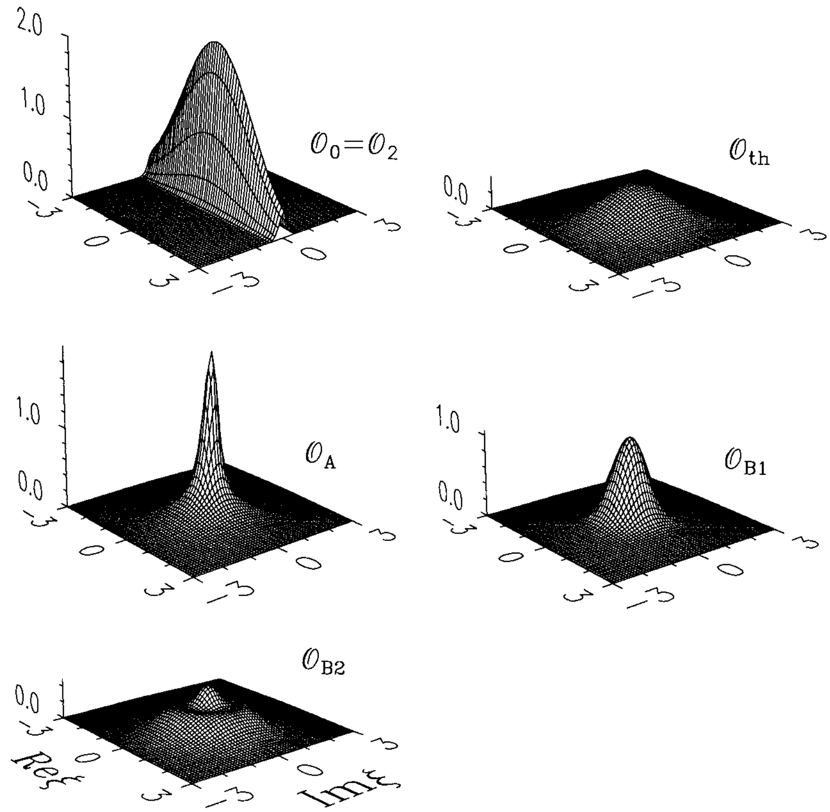

FIG. 1. The reconstructed Wigner functions of the squeezed vacuum state $|\eta\rangle$ with $\overline{n}=2$. We consider the observation levels $\mathcal{O}_0=\mathcal{O}_2$, $\mathcal{O}_1=\mathcal{O}_{\rm th}$, $\mathcal{O}_A$, $\mathcal{O}_{B1}$, and $\mathcal{O}_{B2}$ (see indications in the figure).

are equal to zero, while the higher order symmetrically ordered moments can be expressed in terms of the second order moments.

A. Observation levels $\mathcal{O}_0$ and $\mathcal{O}_2$

The Wigner function of the squeezed vacuum state (5.1) on the complete observation level $\mathcal{O}_0$ is given by Eq. (5.3) and is plotted (in the complex ξ phase space) in Fig. 1 ($\mathcal{O}_0$). This is a Gaussian function, which carries phase information associated with the phase of squeezing. On the thermal observation level $\mathcal{O}_{\rm th}$, which is characterized only by the mean photon number $\overline{n}$, the reconstructed Wigner function of the squeezed vacuum state is a rotationally symmetric Gaussian function centered at the origin of the phase space [see Eq. (4.9) and Fig. 1 ($\mathcal{O}_{\rm th}$)]. On the observation level $\mathcal{O}_1$, the reconstructed Wigner function is the same as on the thermal observation level because the mean amplitudes $\langle\hat{a}\rangle$ and $\langle\hat{a}^\dagger\rangle$ are equal to zero. On the other hand, the Wigner function of the squeezed vacuum can be completely reconstructed on the observation level $\mathcal{O}_2$. To see this we evaluate the entropy S_2 for the squeezed vacuum state. The parameters M and N can be expressed in terms of the squeezing parameter η (we assume η to be real) as

$$N=\frac{\eta^2}{1-\eta^2}, \quad M=\frac{\eta}{1-\eta^2}, \tag{5.4}$$

so that $N(N+1)=M^2$. Consequently, the parameter χ given by Eq. (4.18) is equal to zero, from which it follows that S_2 [see Eq. (4.21)] for the squeezed vacuum is equal to zero.

B. Observation level $\mathcal{O}_A$

The squeezed vacuum state (5.1) is characterized by an oscillatory photon number distribution P_n such that

$$\begin{aligned} P_{2n}&=(1-\eta^2)^{1/2}\frac{(2n)!}{2^{2n}(n!)^2}\eta^{2n}\\ &=\frac{1}{(1+\overline{n})^{1/2}}\frac{(2n)!}{2^{2n}(n!)^2}\left(\frac{\overline{n}}{1+\overline{n}}\right)^n, \quad P_{2n+1}=0. \end{aligned} \tag{5.5}$$

Using Eq. (4.27) we can express the Wigner function $W^{(A)}_{|\eta\rangle}(\xi)$ of the squeezed vacuum on the observation level $\mathcal{O}_A$ as

$$\begin{aligned} W^{(A)}_{|\eta\rangle}(\xi)&=2(1-\eta^2)^{1/2}e^{-2|\xi|^2}\sum_{n=0}^{\infty}\frac{(2n)!\,\eta^{2n}}{2^{2n}(n!)^2}\mathcal{L}_{2n}(4|\xi|^2)\\ &=2\exp\left[-\left(\frac{|\xi|^2}{2\sigma_q^2}+\frac{|\xi|^2}{2\sigma_p^2}\right)\right]I_0\left(\frac{|\xi|^2}{2\sigma_q^2}-\frac{|\xi|^2}{2\sigma_p^2}\right), \end{aligned} \tag{5.6}$$

where $I_0(x)$ is the modified Bessel function. We plot this Wigner function in Fig. 1 ($\mathcal{O}_A$). We see that $W^{(A)}_{|\eta\rangle}(\xi)$ is not negative and that it is much narrower in the vicinity of the origin of the phase space than the Wigner function of the vacuum state. Nevertheless, the total width of the Wigner function $W^{(A)}_{|\eta\rangle}(\xi)$ is much larger than the width of the Wigner function of the vacuum state.

C. Observation level $\mathcal{O}_{B1}$

We can easily reconstruct the Wigner function of the squeezed vacuum state at the observation level $\mathcal{O}_{B1}$. Using general expressions from Sec. IV B we find the following expression for the Wigner function $W^{(B1)}_{|\alpha\rangle}(\xi)$:

$$W^{(B1)}_{|\eta\rangle}(\xi)=\left(P_0-\frac{1-P_0}{\widetilde{n}}\right)W_{|0\rangle}(\xi)+(1-P_0)\frac{\widetilde{n}+1}{\widetilde{n}}W_{\rm th}(\xi), \tag{5.7a}$$

where $P_0=(\overline{n}+1)^{-1/2}$, $W_{|0\rangle}(\xi)$ is the Wigner function of the vacuum state given by Eq. (4.27), and $W_{\rm th}(\xi)$ is the Wigner function of the thermal state (4.9) with an effective number of photons equal to $\widetilde{n}$:

$$\widetilde{n}=\frac{\overline{n}}{1-(1+\overline{n})^{-1/2}}-1. \tag{5.7b}$$

We plot the Wigner function $W^{(B1)}_{|\eta\rangle}(\xi)$ in Fig. 1 ($\mathcal{O}_{B1}$), from which the dominant contribution of the vacuum state is transparent. It is due to the fact that the squeezed vacuum state has a thermal-like photon number distribution.

D. Observation level $\mathcal{O}_{B2}$

If the mean photon number $\overline{n}$ is an integer, then one may consider the observation level $\mathcal{O}_{B2}$ for a nontrivial reconstruction of the Wigner function of the squeezed vacuum state. After some algebra we find that this reconstructed Wigner function reads

$$W^{(B2)}_{|\eta\rangle}(\xi)=\left(1-\frac{1+\overline{n}}{Z_{B2}}\right)W_{|\overline{n}\rangle}(\xi)+\frac{\overline{n}+1}{Z_{B2}}W_{\rm th}(\xi), \tag{5.8}$$

where $W_{|\overline{n}\rangle}(\xi)$ is the Wigner function of the Fock state $|\overline{n}\rangle$ and $W_{\rm th}(\xi)$ is the Wigner function of the thermal state with the mean photon number equal to $\overline{n}$. If $\overline{n}$ is *even* then for $P_{\overline{n}}$ [see Eq. (4.33)] we find

$$P_{\overline{n}}=\frac{\overline{n}!}{2^{\overline{n}}[(\overline{n}/2)!]^2}\,\frac{\overline{n}^{\overline{n}/2}}{(1+\overline{n})^{(1+\overline{n})/2}}. \tag{5.9}$$

We plot this Wigner function in Fig. 1 ($\mathcal{O}_{B2}$). It has a thermal-like character [compare with Fig. 1 ($\mathcal{O}_{\rm th}$)] but the contribution of the Fock state $|\overline{n}=2\rangle$ is more dominant compared with the proper thermal distribution. If $\overline{n}$ is an *odd* integer, then $P_{\overline{n}}=0$ and the corresponding Wigner function can again be reconstructed with the help of Eqs. (5.8) and (4.33).

VI. DETECTION OF QUANTUM COHERENCES

Within the framework of the phase-space formalism one can interpret a reduction of quantum fluctuations as a direct consequence of quantum interference between component (coherent) states [19]. Coherent states form a position-momentum patch of minimum area and may be regarded as the quantum analogue of classical points in phase space. The quantum interference between coherent-state components in phase space (which is intrinsically related to the overcompleteness of the coherent-state basis) is what leads to purely quantum effects.

To be specific, let us represent the squeezed vacuum state (5.1) as a *one*-dimensional superposition of coherent states on a line [20] [for simplicity we assume a real squeezing parameter $\eta\in[0,1)$],

$$|\eta\rangle=\frac{(1-\eta^2)^{1/4}}{\sqrt{2\pi\eta}}\int_{-\infty}^{\infty}d\alpha\exp\left[-\frac{1-\eta}{2\eta}\alpha^2\right]|\alpha\rangle, \tag{6.1}$$

where α is a *real* parameter. The corresponding density operator $\hat{\rho}_{|\eta\rangle}$ in the coherent-state basis can now be expressed as

$$\hat{\rho}_{|\eta\rangle}=\frac{(1-\eta^2)^{1/2}}{2\pi\eta}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta\exp\left[-\frac{1-\eta}{2\eta}\alpha^2-\frac{1-\eta}{2\eta}\beta^2\right]|\alpha\rangle\langle\beta|. \tag{6.2}$$

As follows from the above the off-diagonal elements $|\alpha\rangle\langle\beta|$ of the density operator in the coherent-state basis carry information about the nonclassical properties of quantum states of light, i.e., these elements are associated with quantum-interference effects in phase space. The quantum phase-space interference leads to quadrature squeezing [19] as well as to oscillations in the photon number distribution in the squeezed vacuum state.

For comparison purposes we can consider a statistical mixture associated with the squeezed vacuum state (5.1). This mixture can be represented as a one-dimensional mixture of noninterfering coherent states on a line. The corresponding density operator reads

$$\hat{\rho}_{\rm mix}=\left(\frac{1-\widetilde{\eta}}{\pi\widetilde{\eta}}\right)^{1/2}\int_{-\infty}^{\infty}d\alpha\exp\left[-\frac{1-\widetilde{\eta}}{\widetilde{\eta}}\alpha^2\right]|\alpha\rangle\langle\alpha|. \tag{6.3}$$

The von Neumann entropy of the statistical mixture (6.3) is larger than zero because the density operator (6.3) does not describe a unique state (the information about the interference between coherent components is inevitably lost even on the complete observation level $\mathcal{O}_0$).

The mean photon number $\overline{n}$ in the statistical mixture (6.3) is

$$\overline{n}=\frac{\widetilde{\eta}}{2(1-\widetilde{\eta})}, \tag{6.4}$$

and the thermal-like photon number distribution associated with this mixture reads

$$P_n=(1-\widetilde{\eta})^{1/2}\left(\frac{\widetilde{\eta}}{4}\right)^n\frac{(2n)!}{(n!)^2}=\frac{1}{(1+2\overline{n})^{1/2}}\left(\frac{2\overline{n}}{1+2\overline{n}}\right)^n\frac{(2n)!}{2^{2n}(n!)^2}. \tag{6.5}$$

The variances of the position and momentum operators in the mixture state (6.3) can be expressed as

$$\langle(\Delta\hat{q})^2\rangle=\hbar\widetilde{\sigma}_q^2,\quad \langle(\Delta\hat{p})^2\rangle=\hbar\widetilde{\sigma}_p^2, \tag{6.6a}$$

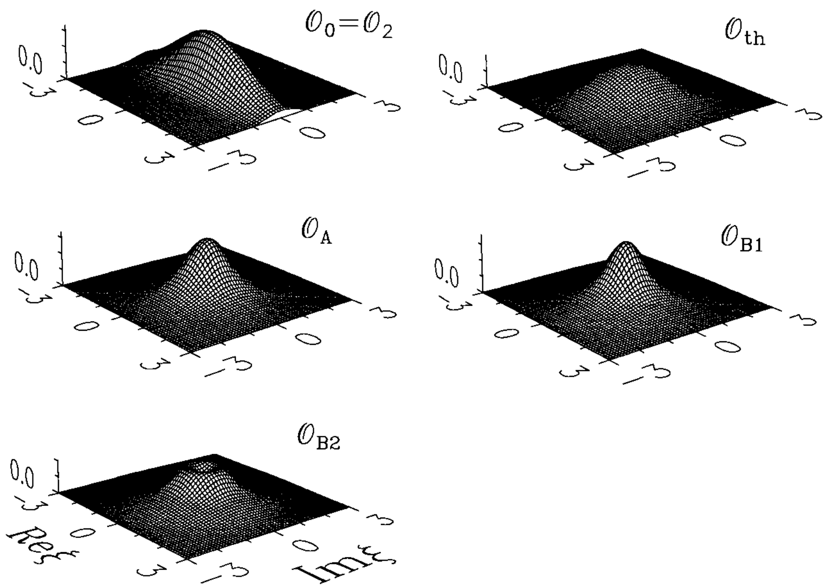

FIG. 2. The reconstructed Wigner functions of the statistical mixture (6.3) with $\overline{n}=2$. We consider the observation levels $\mathcal{O}_0=\mathcal{O}_2$, $\mathcal{O}_1=\mathcal{O}_{\rm th}$, $\mathcal{O}_A$, $\mathcal{O}_{B1}$, and $\mathcal{O}_{B2}$ (see indications in the figure).

where

$$\widetilde{\sigma}_q^2=\frac{1}{2}\left(\frac{1+\widetilde{\eta}}{1-\widetilde{\eta}}\right), \quad \widetilde{\sigma}_p^2=\frac{1}{2}, \tag{6.6b}$$

which means that due to the absence of quantum interference between coherent components the quadrature squeezing in the $\hat{p}$ quadrature is completely deteriorated [compare with Eq. (5.2b)]. The expression for the variance $\widetilde{\sigma}_q^2$ in terms of the parameter $\widetilde{\eta}$ looks exactly the same as in the case of the squeezed vacuum state [see Eq. (5.2b)], but $\widetilde{\sigma}_p^2$ is not reduced below the vacuum-state limit. We can express the variances (6.6a) in terms of the mean photon number $\overline{n}$ given by Eq. (6.4) and we find that

$$\widetilde{\sigma}_q^2=\frac{1+4\overline{n}}{2}, \quad \widetilde{\sigma}_p^2=\frac{1}{2}. \tag{6.6c}$$

Comparing Eqs. (5.2) and (6.6) we see that for the *same* mean photon number the variance $\widetilde{\sigma}_q^2$ (the variance associated with the statistical mixture) is always smaller than σ_q^2 (i.e., the variance in the $\hat{q}$ quadrature of the squeezed vacuum state), which reflects the fact that quantum interference not only reduces fluctuations in the $\hat{p}$ quadrature but, on the other hand, enhance fluctuations in the conjugated $\hat{q}$ quadrature in a very specific way; namely, from the above it follows that the sum of variances σ_q^2 and σ_p^2 for the squeezed vacuum is *equal* to the sum of variances for the corresponding statistical mixture:

$$\sigma_q^2+\sigma_p^2=\widetilde{\sigma}_q^2+\widetilde{\sigma}_p^2=1+2\overline{n}. \tag{6.7}$$

It is interesting to note that both coherent states and squeezed vacuum states belong to the class of minimum uncertainty states (MUS) in the sense that $\sigma_q^2\sigma_p^2=1/4$. Nevertheless, the relation (6.7)shows us that in the class of MUS the coherent states play an exceptional role, because these are the only states for which the sum of the variances σ_q^2 and σ_p^2 takes a minimum value equal to 1.

The Wigner function of the statistical mixture (6.3) on the complete observation level $\mathcal{O}_0$ is the same as for the squeezed vacuum state [see Eq. (5.3b)] except for the variances, which are given by Eq. (6.6), i.e., the statistical mixture (6.3) is described by a Gaussian Wigner function which is not squeezed in the $\hat{p}$ quadrature [see Fig. 2 ($\mathcal{O}_0$)].

From our discussion in Sec. IV it follows that the Gaussian Wigner functions can be completely reconstructed on the observation levels $\mathcal{O}_2$ and the corresponding entropy S_2 [given by Eq. (4.21)] is equal to the von Neumann entropy. The von Neumann entropy of the squeezed vacuum state is equal to zero (this is a pure state), while the von Neumann entropy of the statistical mixture (6.3) is given by Eq. (4.21) with the parameter χ given by the relation

$$\chi=\frac{1}{2}\left(\frac{1+\widetilde{\eta}}{1-\widetilde{\eta}}\right)^{1/2}-\frac{1}{2}=\frac{1}{2}(1+4\overline{n})^{1/2}-\frac{1}{2}. \tag{6.8}$$

The difference between the von Neumann entropy of the squeezed vacuum state (6.1) and of the corresponding statistical mixture (6.3) reflects the presence (absence) of quantum coherences (i.e., the off-diagonal terms in the coherent-state basis in our one-dimensional representation of the squeezed vacuum state) and can be used for a quantification of the degree of quantum interference in the phase space between coherent components of superposition states. In the case of Gaussian states this degree of quantum interference can be completely determined on the observation level $\mathcal{O}_2$. On the other hand, on the observation level $\mathcal{O}_1$ both the squeezed vacuum state and the corresponding mixture are described by the thermal Wigner function (this is due to the fact that for both these states $\langle\hat{a}\rangle=\langle\hat{a}^\dagger\rangle=0$). Consequently, their entropies are equal [see Eq. (4.8)] and therefore, in this particular case, we cannot recover the presence of the quantum-phase-space interference on the observation level $\mathcal{O}_1$.

In Fig. 2 ($\mathcal{O}_{\rm A}$) we plot the Wigner function of the statistical mixture (6.3) reconstructed on the observation level $\mathcal{O}_A$ (for details see Sec. IV). For completeness we plot in Fig. 2 ($\mathcal{O}_{B1}$) and Fig. 2 ($\mathcal{O}_{B2}$) the Wigner functions of the

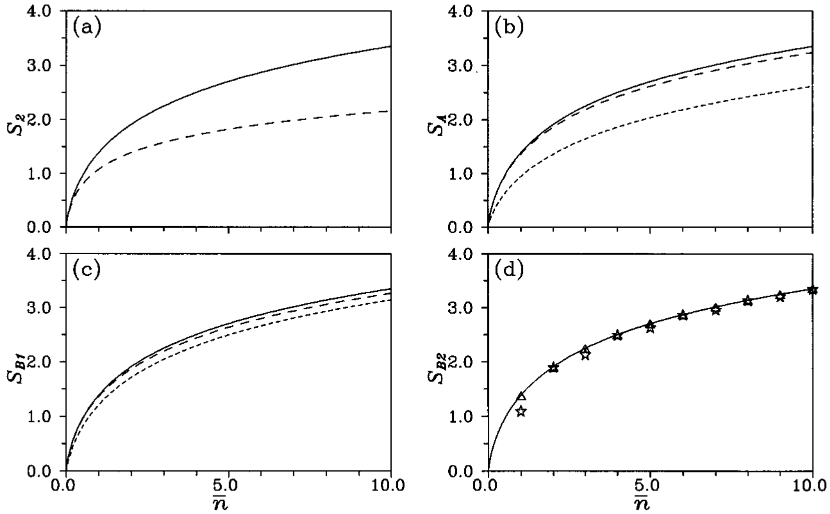

FIG. 3. The entropies $S_X^{|\eta\rangle}(\overline{n})$ (short-dashed line) and $S_X^{\hat{\rho}_{\rm mix}}(\overline{n})$ (long-dashed line) of the squeezed vacuum state and the corresponding statistical mixture, respectively, on different observation levels $\mathcal{O}_X$ as functions of the mean photon number $\overline{n}$: (a) $\mathcal{O}_0 = \mathcal{O}_2$, (b) $\mathcal{O}_A$, and (c) $\mathcal{O}_{B1}$. On the observation level $\mathcal{O}_{B2}$ [see (d)] entropies $S_{B2}^{|\eta\rangle}(\overline{n})$ ($\star$) and $S_{B2}^{\hat{\rho}_{\rm mix}}(\overline{n})$ ($\triangle$) are evaluated only for discrete values of $\overline{n}$. For reference purposes we plot the entropy $S_{\rm th}(\overline{n})$ (solid line) associated with the thermal observation level $\mathcal{O}_{\rm th}$.

statistical mixture (6.3) reconstructed on the observation levels $\mathcal{O}_{B1}$ and $\mathcal{O}_{B2}$, respectively. We see that the shape of $W_{|\eta\rangle}^{(B1)}(\xi)$ and $W_{\hat{\rho}_{\rm mix}}^{(B1)}(\xi)$ are essentially the same except the value of $W_{|\eta\rangle}^{(B1)}(\xi)$ at the origin of the phase space is much larger than the value of $W_{\hat{\rho}_{\rm mix}}^{(B1)}(\xi)$. This is associated with the fact that the contribution of the vacuum state into the squeezed vacuum state is more dominant than into the statistical mixture. The difference between the reconstructed Wigner functions $W_{|\eta\rangle}^{(B2)}(\xi)$, and $W_{\hat{\rho}_{\rm mix}}^{(B2)}(\xi)$ and the Wigner function $W^{(\rm th)}(\xi)$ [see Eq. (4.9)] consists in the contribution $P_2(\overline{n})$ of the Fock state $|2\rangle$ into the given state. To be more specific, for $\overline{n}=2$ we have $P_2^{|\eta\rangle}(\overline{n}) > P_2^{(\rm th)}(\overline{n}) > P_2^{\hat{\rho}_{\rm mix}}(\overline{n})$ [compare Eqs. (5.5), (4.7), and (6.5)].

In what follows we will address the problem of whether we can quantify the degree of quantum interference on the phase-insensitive observation level $\mathcal{O}_A$ with the help of the entropy S_A associated with this observation level. Using the general expression for the Shannon entropy S_A [see Eq. (4.26)] and the expressions for the photon number distribution (5.5) and (6.5) of the squeezed vacuum state and the corresponding statistical mixture, respectively, we find the relation

$$S_A^{|\eta\rangle}(2\overline{n}) = S_A^{\hat{\rho}_{\rm mix}}(\overline{n}), \tag{6.9}$$

where

$$S_A^{\hat{\rho}_{\rm mix}}(\overline{n}) = \frac{k_B}{2}\ln(2\overline{n}+1) + \overline{n}k_B \ln\left(\frac{2\overline{n}+1}{2\overline{n}}\right) + k_B \sum_n P_n(\overline{n}) \ln\left(\frac{2^{2n}(n!)^2}{(2n)!}\right), \tag{6.10}$$

and $P_n(\overline{n})$ is given by Eq. (6.5). From Eq. (6.10) it follows that the entropy $S_A^{\hat{\rho}_{\rm mix}}(\overline{n})$ is an increasing function of $\overline{n}$. Consequently, from Eq. (6.9) it then follows that $S_A^{|\eta\rangle}(\overline{n}) < S_A^{\hat{\rho}_{\rm mix}}(\overline{n})$ which means that the Shannon entropy reflects the presence of quantum interference in phase space. We plot these functions in Fig. 3(b).

Generally speaking, the lower (i.e., less complete) the observation level is, the smaller is the difference between the entropy of the squeezed vacuum state and the corresponding statistical mixture (see Fig. 3). The maximum difference can obviously be found on the complete observation level $\mathcal{O}_0$, while there is no difference on the thermal observation level $\mathcal{O}_{\rm th}$.

A. Decay of quantum coherences

From our previous discussion it follows that the detection of quantum coherences depends on the choice of the observation level. The higher (the more complete) the observation level is, the better we can distinguish between a pure superposition state and the corresponding statistical mixture. This difference can be quantified with the help of the corresponding entropies. In addition to the choice of the observation level, the measurement process can be affected by nonunit efficiency of the measurement apparatus itself, i.e., the measured data are biased by an additional noise in an uncontrollable way. One possibility to model this deterioration of information about quantum-mechanical systems is to consider the interaction of the system under consideration with a large reservoir (heat bath).

For simplicity we will consider a zero-temperature heat bath to model a nonunit efficiency quantum-mechanical measurement. We can interpret this model as the detection of quantum coherences of the single-mode light field decaying into a zero-temperature heat bath. We will analyze the time evolution of entropies associated with different observation levels and we will discuss how the quantum coherences are affected by the presence of the reservoir. In other words, we will study to what extent the entropies under consideration can be used for quantification of the degree of quantum coherence associated with the state.

To be specific, we shall assume that the density operator $\hat{\rho}$ for the field mode obeys a zero-temperature master equation in the Born-Markov approximation. This equation in the interaction picture can be written as

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{\gamma}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}), \tag{6.11}$$

where γ is the decay constant. Following Barnett and Knight [21] we find the time-dependent expression for the density matrix of the squeezed vacuum state (6.2) decaying into the zero-temperature heat bath as

$$\hat{\rho}_{|\eta\rangle}(t)=\frac{(1-\eta^2)^{1/2}}{2\pi\eta}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta\exp\left[-\frac{1-\eta}{2\eta}(\alpha^2+\beta^2)\right]\times\langle\beta|\alpha\rangle^{1-\mu}|\mu^{1/2}\alpha\rangle\langle\mu^{1/2}\beta|, \tag{6.12}$$

where $\mu=\exp(-\gamma t)$.

The Wigner function of the decaying squeezed vacuum state (6.17) reconstructed on the complete observation level $\mathcal{O}_0$ is given by Eq. (5.3b) with the time-dependent parameters $\sigma_q^2(t)$ and $\sigma_p^2(t)$ given by the relations

$$\sigma_q^2(t)=\frac{1}{2}+\mu\frac{\sqrt{\bar{n}}}{\sqrt{1+\bar{n}}-\sqrt{\bar{n}}},\quad \sigma_p^2(t)=\frac{1}{2}-\mu\frac{\sqrt{\bar{n}}}{\sqrt{1+\bar{n}}+\sqrt{\bar{n}}}. \tag{6.13}$$

We see that the decaying squeezed vacuum state is described by a Gaussian Wigner function with time-dependent parameters. We note that the Wigner function of the decaying squeezed vacuum state can be obtained from the Wigner function (5.3b) of the squeezed vacuum state at $t=0$ via a coarse-graining procedure, which can be used to model a nonunit efficiency measurement process [22].

From Eq. (6.13) it follows that at time $t>0$ the decaying squeezed vacuum state is *not* a minimum uncertainty state anymore, i.e.,

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle=\hbar^2\sigma_q^2(t)\sigma_p^2(t)=\frac{\hbar^2}{4}[1+4(\mu-\mu^2)\bar{n}]\geqslant\frac{\hbar^2}{4}. \tag{6.14}$$

The photon number distribution of the decaying squeezed vacuum state (6.12) can be written in the form

$$P_n(\mu)=(1-\eta^2)^{1/2}\sum_{k=[(n+1)/2]}^{\infty}\frac{(2k)!}{(k!)^2}\left(\frac{\mu\eta}{2}\right)^{2k}\times\left(\frac{1-\mu}{\mu}\right)^{2k-n}\binom{2k}{n}. \tag{6.15}$$

The mean photon number $\overline{n(t)}$ evaluated with the help of the distribution (6.15) describes exponential decay induced by the zero-temperature reservoir, i.e., $\overline{n(t)}=\mu\bar{n}=\exp(-\gamma t)\bar{n}$.

Analogously, we find the solution of Eq. (6.11) for the density operator describing the decay of the statistical mixture (6.3):

$$\hat{\rho}_{\rm mix}(t)=\left(\frac{1-\tilde{\eta}}{\pi\tilde{\eta}}\right)^{1/2}\int_{-\infty}^{\infty}d\alpha\times\exp\left[-\frac{1-\tilde{\eta}}{\tilde{\eta}}\alpha^2\right]|\mu^{1/2}\alpha\rangle\langle\mu^{1/2}\alpha|. \tag{6.16}$$

The Wigner function of the decaying statistical mixture reconstructed on the complete observation level $\mathcal{O}_0$ has the form (5.3b) with the time-dependent parameters $\tilde{\sigma}_q^2(t)$ and $\tilde{\sigma}_p^2(t)$ given by the relations

$$\tilde{\sigma}_q^2(t)=\frac{1}{2}+2\mu\bar{n},\quad \tilde{\sigma}_p^2(t)=\frac{1}{2}. \tag{6.17}$$

Here we briefly turn our attention to the fact that the relation (6.7) is valid also in the case when the squeezed vacuum state and the corresponding statistical mixture are decaying into the zero-temperature heat bath, i.e., for any time $t\geqslant 0$ we have

$$\sigma_q^2(t)+\sigma_p^2(t)=\tilde{\sigma}_q^2(t)+\tilde{\sigma}_p^2(t)=1+2\mu\bar{n}=1+2\overline{n(t)}. \tag{6.18}$$

The photon number distribution of the decaying statistical mixture (6.16) reads

$$P_n(\bar{n};t)=\frac{1}{(1+2\mu\bar{n})^{1/2}}\left(\frac{2\mu\bar{n}}{1+2\mu\bar{n}}\right)^n\frac{(2n)!}{2^{2n}(n!)^2}. \tag{6.19}$$

We note that the photon number distribution of the decaying thermal state and the statistical mixture (6.16) can be obtained from their initial $(t=0)$ values by simple rescaling of the mean photon number, i.e., $P_n(\bar{n};t)=P_n(\overline{n(t)};t=0)$, where $\overline{n(t)}=\mu\bar{n}$. This is in sharp contrast with the time evolution of the photon number distribution of the decaying squeezed vacuum state (6.15), because in this case quantum coherences are decaying on a different time scale than the mean photon number (for more details see Ref. [19]).

Both the decaying squeezed vacuum state and the corresponding statistical mixture are described by Gaussian Wigner functions $t>0$ and consequently can be reconstructed in the most reliable way on the observation level $\mathcal{O}_2$. The corresponding entropies $S_2^{|\eta\rangle}(t)$ and $S_2^{\hat{\rho}_{\rm mix}}(t)$ (which in the present case are equal to the von Neumann entropy) are given by Eq. (4.21), where the time-dependent parameters for the squeezed vacuum state $\chi_{|\eta\rangle}(t)$ and for the corresponding mixture $\chi_{\hat{\rho}_{\rm mix}}(t)$ read

$$\chi_{|\eta\rangle}(t)=[(\mu-\mu^2)\bar{n}+1/4]^{1/2}-1/2, \tag{6.20a}$$

$$\chi_{\hat{\rho}_{\rm mix}}(t)=[\mu\bar{n}+1/4]^{1/2}-1/2, \tag{6.20b}$$

respectively.

The von Neumann entropy of the statistical mixture decaying into the zero-temperature heat bath is a monotonically decreasing function, while the von Neumann entropy of the squeezed vacuum state decaying into the zero-temperature heat bath increases during the first period of its time evolution and after reaching its maximum starts to decrease. To find the moment at which the entropy $S_2^{|\eta\rangle}(t)$ reaches its maximum value we solve the equation

$$\frac{\partial}{\partial t}S_2^{|\eta\rangle}(t)=k_B\frac{\partial\chi_{|\eta\rangle}(t)}{\partial t}\ln\left(\frac{1+\chi_{|\eta\rangle}(t)}{\chi_{|\eta\rangle}(t)}\right)=0. \tag{6.21a}$$

Using the explicit expression for $\chi_{|\eta\rangle}(t)$ we find that

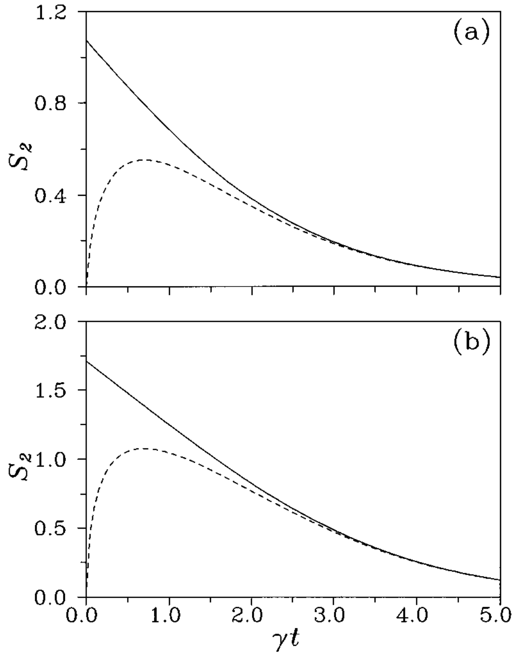

FIG. 4. The time evolution of the von Neumann entropies $S_2^{|\eta\rangle}(t)$ (dashed line) and $S_2^{\hat{\rho}_{\rm mix}}(t)$ (solid line) of the squeezed vacuum state and the corresponding statistical mixture, respectively, on the observation level $\mathcal{O}_0=\mathcal{O}_2$ for the initial mean photon number $\overline{n}=1$ (a) and $\overline{n}=4$ (b).

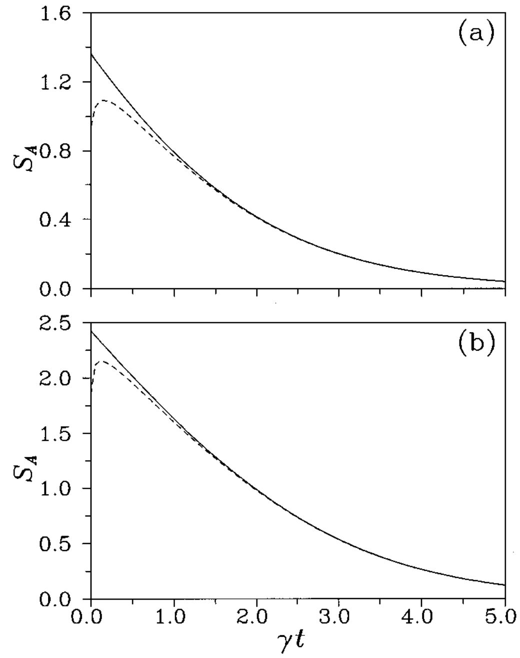

FIG. 5. The time evolution of the Shannon entropies $S_A^{|\eta\rangle}(t)$ (dashed line) and $S_A^{\hat{\rho}_{\rm mix}}(t)$ (solid line) of the squeezed vacuum state and the corresponding statistical mixture, respectively, on the observation level $\mathcal{O}_A$ for the initial mean photon number $\overline{n}=1$ (a) and $\overline{n}=4$ (b).

$$\frac{\partial \chi_{|\eta\rangle}(t)}{\partial t}=-\frac{\gamma\mu\overline{n}(1-2\mu)}{2\sqrt{(\mu-\mu^2)\overline{n}+1/4}}=0 \qquad (6.21b)$$

for $\mu=1/2$ *irrespective* of the initial mean photon number $\overline{n}$ of the squeezed vacuum state.

The other important property of the von Neumann entropies $S_2^{|\eta\rangle}(t)$ and $S_2^{\hat{\rho}_{\rm mix}}(t)$ is that $S_2^{|\eta\rangle}(t)<S_2^{\hat{\rho}_{\rm mix}}(t)$ for any $t>0$ and $\overline{n}>0$, that is, using the entropy associated with the observation level $\mathcal{O}_2$ we can discriminate between the pure squeezed vacuum state and the corresponding mixture, i.e., we can "detect" the presence of quantum coherences even in the case of nonunit efficiency measurement. It is interesting to note that at the moment when the von Neumann entropy of the decaying squeezed vacuum states reaches its maximum value (i.e., at $\mu=1/2$) the parameters $\chi_{|\eta\rangle}(t)$ and $\chi_{\hat{\rho}_{\rm mix}}(t)$ read

$$\chi_{|\eta\rangle}(t)|_{\mu=1/2}=[\overline{n}/4+1/4]^{1/2}-1/2,$$

$$\chi_{\hat{\rho}_{\rm mix}}(t)|_{\mu=1/2}=[\overline{n}/2+1/4]^{1/2}-1/2, \qquad (6.22)$$

which means that at this moment the von Neumann entropy of the decaying squeezed vacuum state with the initial mean photon number equal to $\overline{n}$ is *equal* to the von Neumann entropy of the decaying statistical mixture with the initial mean photon number equal to $\overline{n}/2$. In other words, even at $\mu=1/2$ there is a significant difference between $S_2^{|\eta\rangle}(t)$ and $S_2^{\hat{\rho}_{\rm mix}}(t)$. In Fig. 4 we plot the time evolution of these entropies for $\overline{n}=1$ [Fig. 4(a)] and $\overline{n}=4$ [Fig. 4(b)]. We see that on the observation level $\mathcal{O}_2$ we can clearly "detect" the presence of quantum coherence associated with *Gaussian* states at least for the detector efficiency $\mu\geqslant 1/2$.

As we have said earlier, the lower (i.e., the less complete) the observation level, the smaller is the difference between the entropy of the pure squeezed vacuum state and that of the corresponding statistical mixture (see Fig. 3). This difference, which reflects the presence of quantum coherences, is even smaller when a nonunit efficiency measurement is under consideration. In particular, in Fig. 5 we plot the time evolution of the Shannon entropies $S_A^{|\eta\rangle}(t)$ and $S_A^{\hat{\rho}_{\rm mix}}(t)$ of the decaying squeezed vacuum state and of the decaying statistical mixture with the initial mean photon number $\overline{n}$ equal to 1 [Fig. 5(a)] and 4 [Fig. 5(b)], respectively. Here we stress that for Gaussian states the observation level $\mathcal{O}_2$ is identical to the complete observation level $\mathcal{O}_0$; consequently, $\mathcal{O}_A$ is reduced with respect to $\mathcal{O}_2$.

B. Decay of quantum coherences of a superposition of two coherent states

From our previous discussion it follows that quantum coherence observed on $\mathcal{O}_2$, which is responsible for quadrature squeezing of the squeezed vacuum state, is very *robust* with respect to dissipative processes. This robustness is reflected by a significant difference between entropies $S_2^{|\eta\rangle}(t)$ and $S_2^{\hat{\rho}_{\rm mix}}(t)$ even at time $\gamma t=\ln 2$ when $S_2^{|\eta\rangle}(t)$ reaches its maximum value irrespective of the initial intensity of the squeezed vacuum state.

To illuminate this property more clearly, we will consider now superposition of just two coherent states $|\alpha\rangle$ and

$|-\alpha\rangle$ (we will assume the amplitude α to be real). It is well known that for a proper choice of the relative phase between coherent components the corresponding superposition state exhibits significant squeezing [19]. This effect appears as a consequence of quantum interference between $|\alpha\rangle$ and $|-\alpha\rangle$. To be more specific, let us consider the so-called even coherent state [23] described by the density operator in the coherent-state basis as

$$\hat{\rho}_{|\alpha_e\rangle}=\mathcal{N}_e\{|\alpha\rangle\langle\alpha|+|-\alpha\rangle\langle-\alpha|+|\alpha\rangle\langle-\alpha|+|-\alpha\rangle\langle\alpha|\},$$

$$\mathcal{N}_e=\frac{1}{2[1-\exp(-2\alpha^2)]}. \tag{6.23a}$$

The mean photon number $\overline{n}$ in the even coherent state (6.23a) is given by the relation $\overline{n}=\alpha^2\tanh\alpha^2$, which in the limit of large α is equal to α^2. The quantum interference terms in the superposition state (6.23a) are described by the off-diagonal elements $|\alpha\rangle\langle-\alpha|$ and $|-\alpha\rangle\langle\alpha|$.

A statistical mixture corresponding to the pure superposition state (6.23a) is described by the density operator $\hat{\rho}_{\text{mix}}$,

$$\hat{\rho}_{\text{mix}}=\frac{1}{2}\{|\alpha\rangle\langle\alpha|+|-\alpha\rangle\langle-\alpha|\}. \tag{6.23b}$$

The von Neumann entropy of the statistical mixture (6.23b) is nonzero for $\alpha\neq 0$. The maximum value of the von Neumann entropy of the mixture of two coherent states equals $k_B\ln 2$ in the limit $\alpha\to\infty$, which corresponds to the entropy of a "two-state" quantum-mechanical system with equal probabilities of population of each state.

Density operators describing the decay of the even coherent state and the corresponding statistical mixture into the zero-temperature heat bath [i.e., the solutions of Eq. (6.11) with the initial conditions (6.23a) and (6.23b), respectively] read

$$\hat{\rho}_{|\alpha_e\rangle}(t)=\mathcal{N}_e\sum_{k,l=0}^{1}\langle(-1)^k\alpha|(-1)^l\alpha\rangle^{1-\mu}|(-1)^k\mu^{1/2}\alpha\rangle \times\langle(-1)^l\mu^{1/2}\alpha|, \tag{6.24a}$$

and

$$\hat{\rho}_{\text{mix}}(t)=\frac{1}{2}\left\{\sum_{k=0}^{1}|(-1)^k\mu^{1/2}\alpha\rangle\langle(-1)^k\mu^{1/2}\alpha|\right\}. \tag{6.24b}$$

To describe the deterioration of quantum coherence of the even coherent state due to the interaction with the zero-temperature heat bath, we evaluate the von Neumann entropy $S_0^{|\alpha_e\rangle}(t)$ of this state at time t and compare it with the von Neumann entropy of the corresponding statistical mixture $S_0^{\text{mix}}(t)$. The von Neumann entropy in the case of the decaying even coherent state (6.24a) can be expressed as

$$S_0^{|\alpha_e\rangle}(t)=-k_B\sum_{j=1}^{2}\Pi_j^{|\alpha_e\rangle}(t)\ln\Pi_j^{|\alpha_e\rangle}(t), \tag{6.25}$$

where $\Pi_j^{|\alpha_e\rangle}(t)$ $(j=1,2)$ are the eigenvalues of the density operator $\hat{\rho}_{|\alpha_e\rangle}(t)$ and they read

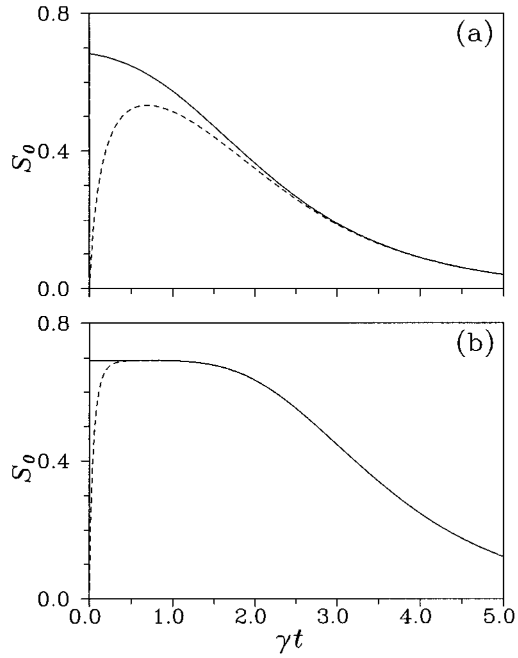

FIG. 6. The time evolution of the von Neumann entropies $S_0^{|\alpha_e\rangle}(t)$ (dashed line) and $S_0^{\hat{\rho}_{\text{mix}}}(t)$ (solid line) of the decaying even coherent state and the corresponding statistical mixture, respectively, for the initial mean photon number $\overline{n}=1$ (a) and $\overline{n}=4$ (b).

$$\Pi_j^{|\alpha_e\rangle}(t)=\frac{1}{2}\left[1\pm\frac{e^{-2\mu\alpha^2}+e^{-2(1-\mu)\alpha^2}}{1+e^{-2\alpha^2}}\right], \quad j=1,2. \tag{6.26a}$$

The von Neumann entropy $S_0^{\hat{\rho}_{\text{mix}}}(t)$ of the decaying statistical mixture (6.24b) has the form (6.25) but with the eigenvalues $\Pi_j^{\hat{\rho}_{\text{mix}}}(t)$ of the density operator $\hat{\rho}_{\text{mix}}(t)$, which read

$$\Pi_j^{\hat{\rho}_{\text{mix}}}(t)=\frac{1}{2}[1\pm e^{-2\mu\alpha^2}], \quad j=1,2. \tag{6.26b}$$

The entropy $S_0^{\hat{\rho}_{\text{mix}}}(t)$ of the statistical mixture is a monotonically decreasing function of time. On the other hand, the entropy $S_0^{|\alpha_e\rangle}(t)$ of the decaying even coherent state rapidly increases during the first instants of the time evolution, and after reaching its maximum at time $\gamma t=\ln 2$ it starts to decrease. We have to stress here that, unlike the von Neumann entropy of the squeezed state, the increase of the von Neumann entropy of the even coherent state during the first instants of the time evolution depends on the intensity of the mean photon number in the field. To be more specific, the larger the mean photon number is, the faster the entropy $S_0^{|\alpha_e\rangle}(t)$ increases, and after a very short time its value becomes essentially equal to the entropy $S_0^{\hat{\rho}_{\text{mix}}}(t)$ of the corresponding statistical mixture (see Figs. 6). In particular, at time $\gamma t=\ln 2$ the eigenvalues $\Pi_j^{|\alpha_e\rangle}(t)$ and $\Pi_j^{\hat{\rho}_{\text{mix}}}(t)$ are related as

$$\Pi_j^{|\alpha_e\rangle}(t)\Big|_{\mu=1/2}=\frac{1\pm e^{-\alpha^2}}{1+e^{-2\alpha^2}}\Pi_j^{\hat{\rho}_{\rm mix}}(t)\Big|_{\mu=1/2,}\qquad j=1,2, \tag{6.27}$$

from which it follows that for α large enough the corresponding eigenvalues are almost equal, and, consequently, the entropies $S_0^{\hat{\rho}_{\rm mix}}(t)$ and $S_0^{|\alpha_e\rangle}(t)$ are equal as well. This is in contrast with the case of the squeezed vacuum state and its corresponding statistical mixture [see the discussion following Eq. (6.22) and Fig. 4].

From the above we can conclude that quantum coherences which are established due to quantum interference between *two* coherent components $|\alpha\rangle$ and $|-\alpha\rangle$ of the superposition state (6.23) deteriorate very rapidly under the influence of the zero-temperature heat bath. To be more specific, quantum coherences deteriorate with a rate proportional to $\gamma\alpha^2$ (for more details see the review article by Bužek and Knight [19] and references therein). This deterioration is clearly seen already on the observation level $\mathcal{O}_0$. One of the consequences of this fact is that it is almost impossible to reconstruct a Wigner function of the even coherent state on the observation level $\mathcal{O}_0$, providing a nonunit efficiency measurement is considered (obviously, in the case of reduced observation levels the situation is even worse). As we said, this is in a sharp contrast with the case when the squeezed vacuum state of the same intensity is considered. On the other hand, we have to underline that the squeezed vacuum state can be expressed as an *infinite* sum of ''interfering'' pairs of coherent states of the form (6.23). We have seen that quantum coherences established between two coherent states deteriorate very rapidly under the influence of the zero-temperature heat bath. Nevertheless, the *total* quantum coherence of the squeezed vacuum state is *robust* with respect to the decay. This robustness is in a sense a ''collective'' effect of an infinite number of mutually interfering coherent components of the squeezed vacuum state.

We note that a very similar effect can be observed in the case of a Fock state $|n\rangle$, which can be expressed as a one-dimensional superposition of coherent states on a circle. One can find that the Shannon entropy (which in this case is equal to the von Neumann entropy, i.e., for Fock states the observation level $\mathcal{O}_A$ is identical to the complete observation level $\mathcal{O}_0$) of the decaying Fock state reaches its maximum at $\gamma t=\ln 2$, and its value is significantly different from the entropy of the corresponding statistical mixture. In this case again a collective interference between an infinite number of coherent components preserves the global quantum coherence associated with the Fock state [24].

To complete our discussion, we briefly note that on the observation level $\mathcal{O}_2$ the reconstructed Wigner function of the even coherent state (6.23) has a Gaussian form (5.3b) with the parameters σ_q^2 and σ_p^2 given by the relations

$$\sigma_q^2=(\overline{n}+1/2)+\alpha^2,\qquad \sigma_p^2=(\overline{n}+1/2)-\alpha^2 \tag{6.28a}$$

(here we recall that the mean photon number in the even coherent state is $\overline{n}=\alpha^2\tanh\alpha^2$). The Wigner function of the corresponding statistical mixture reconstructed on the observation level $\mathcal{O}_2$ also has a Gaussian shape [see Eq. (5.3b)] with the variances $\tilde{\sigma}_q^2$ and $\tilde{\sigma}_p^2$, which read

$$\tilde{\sigma}_q^2=2\alpha^2+1/2=2\overline{n}+1/2,\qquad \tilde{\sigma}_p^2=1/2. \tag{6.28b}$$

It is interesting to note that the Wigner functions of statistical mixtures corresponding to the squeezed vacuum state and the even coherent state, respectively, are on the observation level $\mathcal{O}_2$ *identical* [compare Eqs. (6.6c) and (6.28b) for the variances characterizing the corresponding Gaussian Wigner functions].

Using the general expression (4.21) we can evaluate the entropies $S_2^{|\alpha_e\rangle}(t)$ and $S_2^{\hat{\rho}_{\rm mix}}(t)$ associated with the even coherent state and the corresponding statistical mixture on the observation level $\mathcal{O}_2$, where the parameters $\chi_{|\alpha_e\rangle}(t)$ and $\chi_{\hat{\rho}_{\rm mix}}(t)$ are given by the relations

$$\chi_{|\alpha_e\rangle}(t)=\left[\mu\overline{n}-\mu^2\frac{\alpha^4}{\cosh^2\alpha^2}+1/4\right]^{1/2}-1/2 \tag{6.29a}$$

and

$$\chi_{\hat{\rho}_{\rm mix}}(t)=[\mu\overline{n}+1/4]^{1/2}-1/2, \tag{6.29b}$$

respectively. From Eqs. (4.21) and (6.29) it follows that the entropy $S_2^{|\alpha_e\rangle}(t)$ of the even coherent state on the observation level $\mathcal{O}_2$ is nonzero even at $t=0$, which simply reflects the fact that this is a non-Gaussian state. Moreover, in the limit of large α when $\tanh\alpha^2\to 1$, the entropies $S_2^{|\alpha_e\rangle}(t)$ and $S_2^{\hat{\rho}_{\rm mix}}(t)$ are equal for any time t. This means that for intensities large enough we are not able on the observation level $\mathcal{O}_2$ to distinguish between the even coherent state and the corresponding statistical mixture (even in the case of an ideal measurement, which in our model corresponds to $\gamma=0$).

VII. CONCLUSIONS

We have presented a universal method for reconstruction of Wigner functions of quantum-mechanical states of light. This method allows us to reconstruct Wigner functions with a certain degree of credibility (quantified with the help of the entropies) from a set of measured values of system observables. This set of observables defines a given observation level. We have to stress that the concept of observation levels plays a very important role in our attempt to measure and understand nonclassical effects associated with quantum states of light. In particular, a measurement of the second order quadrature squeezing is implicitly associated with the observation level $\mathcal{O}_2$. We know that reduction of quantum fluctuations (i.e., quadrature squeezing) has its origin in quantum interference between coherent components of superposition states of light. We have shown in our paper that the entropies S_k associated with quantum-mechanical states reconstructed on the observation level $\mathcal{O}_k$ can be used for quantification of the degree of quantum coherence, which has its origin in the phase-space interference between coherent components of superposition states. We have discussed in detail the role of nonunit measurement efficiency modeled as the decay of a quantum-mechanical state into a zero-temperature heat bath. We have shown that, in spite of the fact that the quantum coherence between two interfering coherent states deteriorates very rapidly, the global coherence associated with the squeezed vacuum state is very robust and can be easily ''detected'' on the observation level $\mathcal{O}_2$, which is identical to the complete observation level for *Gaussian* states.

ACKNOWLEDGMENTS

We acknowledge support by the East-West Program of the Austrian Academy of Sciences under Contract No. 45.367/1-IV/6a/94 of the Österreichisches Bundesministerium für Wissenschaft und Forschung.

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