# Displaced squeezed number states: Position space representation, inner product, and some applications

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For some applications the overall phase of a quantum state is crucial. For the so-called *displaced squeezed number state* (DSN), which is a generalization of the well-known squeezed coherent state, we obtain the position space representation with the correct overall phase, from the dynamics in a harmonic potential. The importance of the overall phase is demonstrated in the context of characteristic or moment generating functions. For two special cases the characteristic function is shown to be computable from the inner product of two different DSNs. [S1050-2947(96)06212-9]

## I. INTRODUCTION

For many purposes the overall phase of a quantum state is unimportant—for instance, when calculating transition probabilities or expectation values. However, in some applications the inclusion of the correct overall phase is crucial. A special class of such applications is the computation of expectation values using characteristic or generating functions [1–3] where the overall phase may depend on the parameter with respect to which differentiation is performed, as demonstrated in this paper for two specific cases.

In the present work we are concerned with various properties of a generalization of the squeezed coherent state (SCS) of the harmonic oscillator [4–8] (for reviews on squeezed states see, e.g., [9,10]) which we call the *displaced squeezed number state* (DSN). This state is equivalent to the *generalized harmonic-oscillator state* (GHO) [11,12], the only difference being that the DSN is defined using ''squeezed state terminology.''

In particular, we are interested in the characteristic function (CF) for Cahill-Glauber ordered products [13] of the canonical annihilator and creator $a$ and $a^\dagger$, respectively, and in the CF for powers of the number operator $a^\dagger a$, and it is shown that the CFs in these cases may be computed as inner products of two DSNs.

The inner product is most conveniently evaluated in position space and for this purpose we present an alternative way of deriving the correct position space representation of the DSN—correct in the sense that we obtain the correct overall phase. Our approach differs from previously reported methods for the SCS [6,14,15] in exploiting the relation between rotations in phase space and time evolution in a harmonic potential, thereby rendering our procedure physically more intuitive.

The paper is organized as follows: In Sec. II the basic framework is put forward along with the definition of a DSN. Section III derives the correct position space representation of the DSN and the general expression for the inner product is presented. In Sec. IV the above mentioned CFs are obtained as inner products from group-theoretical arguments and for two special cases they are given explicitly. Finally,

two applications of the CF are presented in Sec. V, and concluding remarks are found in Sec. VI.

## II. DEFINITION OF A DISPLACED SQUEEZED NUMBER STATE

Since the displacement and squeezing operators may be defined in a variety of ways we find it useful to present our definitions along with notational aspects.

With $a$ and $a^\dagger$ as the *canonical* annihilator (see, e.g., [16]) and creator, respectively,

$$a = \frac{1}{\sqrt{2\hbar}}(q+ip), \quad a^\dagger = \frac{1}{\sqrt{2\hbar}}(q-ip), \quad (1)$$

obeying the canonical commutator

$$[a,a^\dagger]=1, \quad (2)$$

we define the *reference* harmonic-oscillator Hamiltonian, with unity mass and frequency, as

$$H_{\text{H.O.}} = \hbar(a^\dagger a + 1/2) = \tfrac{1}{2}(p^2+q^2). \quad (3)$$

As defined in Eq. (1) $a$ and $a^\dagger$ have the standard properties

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad (4)$$

with $|n\rangle$ as the $n$th eigenstate of the Hamiltonian Eq. (3).

In accordance with Hollenhorst [7] and Caves [8] we define the displacement and squeezing operators as

$$D(z) = \exp(za^\dagger - z^*a)$$

$$S(\zeta) = \exp\{\tfrac{1}{2}(\zeta a^{\dagger 2} - \zeta^* a^2)\}, \quad (5)$$

respectively. Here, $z=|z|e^{i\phi}$ and $\zeta=re^{i\theta}$ are complex numbers and $z$ is related to the real numbers $q_z, p_z$ by

$$z = \frac{1}{\sqrt{2\hbar}}(q_z+ip_z) \quad (6)$$

or

$$q_z = \sqrt{2\hbar}|z|\cos\phi, \quad p_z = \sqrt{2\hbar}|z|\sin\phi. \quad (7)$$

One may also introduce a unitary rotation operator as $R(\lambda) = \exp[i\lambda(a^\dagger a + 1/2)]$, where $\lambda$ is real valued. Then, from the relation $R(\lambda) a R^\dagger(\lambda) = a \exp(-i\lambda)$ and by use of the unitary property of $R$ we have that (see also the appendix of [7])

$$R(\lambda) D(z) R^\dagger(\lambda) = D(z e^{i\lambda}),$$

$$R(\lambda) S(\zeta) R^\dagger(\lambda) = S(\zeta e^{2i\lambda}). \quad (8)$$

These relations serve as a basis for the following.

With the displacement and squeezing operators (5) we now *define* a DSN by

$$|z, \zeta, n\rangle = D(z) S(\zeta) |n\rangle. \quad (9)$$

We immediately recover the *squeezed state* of Caves [8] as $|z, \zeta\rangle \equiv |z, \zeta, 0\rangle$. At this point it should be noted that Yuen [6] introduced the squeezed state as a squeezed coherent state (SCS), i.e., as $S(\zeta) D(z) |0\rangle$. However, it follows from the definitions in Eq. (5) that (see, e.g., [9])

$$D(z) S(\zeta) = S(\zeta) D(\tilde{z}), \quad (10)$$

where

$$\tilde{z} = z \cosh r - z^* e^{i\theta} \sinh r, \quad (11)$$

and one may therefore easily get from one definition to the other. For this reason we also refer to $|z, \zeta\rangle$ as a SCS.

As for the reference oscillator we may introduce an annihilator and a creator for the DSNs. The DSN annihilator $A$ is defined through the requirement

$$A |z, \zeta, n\rangle = A D(z) S(\zeta) |n\rangle = D(z) S(\zeta) a |n\rangle. \quad (12)$$

From this we immediately obtain

$$A = D(z) S(\zeta) a S^\dagger(\zeta) D^\dagger(z)$$
$$= (a - z)\cosh r - (a^\dagger - z^*) e^{i\theta} \sinh r,$$

$$A^\dagger = D(z) S(\zeta) a^\dagger S^\dagger(\zeta) D^\dagger(z)$$
$$= (a^\dagger - z^*)\cosh r - (a - z) e^{-i\theta} \sinh r, \quad (13)$$

resulting in $A$ and $A^\dagger$ satisfying $[A, A^\dagger] = 1$ and having the following properties, analogous to Eq. (4) for $a$ and $a^\dagger$:

$$A |z, \zeta, n\rangle = \sqrt{n} |z, \zeta, n-1\rangle,$$

$$A^\dagger |z, \zeta, n\rangle = \sqrt{n+1} |z, \zeta, n+1\rangle. \quad (14)$$

In essence, for a fixed pair $(z', \zeta')$ the set of DSNs with $n = 0, 1, \ldots$ are the eigenstates of another oscillator with a (complex-valued) frequency different from unity carrying the same properties as the eigenstates of the reference oscillator [17].

### III. POSITION SPACE REPRESENTATION AND INNER PRODUCT

#### A. DSN in position space

In this section we derive an explicit expression for a DSN in position space, i.e.,

$$\langle q | z, \zeta, n \rangle = \langle q | D(z) S(\zeta) | n \rangle. \quad (15)$$

To our knowledge such an expression does not exist—not even for the ground state. It should be noted, though, that due to the relation in Eq. (11) an expression for $\langle q | D(z) S(\zeta) | n \rangle$ can be obtained from an expression for $\langle q | S(\zeta) D(z) | n \rangle$. An expression for the latter (although only for $n = 0$) has already been given by Yuen [6] and it has been rederived later in different ways by several authors [14,15]. However, it is common to all these derivations that they involve evaluation of complicated integrals which is not the case in the approach we present below. Furthermore, our approach treats squeezing and displacement equally for all $n$.

It has previously been argued that the coordinate representation of a DSN can be written on the form of a generalized harmonic-oscillator state (GHO) [11,12] which takes the form

$$\langle q | \phi_n \rangle = \frac{1}{\sqrt{2^n n!}} \left( \frac{1}{\pi \hbar} \right)^{1/4} H_n[\sqrt{\kappa}(q - q')] G(q) e^{-in\beta}, \quad (16)$$

where

$$G(q) = \exp\left\{ \frac{i}{\hbar} [\alpha(q - q')^2 + p_z(q - q') + \gamma] \right\}, \quad (17)$$

and $\kappa = 2 \operatorname{Im} \alpha / \hbar$. In this expression $\alpha$ and $\gamma$ may be complex while the other parameters are real. With the parameter values $\alpha = i/2$, $\beta = \gamma = 0$, and $q' = p' = 0$, the $n$th GHO equals the coordinate representation of harmonic-oscillator eigenstate $|n\rangle$. It was shown in [12] that a GHO with given parameters is proportional to a DSN and the $z$ and $\zeta$ parameters were determined in terms of the parameters in the GHO. However, the proportionality constant was left undetermined. In the present work we determine uniquely the parameters in Eq. (16) so that

$$\langle q | z, \zeta, n \rangle = \langle q | \phi_n \rangle. \quad (18)$$

To evaluate the effect of the squeezing operator with the complex argument $\zeta$ we decompose $S(\zeta)$, as in [15], into a product of rotations and a squeezing with a real parameter as given by Eq. (8),

$$S(\zeta) = S(r e^{i\theta}) = R(\theta/2) S(r) R^\dagger(\theta/2). \quad (19)$$

Then the squeezing part is easy to evaluate since the squeezing operator with a real argument is just a scale transformation with the scale factor $\exp(-r)$ [5,7], that is,

$$\langle q | S(r) | \psi \rangle = e^{-r/2} \langle e^{-r} q | \psi \rangle. \quad (20)$$

To evaluate the effect of the rotation operators we make use of the relation between rotations in phase space and time evolution in a harmonic potential. The classical time evolution in a harmonic potential is a rigid rotation in phase space about the origin and since the dynamics in a harmonic potential is essentially the same in classical and quantum mechanics the same is to be expected in quantum mechanics. In fact, it follows from the previous section that

$$R(\theta/2) = U_{\text{H.O.}}(-\theta/2), \qquad (21)$$

where

$$U_{\text{H.O.}}(t) = e^{-iH_{\text{H.O.}}t/\hbar} \qquad (22)$$

is the time evolution operator for the harmonic oscillator. Thus

$$S(\zeta) = U_{\text{H.O.}}(-\theta/2)S(r)U_{\text{H.O.}}(\theta/2). \qquad (23)$$

The form of a GHO is well known to be preserved under time evolution in a harmonic potential [12,18], the parameters $\alpha, \beta, \gamma$ satisfying

$$\alpha_t = \frac{1}{2}\left(\frac{2\alpha_0\cos t - \sin t}{2\alpha_0\sin t + \cos t}\right),$$

$$\beta_t = \beta_0 + 2\int_0^t dt'\,\text{Im}\alpha_{t'},$$

$$\gamma_t = \gamma_0 + \tfrac{1}{2}(q_t p_t - q_0 p_0)$$

$$+ \frac{i\hbar}{2}\ln(2\alpha_0\sin t + \cos t). \qquad (24)$$

Here $\alpha_0$, $\beta_0, \gamma_0$ are initial values and $q_t, p_t$ are given by classical evolution of their initial values $q_0, p_0$.

The displacement operator $D(z)$ displaces the expectation value of the position by $q_z$ and the expectation values of the momentum by $p_z$ according to

$$\langle q|D(z)|\psi\rangle = \langle q - q_z|\psi\rangle e^{iqp_z/\hbar}e^{-iq_z p_z/(2\hbar)}. \qquad (25)$$

Putting it all together we therefore find that a DSN can be written

$$\langle q|z,\zeta,n\rangle = \langle q|\phi_n\rangle, \qquad (26)$$

where the right-hand side is a GHO defined in Eq. (16) with the parameters given by

$$\alpha = \frac{i}{2}\left(\frac{\cosh r - e^{i\theta}\sinh r}{\cosh r + e^{i\theta}\sinh r}\right), \quad \beta = \frac{i}{2}\ln\left(\frac{\cosh r + e^{-i\theta}\sinh r}{\cosh r + e^{i\theta}\sinh r}\right),$$

$$\gamma = \frac{i\hbar}{2}\ln(\cosh r + e^{i\theta}\sinh r) + \frac{q_z p_z}{2}, \qquad (27)$$

and $q' = q_z$, $p' = p_z$. The uncertainties in position and momentum of a DSN are [12]

$$(\Delta q)_n^2 = (1 + 2n)\frac{\hbar}{4\,\text{Im}\alpha} = (1 + 2n)\frac{\hbar}{2\,\text{Re}s_\theta},$$

$$(\Delta p)_n^2 = (1 + 2n)\frac{\hbar|\alpha|^2}{\text{Im}\alpha} = (1 + 2n)\frac{\hbar\,|s_\theta|^2}{2\,\text{Re}s_\theta}, \qquad (28)$$

where we have introduced the complex squeezing parameter $s_\theta = -2i\alpha$ of Schleich and Wheeler [19]. We see that the uncertainty product equals the uncertainty product for a harmonic-oscillator eigenstate if $s_\theta$ is real, that is, if $\theta = l\pi$ with $l$ integral.

As a special case we have the coordinate representation of a SCS

$$\langle q|z,\zeta\rangle = \left(\frac{1}{\pi\hbar}\right)^{1/4}(\cosh r + e^{i\theta}\sinh r)^{-1/2}$$

$$\times\exp\left\{-\frac{1}{2\hbar}\left(\frac{\cosh r - e^{i\theta}\sinh r}{\cosh r + e^{i\theta}\sinh r}\right)(q - q_z)^2\right.$$

$$\left. + \frac{i}{\hbar}p_z(q - q_z/2)\right\}, \qquad (29)$$

where $q_z = \sqrt{2\hbar}\,\text{Re}z$ and $p_z = \sqrt{2\hbar}\,\text{Im}z$. Invoking Eq. (11) we find that this expression coincides with Eq. (3.24) of Yuen [6].

We end this section by mentioning that the momentum space and the Wigner phase-space representations of a GHO were found in [12] and with Eq. (27) the expressions in [12] also give the momentum space and the Wigner phase-space representations of a DSN.

### B. Inner products of DSNs

As mentioned previously, a set of DSNs with $n = 0,1,\ldots$ for a fixed pair $(z',\zeta')$ has the same properties as the reference oscillator. This means that for each pair of $(z',\zeta')$ the set of DSNs constitutes a complete basis (the harmonic-oscillator eigenbasis being a special case). The inner product of two DSNs characterized by $(z_1,\zeta_1)$ and $(z_2,\zeta_2)$ therefore gives the transition amplitude between basis states belonging to different bases. State expansion into a basis of DSNs is particularly convenient in the study of dynamics in quadratic potentials [11,12,20,21]. For the present purpose the inner product of two DSNs is desired in order to evaluate various characteristic functions and their derivatives as shown in Sec. IV.

Since we have the coordinate representation of the DSNs, the obvious way to calculate the inner product would be to evaluate the integral

$$\int dq\langle z_1,\zeta_1,n_1|q\rangle\langle q|z_2,\zeta_2,n_2\rangle. \qquad (30)$$

This has been done for several special cases in the literature (in connection with the evaluation of Franck-Condon factors, see, e.g., [22,23]) but is rather cumbersome in the general case. A different approach using annihilators and creators was used by Meyer [11] in the evaluation of the inner product of a harmonic-oscillator eigenstate and a DSN. Meyer takes advantage of the linearity of the transformation between $A, A^\dagger$ and $a, a^\dagger$ as given in Eq. (13). This approach can easily be generalized to the evaluation of the inner product of two DSNs since, in general, the transformation between $A_1, A_1^\dagger$ and $A_2, A_2^\dagger$ is a linear canonical transformation, given by

$$\begin{pmatrix} A_2 \\ A_2^\dagger \\ 1 \end{pmatrix} = \begin{pmatrix} \sigma_{21} & \delta_{21} & -\eta_{21} \\ \delta_{21}^* & \sigma_{21}^* & -\eta_{21}^* \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_1^\dagger \\ 1 \end{pmatrix}, \qquad (31)$$

where

$$\sigma_{kl} = \cosh r_k \cosh r_l - e^{i(\theta_k - \theta_l)} \sinh r_k \sinh r_l,$$

$$\delta_{kl} = e^{i\theta_l} \sinh r_l \cosh r_k - e^{i\theta_k} \sinh r_k \cosh r_l,$$

$$\eta_{kl} = (z_k - z_l) \cosh r_k - (z_k^* - z_l^*) e^{i\theta_k} \sinh r_k. \qquad (32)$$

Having left the details for the Appendix, we find that the inner product of two DSNs can be written as

$$\langle z_1, \zeta_1, n_1 | z_2, \zeta_2, n_2 \rangle$$

$$= \frac{1}{\sqrt{\sigma_{21}}} \exp\left\{ \frac{\eta_{21}\eta_{12}^*}{2\sigma_{21}} + \frac{1}{2}(z_2 z_1^* - z_2^* z_1) \right\} \left( \frac{\eta_{21}}{\sigma_{21}} \right)^{n_1} \left( \frac{\eta_{12}^*}{\sigma_{21}} \right)^{n_2}$$

$$\times \sum_{j=0}^{[n_1/2]} \sum_{k=0}^{[n_2/2]} \sum_{l=0}^{\min(m_1,m_2)} \frac{(n_1! n_2!)^{1/2}}{j! k! l! (m_1-l)! (m_2-l)!}$$

$$\times \left( \frac{-\delta_{21}\sigma_{21}}{2(\eta_{21})^2} \right)^j \left( \frac{\delta_{21}^*\sigma_{21}}{2(\eta_{12}^*)^2} \right)^k \left( \frac{\sigma_{21}}{\eta_{21}\eta_{12}^*} \right)^l, \qquad (33)$$

where $m_1 = n_1 - 2j$ and $m_2 = n_2 - 2k$.

## IV. CHARACTERISTIC FUNCTIONS AS INNER PRODUCTS

In this section we consider how the general inner product obtained in Eq. (33) may be employed to facilitate the evaluation of certain transition matrix elements using characteristic functions (CFs). Specifically, we are interested in the transition matrix elements of the following two types of operators:

(i) Cahill-Glauber [13] ordered products $\{a^{\dagger k} a^l\}_s$,

(ii) powers of the number operator $(a^\dagger a)^k$.

Here, $s$ is an ''ordering'' parameter where $s = (-)1$ means (anti-)normal ordering and $s = 0$ is symmetrical or Weyl ordering (see, e.g., [24]).

In principle, we only need the transition matrix elements of the first of the above types of operators since the powers of the number operator can be expressed as a function of Cahill-Glauber type products [25]. However, because the number operator is ''special'' we find it appropriate to consider the powers of it separately.

In this section we generalize the notion of a ''characteristic function,'' in that both expectation values—as usual—and transition matrix elements should be computable from it. Hence, in the general case the CF $C_\Omega(\epsilon; \Psi, \Psi')$ for the ''transition moments'' of an operator $\Omega$ between the states $|\Psi\rangle$ and $|\Psi'\rangle$ is chosen such that $\langle \Psi | \Omega^k | \Psi' \rangle$ is given by

$$\langle \Psi | \Omega^k | \Psi' \rangle = \lim_{\epsilon \to 0} \partial_\epsilon^k C_\Omega(\epsilon; \Psi, \Psi'), \qquad (34)$$

which may be accomplished by defining the CF as

$$C_\Omega(\epsilon; \Psi, \Psi') = \langle \Psi | e^{\epsilon\Omega} | \Psi' \rangle. \qquad (35)$$

In the CF $\epsilon$ is a ''dummy'' parameter used for the temporary differentiations, according to Eq. (34). Equation (35) is a straightforward generalization of the expectation value CF. In the following the expectation value CF is referred to as the diagonal CF.

For the cases of current interest we define the CFs, for future convenience, slightly differently by

$$C^s(\xi, \xi^*) = e^{s\xi\xi^*/2} \langle z, \zeta, n | \exp(\xi a^\dagger - \xi^* a) | z', \zeta', n' \rangle,$$

$$C(\chi) = \langle z, \zeta, n | \exp(i\chi a^\dagger a) | z', \zeta', n' \rangle, \qquad (36)$$

where the ''dummy'' parameters $\xi$ and $\chi$ are complex valued and real valued, respectively. $\xi$ and $\xi^*$ are considered independent during differentiation. From this the transition moments are given by

$$\langle z, \zeta, n | \{a^{\dagger k} a^l\}_s | z', \zeta', n' \rangle = \lim_{\xi \to 0} \partial_\xi^k \partial_{-\xi^*}^l C^s(\xi, \xi^*),$$

$$\langle z, \zeta, n | (a^\dagger a)^k | z', \zeta', n' \rangle = \lim_{\chi \to 0} \partial_{i\chi}^k C(\chi). \qquad (37)$$

To evaluate the CFs Eq. (36) we invoke the properties of the so-called *two-photon* Lie group $H_6$ and its corresponding Lie algebra $h_6$ [26]. The generators of the algebra $h_6$ are $\{a^\dagger a + \frac{1}{2}, a^{\dagger 2}, a^2, a^\dagger, a, I\}$ and the elements of the corresponding group, $H_6$, may be obtained by exponentiation of the $h_6$ generators (see, e.g., [27]). A general *unitary* element $h$ of the group $H_6$ can be expressed as [26]

$$h = DS R e^{i\tau I}, \qquad (38)$$

where $D$, $S$, and $R$ are the operators introduced in Sec. II, $I$ is the identity, and $\tau$ is a real-valued parameter. Hence, the most general state that can be formed by action of a unitary element of $H_6$ on a number state $|n\rangle$ is $DSRe^{i\tau I}|n\rangle$. However, since $|n\rangle$ is an eigenstate of the operator $R$ the most general state reduces to $DS|n\rangle e^{i\tau'}$. In other words, acting on a number state $|n\rangle$ with a unitary element of the Lie group $H_6$ results in a DSN multiplied with a phase factor.

Keeping this in mind, we then return to the definition of the CFs in Eq. (36). The exponential operators appearing there are also seen to be unitary elements of $H_6$. Hence, we have that

$$\text{CF} = \text{const} \times \langle z, \zeta, n | h_1 | z', \zeta', n' \rangle$$

$$= \text{const} \times \langle n | h^{-1} h_1 h' | n' \rangle$$

$$= \text{const} \times \langle z, \zeta, n | z, ''\zeta, ''n' \rangle, \qquad (39)$$

where ''CF'' means any of the two CFs of present interest [28]. Thus, the CFs are nothing but the inner product of two different DSNs, multiplied with a constant. Below, the diagonal CFs for the moments are given explicitly.

### A. CF for Cahill-Glauber products

The diagonal CF for Weyl ordering, in general terms, is perhaps one of the most well-known CFs in quantum mechanics, since it is the Fourier transform of the celebrated Wigner distribution function (see, e.g., [1,24,29,30]). Also the diagonal CFs for normal and antinormal ordering have been discussed extensively in the literature [1,30] and accordingly we shall not go into great detail with these.

### 1. General case

For a DSN the diagonal CF for Weyl ordering has been obtained by Zhang, Feng, and Gilmore [26] in the special case $|z,\zeta,0\rangle$ [31]. The extension to the general case with DSNs is straightforward using the result above together with the inner product Eq. (33). Using the Baker-Campbell-Hausdorff relation for displacement operators (see, e.g., [26,27]) we immediately have

$$C^s(\xi,\xi^*)=\langle z,\zeta,n|\xi+z',\zeta',n'\rangle e^{(\xi z'^*-\xi^*z')/2}e^{s|\xi|^2/2}.$$
(40)

### 2. Diagonal case

In the diagonal case of the above CF the parameters Eq. (32) for the inner product are particularly simple,

$$\sigma_{21}=1, \quad \delta_{21}=0,$$

$$\eta_{21}=-\eta_{12}=\xi\cosh r-\xi^*e^{i\theta}\sinh r\equiv\widetilde{\xi},$$
(41)

and since $\delta_{21}=0$ the only contribution from the $j$ and $k$ sums in Eq. (33) is unity for $j=k=0$. This, together with $n_1=n_2=n$ resulting in $m_1=m_2=n$, gives us the final expression

$$C^s_{\text{diag}}(\xi,\xi^*)=\exp[\xi\,z^*-\xi^*z-(|\widetilde{\xi}|^2-s|\,\xi|^2)/2]$$

$$\times\sum_{k=0}^{n}\binom{n}{k}\frac{(-|\widetilde{\xi}|^2)^k}{k!},$$
(42)

where we have replaced the summation index $n-l$ in Eq. (33) by $k$.

### B. CF for powers of the number operator

### 1. General case

The CF $C(\chi)$ for powers of the number operator, $(a^\dagger a)^k$, is obtained in a manner similar to the previous one. Hence, we immediately obtain

$$C(\chi)=\langle z,\zeta,n|R(\chi)D(z')S(\zeta')R^\dagger(\chi)R(\chi)|n'\rangle e^{-i\chi/2}$$

$$=\langle z,\zeta,n|z'e^{i\chi},\zeta'e^{2i\chi},n'\rangle e^{in'\chi},$$
(43)

where we have used that $|n\rangle$ is an eigenstate of $R$.

### 2. Diagonal case

The parameters Eq. (32) for the diagonal CF are in this situation all nonvanishing,

$$\sigma_{21}=\cosh^2r-e^{2i\chi}\sinh^2r$$

$$\delta_{21}=e^{i\theta}\sinh r\cosh r(1-e^{2i\chi}),$$

$$\eta_{21}=|z|e^{i\theta/2}(e^{i\chi}-1)\{\cos\phi'(\cosh r+e^{i\chi}\sinh r)$$

$$+\sin\phi'(\cosh r-e^{i\chi}\sinh r)\},$$

$$\eta_{12}^*=-|z|e^{-i\theta/2}(e^{-i\chi}-1)\{\cos\phi'(\cosh r+e^{i\chi}\sinh r)$$

$$-\sin\phi'(\cosh r-e^{i\chi}\sinh r)\},$$
(44)

where $\phi'=\phi-\theta/2$. This results in the more complicated final expression

$$C_{\text{diag}}(\chi)=e^{in\chi}(\cosh^2r-e^{2i\chi}\sinh^2r)^{-(n+1/2)}\times\exp\{-|z|^2(1-\cos\chi)(s_\chi^{-1}\cos^2\phi'-s_\chi\sin^2\phi')+i|z|^2\sin\chi\}$$

$$\times\sum_{j=0}^{[n/2]}\sum_{k=0}^{[n/2]}\sum_{l=0}^{\min(m_1,m_2)}\frac{n!}{j!k!l!(m_1-l)!(m_2-l)!}\left(\frac{(e^{2i\chi}-1)\sinh r\cosh r}{2|z|^2(e^{i\chi}-1)^2(s_\chi^{-1}\cos^2\phi'+s_\chi\sin^2\phi'+2\cos\phi'\sin\phi')}\right)^j$$

$$\times\left(\frac{(1-e^{-2i\chi})\sinh r\cosh r}{2|z|^2(1-e^{-i\chi})^2(s_\chi^{-1}\cos^2\phi'+s_\chi\sin^2\phi'-2\cos\phi'\sin\phi')}\right)^k[|z|^2(1-\cos\chi)(s_\chi^{-1}\cos^2\phi'-s_\chi\sin^2\phi')]^{n-l},$$
(45)

where $s_\chi$ is given by

$$s_\chi=\frac{\cosh r-e^{i\chi}\sinh r}{\cosh r+e^{i\chi}\sinh r}.$$
(46)

In the case of a SCS $|z,\zeta\rangle$ the sums in Eq. (45) reduce to unity, simplifying the CF considerably.

From the expressions of the CFs, Eq. (45) and Eq. (42), and the prescriptions for the evaluation of the moments, Eq. (37), it is evident why the overall phase of a DSN is crucial in this context. The parameters with respect to which we differentiate in Eq. (37) enters the DSN together with the

displacement and squeezing angles. Hence, any part of the DSN depending on these angles is to be differentiated—even the overall phase.

## V. APPLICATIONS

Different aspects of the results obtained in the previous sections may be applied in different contexts, which in this section is demonstrated by two examples.

### A. Energy transfer in quadratic potentials

A simple application of the moments of the number operator is in the study of vibrational energy transfer in heavy

particle scattering. A model for this is a forced harmonic oscillator with time-dependent frequency [11]. In scaled coordinates the model Hamiltonian reads

$$H(t) = \tfrac{1}{2}[p^2 + \omega(t)q^2] - qF(t), \qquad (47)$$

where $F(t \to -\infty) = F(t \to \infty) = 0$ and $\omega(t \to -\infty) = \omega(t \to \infty)\omega(t \to -\infty) = \omega(t \to \infty) = 1$. If the initial state $(t \to -\infty)$ is expanded in eigenstates of the harmonic oscillator $H = (p^2 + q^2)/2$, i.e.,

$$|\Psi_i\rangle = \sum_n c_n|n\rangle, \qquad (48)$$

the final state $(t \to \infty)$ is a sum of DSNs with the same expansion coefficients [11,12]

$$|\Psi_f\rangle = \sum_n c_n|z,\zeta,n\rangle. \qquad (49)$$

The energy transfer is given by

$$\Delta E = \langle a^\dagger a\rangle_f - \langle a^\dagger a\rangle_i. \qquad (50)$$

Similarly, the variance in the energy transfer can be defined as

$$\Delta^2 = \langle (a^\dagger a)^2\rangle_f - \langle a^\dagger a\rangle_f^2 + \langle (a^\dagger a)^2\rangle_i - \langle a^\dagger a\rangle_i^2$$
$$- 2\,\text{Re}\langle (a^\dagger a)_f(a^\dagger a)_i\rangle + 2\langle a^\dagger a\rangle_f\langle a^\dagger a\rangle_i. \qquad (51)$$

Thus the energy transfer and the variance are expressed in terms of the moments

$$\langle (a^\dagger a)^k\rangle_i = \sum_n |c_n|^2 n^k,$$

$$\langle (a^\dagger a)^k\rangle_f = \sum_{m,n} c_m^* c_n\langle z,\zeta,m|(a^\dagger a)^k|z,\zeta,n\rangle,$$

$$\langle (a^\dagger a)_f(a^\dagger a)_i\rangle = \sum_{m,n} n c_m^* c_n\langle z,\zeta,m|(a^\dagger a)^k|z,\zeta,n\rangle,$$
$$\qquad (52)$$

where $k = 1,2$. One way to evaluate the moments in the second line is to rewrite $(a^\dagger a)^k$ in terms of $A$ and $A^\dagger$ and make use of Eq. (14) as done in [11] but an alternative way would be to evaluate them using the CF method discussed in Sec. IV [32].

### B. Atomic inversion in the Jaynes-Cummings model

Again in this example we are concerned about moments of the number operator. However, unlike the preceding example, where one could argue that the CF method to a certain degree is ''overkill,'' it is in this case essential for practical purposes.

On exact one-photon resonance, the atomic inversion in the Jaynes-Cummings model (JCM) [33], with the atom initially in its ground state, is customarily given by

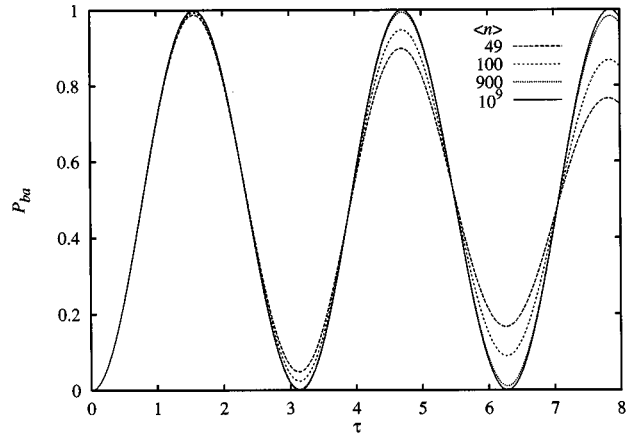$$P_{ba}(t) = \sum_{n=0}^{\infty} P_n\sin^2(\omega_n t/2), \qquad (53)$$



FIG. 1. Atomic inversion for an initial CS field as a function of the dimensionless time $\tau$ defined in the text. At low intensity the familiar collapse is apparent whereas at high intensities the inversion approximates the classical Rabi oscillation.

where $\omega_n = 2g\sqrt{n}/\hbar$ is the $n$-photon quantum Rabi frequency [34], $g$ is the dipole coupling matrix element, and $P_n$ is the photon number distribution associated with the initial field state.

Since Eq. (53) is a sum over $n$, the standard expression is inconvenient for high-intensity fields (see the example below). However, by a Taylor expansion of the square of the sine in the above expression, and by employing the fact that $\sum_{n=0}^{\infty} P_n n^k = \langle n^k\rangle$, we may obtain the following alternative expression for the inversion:

$$P_{ba}(t) = -\sum_{k=1}^{\infty} \frac{(-4)^k}{2(2k)!} \tau^{2k} \frac{\langle n^k\rangle}{\bar{n}^k}, \qquad (54)$$

where $\tau$ is a dimensionless time given by $\tau = gt\sqrt{\bar{n}}/\hbar$ and $\bar{n} = \langle n\rangle$. This expression is more suitable than Eq. (53) for comparatively large field intensities, since the sum is independent of $n$. A thorough discussion of the expression Eq. (54), along with an expression for the case of nonvanishing detuning, is to appear elsewhere [35].

For all practical purposes the series Eq. (54) must, of course, be truncated and in this case it is only valid for short times. Hence, for the $k$th order truncation we need the moments up to order $k$. However, for $k$ much larger than two the direct evaluation of the moments becomes an enormous task. This is where the CF method comes into play [36].

To illustrate the usefulness of Eq. (54) we have calculated the inversion numerically for two different initial fields at four different intensities:

(i) CS with $\bar{n} = 49,100,900,10^9$ (Fig. 1),

(ii) SCS with $\bar{n} = 49,100,900,10^9$, $\lambda = 0.01$, and $\theta = 0$ (Fig. 2),

where $\lambda$ is an alternative parameter for the squeezing magnitude $r$ [37]. In all cases we have truncated Eq. (54) at $k = 50$. In comparison, using the standard expression for, e.g., a CS one should at least include $\sqrt{\bar{n}}$ terms which in the case of $\bar{n} = 10^9$ would amount to approximately 30 000 terms.
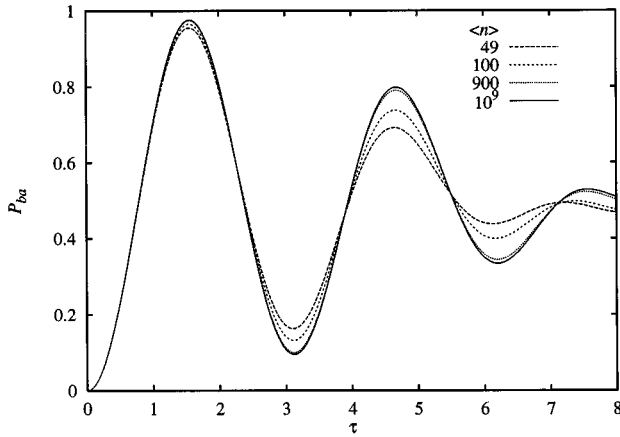
FIG. 2. Atomic inversion for an initial SCS field as a function of the dimensionless time $\tau$ defined in the text. The collapse of the inversion is seen to persist at even high intensities, which manifests squeezed light as being truly nonclassical.

Figure 1 shows the results for the CS and the well-known fact that the inversion for a CS at high intensities approximates the classical Rabi oscillation is recovered. In Fig. 2 the inversion for the SCS is shown. Unlike the CS case, the inversion does not approximate the classical Rabi oscillation at high intensities which, to the best of the authors' knowledge, has not been demonstrated previously.

## VI. CONCLUSION

The purpose of the present work has been twofold: (a) We have obtained the position representation of a displaced squeezed number state with the correct overall phase using the relations between geometric operations in phase space and dynamics, and (b) we have shown by group-theoretical arguments that certain characteristic or moment generating functions are obtained as inner products between different DSNs.

The position representation with the correct overall phase has been given previously by several authors in the special case of a squeezed state. However, these derivations all involved the evaluation of complicated integrals and for the general DSN one has not previously been concerned about the overall phase. The scheme put forward in this paper for obtaining the position representation for the general DSN avoids the evaluation of complicated integrals by exploiting that rotations in harmonic-oscillator phase space are nothing but time translations in the corresponding harmonic potential. Hence, the decomposition of the general squeezing operator into a product of rotations in phase space and a squeezing along the $q$ axis enables the construction of a DSN to be regarded as two geometrical operations and two dynamical operations: a forward time evolution, a squeeze along the $q$ axis, a backward time evolution, and finally a displacement. In this way the construction of a DSN becomes physically very intuitive.

The reason for worrying about the overall phase, at all, became very clear in connection with the considered CFs. Because of the group relation between the displacement, squeezing, and rotation (time-evolution) operators we were able to express the CFs for Cahill-Glauber ordered products and powers of the number operator as inner products of DSNs. Since the parameters, with respect to which we have to differentiate the CFs when evaluating moments, enter the wave function together with the squeezing and displacement angles and since the overall phase *depends* on these angles it is obvious that omitting the overall phase in this context would lead to erroneous results for the moments.

The usefulness of the CFs in calculating moments was demonstrated by two examples: (a) calculation of mean value and variance of the energy transfer between states in a quadratic potential, and (b) calculation of the atomic inversion in the Jaynes-Cummings model for high-intensity quantized fields. The results of the first example was simplified by the CF method but could as well have been obtained by standard operator reordering, as done by Meyer for a special case. In the second example, however, the power of the CF method emerged. In that case we obtained the well-known result that the atomic inversion approximates the classical Rabi oscillations for high-intensity coherent fields. More interestingly, we were able to show that the atomic inversion does not become a classical Rabi oscillation when a high intensity *squeezed* field is applied.

## APPENDIX

In this appendix we briefly review the method used by Meyer [11] to find the inner product of two generalized number states where the annihilator and creator defining one number state can be obtained by a linear canonical transformation of the annihilator and creator defining the other. The essence of Meyers approach is to use the annihilator and creator for the two DSNs to relate the inner product of these to the inner product of the corresponding SCSs. This can be done invoking the relation

$$\sum_n \frac{t^n}{\sqrt{n!}}|z,\zeta,n\rangle = e^{tA^{\dagger}}|z,\zeta,0\rangle = e^{tA^{\dagger}}|z,\zeta\rangle \tag{A1}$$

in order to write

$$\sum_{n_1,n_2} \frac{s^{n_1}t^{n_2}}{\sqrt{n_1!n_2!}}\langle z_1,\zeta_1,n_1|z_2,\zeta_2,n_2\rangle$$
$$= \langle z_1,\zeta_1|e^{sA_1}e^{tA_2^{\dagger}}|z_2,\zeta_2\rangle. \tag{A2}$$

Using Eq. (31) and $[A_k,A_k^{\dagger}]=1$, the right-hand side of Eq. (A2) can be expanded into a power series in $s$ and $t$ times the inner product $\langle z_1,\zeta_1|z_2,\zeta_2\rangle$ (see [11] for details). Comparing like powers of $s$ and $t$ then gives

$$\langle z_1,\zeta_1,n_1|z_2,\zeta_2,n_2\rangle = \langle z_1,\zeta_1|z_2,\zeta_2\rangle \left(\frac{\eta_{21}}{\sigma_{21}}\right)^{n_1}\left(\frac{\eta_{12}^*}{\sigma_{21}}\right)^{n_2}\sum_{j=0}^{[n_1/2]}\sum_{k=0}^{[n_2/2]}\sum_{l=0}^{\min(m_1,m_2)}$$

$$\times\frac{(n_1!n_2!)^{1/2}}{j!k!l!(m_1-l)!(m_2-l)!}\left(\frac{-\delta_{21}\sigma_{21}}{2(\eta_{21})^2}\right)^j\left(\frac{\delta_{21}^*\sigma_{21}}{2(\eta_{12}^*)^2}\right)^k\left(\frac{\sigma_{21}}{\eta_{21}\eta_{12}^*}\right)^l, \tag{A3}$$

where $m_1=n_1-2j$ and $m_2=n_2-2k$. Since the SCSs are Gaussians in position space one easily gets

$$\langle z_1,\zeta_1|z_2,\zeta_2\rangle = \int dq\langle z_1,\zeta_1|q\rangle\langle q|z_2,\zeta_2\rangle = (\cosh r_2\cosh r_1 - e^{i(\theta_2-\theta_1)}\sinh r_2\sinh r_1)^{-1/2}e^{-i(q_{z_2}p_{z_2}-q_{z_1}p_{z_1})/(2\hbar)}$$

$$\times\exp\left\{\frac{\alpha_2\alpha_1^*(q_{z_2}-q_{z_1})^2+(p_{z_2}-p_{z_1})^2/4-(p_{z_2}-p_{z_1})(\alpha_2q_{z_2}-\alpha_1^*q_{z_1})}{i\hbar(\alpha_2-\alpha_1^*)}\right\}, \tag{A4}$$

where again $q_z=\sqrt{2\hbar}\,\mathrm{Re}z$, $p_z=\sqrt{2\hbar}\,\mathrm{Im}z$ and $\alpha$ is given in Eq. (27). Introducing the parameters in Eq. (32) the inner product $\langle z_1,\zeta_1|z_2,\zeta_2\rangle$ can be written in a more compact form as

$$\langle z_1,\zeta_1|z_2,\zeta_2\rangle = \frac{1}{\sqrt{\sigma_{21}}}\exp\left\{\frac{\eta_{21}\eta_{12}^*}{2\sigma_{21}}+\frac{1}{2}(z_2z_1^*-z_2^*z_1)\right\}. \tag{A5}$$

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[37] $\lambda$—called the "squeezer"—is defined through the relation $\lambda\bar{n}=\sinh^2 r$. By varying $\lambda$ the squeezing magnitude $r$ is altered without altering $\bar{n}$. Hence, keeping the "squeezer" constant means fixing the ratio between "coherent" and "squeezed" photons. The significance of $\lambda$ is to be discussed further in [35].