# Non-Markovian dynamics of the micromaser due to discrete and continuous non-Poissonian pumping

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We investigate the effect of non-Poissonian pump fluctuations on the micromaser dynamics. Non-Poissonian micromaser pumping has been described by two, seemingly unrelated, theoretical approaches. The first employs a discrete pumping process where the pump atoms are allowed to arrive, with certain probability $p$, only at regularly spaced instants of time, whereas the second refers to continuously distributed arrival times of the pump atoms. Based on a generalization of the latter, we present a unified approach that can handle both situations on an equal footing. We find that for any kind of non-Poissonian pumping the resulting dynamics of the micromaser field is a non-Markovian one. For a micromaser with discrete non-Poissonian pumping, we show the equivalence of ensemble averaging and time averaging, providing a rare example where the validity of the ergodic hypothesis can be explicitly demonstrated. Moreover, we investigate the time-delayed field-field correlation function and a generalized $k$-photon spectrum of the cavity field, which for $k=1$ corresponds to the usual power spectrum. For the case that the micromaser is operated under the $k$-photon trapping condition, we derive exact analytical expressions for the $k$-photon spectrum and the corresponding correlation functions that result from the exact solution of a non-Markovian evolution problem. Provided that $p>\frac{1}{2}$, the spectrum is found to be split into several equidistant peaks for certain values of the interaction parameters. [S1050-2947(96)03312-4]

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## I. INTRODUCTION

In the one-atom micromaser a beam of Rydberg atoms interacts with the radiation field in a microwave cavity of high quality in such a way that, at most, one atom at a time is present in the cavity [1]. Usually the atoms in the beam are excited to the upper level of a resonant microwave transition. In the conventional micromaser, the injected atoms that pump the cavity are statistically independent, the pump statistics therefore being described by a Poissonian process. Due to the atom-field interaction, nonclassical properties of the radiation field such as sub-Poissonian photon statistics may arise [2]. The steady-state properties of the field as well as the field dynamics depend on the character of the pump statistics. In particular, the nonclassical character of the field can be enhanced by sub-Poissonian pumping. Therefore, the effect of non-Poissonian pump fluctuations on the micromaser field has attracted a great deal of interest in previous years [3–10]. It has been investigated by two, seemingly unrelated, theoretical approaches. The first employs a discrete pumping model where the atoms are allowed to arrive, with certain probability, only at distinct instants of time that are separated by a constant time interval [3–8]. In contrast to this, in the second approach the arrival times of the atoms are continuously distributed in time. This has been treated with the help of a quantum-field model of the injected atomic beam [9] and by applying the theory of stochastic point processes [10]. Both treatments are somewhat related to a special model of optical pumping in lasers [11].

In this paper we present a unified approach to discrete and continuous non-Poissonian micromaser pumping that can handle both situations on an equal footing and is based on a generalization of the model of continuous pumping. With the help of our method, for a micromaser with discrete non-Poissonian pumping, the equivalence of ensemble averaging and time averaging is explicitly shown, providing a rare example where the validity of the ergodic hypothesis can be explicitly demonstrated. Moreover, we show that for all kinds of non-Poissonian pumping, the resulting dynamics of the micromaser field is a non-Markovian one. In each case it is easily possible to postulate a master equation that can serve as a Markovian approximation to the non-Markovian evolution equation of the field density matrix and yields the correct stationary state. In particular, for discrete non-Poissonian pumping, the approximation is equivalent to a recently proposed ''macroscopic'' master equation [8]. It should be stressed, however, that this approximation is strictly justified only as a means to find the steady state. When the dynamical properties are of interest, it turns out that the results obtained with this Markovian master equation may differ from the smooth approximation to the ensuing non-Markovian dynamics.

To apply our method we calculate the time-dependent field-field correlation function and the spectrum of a micromaser with discrete non-Poissonian pumping where the atoms arrive with probability $p$ at equally spaced time instants. This has been done already previously [6,7] using the conventional mapping model of discrete pumping combined with a subsequent time-averaging procedure. In contrast to this, our unified treatment of non-Poissonian micromaser pumping rests on a pure ensemble-averaging approach. It allows for a physical interpretation of the underlying mecha-

 

nism and in particular explains the fact that the spectral line splitting that is observed for certain interaction parameters vanishes when $p<\frac{1}{2}$. We calculate a so-called $k$-photon spectrum that corresponds to the usual power spectrum when $k=1$. For a micromaser operated under the $k$-photon trapping condition, we find exact analytical results for the $k$-photon spectrum and the corresponding correlation functions. It should be stressed that these results are obtained from the exact analytical solution of a non-Markovian problem.

The paper is organized as follows. In Sec. II we outline the basic ideas that permit a unified treatment of discrete and continuous non-Poissonian pumping. We derive the non-Markovian evolution equation for the density operator of the field and we show how two-time expectation values of the field operators can be calculated in the stationary state. The results are applied to a micromaser with discrete non-Poissonian pumping in Sec. III. First, we discuss the statistics of the pump atoms in the frame of ensemble averaging. Then, the density-operator equation of the cavity field is derived and the equivalence of ensemble averaging and time averaging is explicitly shown. Finally, we investigate the non-Markovian behavior that is displayed in the time-delayed field-field correlation functions and the spectrum of the micromaser.

## II. UNIFIED TREATMENT OF DISCRETE AND CONTINUOUS NON-POISSONIAN PUMPING

### A. Evolution equation for the density operator of the field

We start with repeating some important results from the stochastic-process treatment of the pump statistics [10]. It has been found that one has to discriminate carefully between the unconditioned density operator $\rho$ of the cavity field and so-called injection-time conditioned density operators. The latter refer to subensembles of the whole quantum-mechanical ensemble and represent the state of the field at certain time instants that are defined with respect to the arrival times of the atoms. In particular, when the level-selective statistics of the atoms leaving the cavity is to be investigated, one has to know the time evolution of the specific injection-time conditioned field density operator $\varrho^c$ that refers to the state of the field immediately prior to the transit of an atom [10]. In contrast to the injection-time conditioned density operator, the unconditioned field density operator $\varrho$ describes the whole quantum-mechanical ensemble. As a consequence of the ergodic theorem, with the help of its steady-state solution $\overline{\rho}$ the time-averaged value of any field variable can be calculated as the quantum-mechanical expectation value of the operator corresponding to this variable. The most general equation for the evolution of the unconditioned density operator of the cavity field from the initial time $t=0$ to the final time $t=\tau$ takes the form

$$\rho(\tau)=V_s(\tau,0)\rho(0), \tag{2.1}$$

where [10]

$$V_s(\tau,0)=\int_\tau^\infty dt_1\int_{-\infty}^0 dt_0 Q_2(t_0,t_1)e^{L\tau}$$
$$+\sum_{k=1}^{\infty}\int_\tau^\infty dt_{k+1}\int_0^\tau dt_k\int_0^{t_k} dt_{k-1}\cdots\int_0^{t_2} dt_1$$
$$\times\int_{-\infty}^0 dt_0 Q_{k+2}(t_0,t_1,\ldots,t_{k+1})e^{L(\tau-t_k)}$$
$$\times M_{\mathrm{tr}}e^{L(t_k-t_{k-1})}\cdots M_{\mathrm{tr}}e^{L(t_2-t_1)}M_{\mathrm{tr}}e^{Lt_1}, \tag{2.2}$$

with $L$ being the usual damping operator of the cavity field and the operator $M_{\mathrm{tr}}$ describing the effect of the transit of a single atom on the density operator of the field according to the equation

$$\rho(t+t_{\mathrm{int}})=M_{\mathrm{tr}}\rho(t)\ . \tag{2.3}$$

In Eq. (2.2) it has been assumed that the damping of the cavity field can be neglected over the transit time $t_{\mathrm{int}}$ of a single atom that is negligibly short in our model. The functions $Q_{k+2}(t_0,t_1,\ldots,t_{k+1})$ $(k=0,1,\ldots)$ are the exclusive probability densities for the injection of exactly $k+2$ atoms into the cavity at the time instants $t_0,t_1,\ldots,t_{k+1}$ without any other atom being injected in between. From the functions $Q_{k+2}(t_0,t_1,\ldots t_{k+1})$ we find the probabilities $W_k(\tau)$ that the total number of atoms injected into the cavity over an arbitrary interval of length $\tau$ is equal to $k$ $(k=0,1,\ldots)$. These atom-number probabilities $W_k(\tau)$ read [10]

$$W_k(\tau)=\int_\tau^\infty dt_{k+1}\int_0^\tau dt_k\int_0^{t_k} dt_{k-1}\cdots\int_0^{t_2} dt_1$$
$$\times\int_{-\infty}^0 dt_0 Q_{k+2}(t_0,t_1,\ldots,t_{k+1})\ . \tag{2.4}$$

The stationary stochastic process describing the statistics of the injection of atoms into the cavity is completely characterized by the whole set of the exclusive probability densities $Q_{k+2}(t_0t_1,\ldots,t_{k+1})$ $(k=0,1,\ldots)$. An important class of stationary stochastic processes are the stationary renewal processes [12], where the functions $Q_{k+2}(t_0,t_1,\ldots,t_{k+1})$ factorize in the form

$$Q_{k+2}(t_0,t_1,\ldots,t_{k+1})=r\prod_{i=1}^{k+1} f(t_i-t_{i-1}), \tag{2.5}$$

with $r$ being the injection rate of the atoms, which can be calculated with the help of the equation

$$r^{-1}=\int_0^\infty tf(t)dt\ . \tag{2.6}$$

Thus the injection statistics is uniquely defined by the function $f(t)$, which has the meaning of the waiting-time distribution between consecutive injected atoms, i.e., of the probability density that an atom is injected at time $t'+t$ provided that the previous atom had been injected at time $t'$. It has been shown [10] that for injection of atoms according to a renewal process, a simple mapping equation exists that determines the steady-state solution of the specific injection-time conditioned density operator $\rho^c$ of the cavity field,

which refers to the time instants immediately prior to the transit of an atom. This mapping condition is given by

$$\overline{\rho^c}=\int_0^\infty dt\ f(t)e^{Lt}M_{\rm tr}\overline{\rho^c}\ . \tag{2.7}$$

Moreover, in this case the steady-state solution $\overline{\rho}=\lim_{\tau\to\infty}\rho(\tau)$, following from Eq. (2.1), is connected with the operator $\overline{\rho^c}$ by the simple equation [10]

$$r(M_{\rm tr}-1)\,\overline{\rho^c}+L\,\overline{\rho}=0\ . \tag{2.8}$$

From Eqs. (2.7) and (2.8) it becomes obvious that the steady-state solution of Eq. (2.1) does not depend on the initial operator $\rho(0)$ provided that the factorization condition (2.5) is fulfilled. We now derive an operator equation for $\overline{\rho}$ by first writing Eq. (2.7) in the form

$$M_{\rm tr}\overline{\rho^c}=\left[\int_0^\infty dt\ f(t)e^{Lt}\right]^{-1}\overline{\rho^c}, \tag{2.9}$$

which yields, together with Eq. (2.8),

$$r\,\overline{\rho^c}=\frac{L\,\overline{\rho}}{1-\left[\int_0^\infty dt\ f(t)e^{Lt}\right]^{-1}}\ . \tag{2.10}$$

Therefore, Eq. (2.8) can be written as

$$(M_{\rm tr}-1)\frac{L}{1-\left[\int_0^\infty dt\,f(t)e^{Lt}\right]^{-1}}\,\overline{\rho}+L\,\overline{\rho}=0\ . \tag{2.11}$$

The operator equation (2.11) is interesting for formal reasons. For practically calculating the field density matrix, however, it seems easier to use Eqs. (2.7) and (2.8), as has been done in [10] where the waiting-time distribution has been modeled as

$$f(t)=\frac{\lambda_1\lambda_2}{\lambda_2+\alpha\lambda_1}(e^{-\lambda_1 t}+\alpha e^{-\lambda_2 t})\ . \tag{2.12}$$

Here $\alpha$, $\lambda_1$, and $\lambda_2$ are real parameters with $\alpha\geqslant-1$ and $\lambda_1,\lambda_2\geqslant 0$. The renewal process characterized by the above equation describes continuous non-Poissonian injection with bunching or antibunching of the injected atoms. The special case of Poissonian injection is contained in the ansatz (2.12) when $\alpha=0$ and $\lambda_1=r$. On the other hand, discrete injection of atoms with fixed time distance $T$ between consecutive atoms, i.e., regular injection, is described by a stationary renewal process having the waiting-time distribution [9]

$$f(t)=\delta(t-T), \tag{2.13}$$

with $\delta$ being the Dirac delta function. With the help of Eq. (2.6) we obtain the injection rate $r=T^{-1}$. Thus, by substituting Eq. (2.13) into Eqs. (2.5)–(2.7), continuous and discrete non-Poissonian pumping can be treated on an equal footing.

### B. Random choice of pump atoms

Without the restrictive assumption that the atoms injected into the cavity are distributed according to a renewal process, the exploitation of the general density-matrix equation (2.1) becomes extremely complicated or even impossible and one would have to resort to numerical simulations. However, the variety of different pump statistics that are tractable with this assumption can be considerably enlarged by introducing an additional degree of freedom. For this purpose we suppose that not all of the injected atoms interact with the field during their transit through the cavity because some of them are out of resonance. When $p$ with $0\leqslant p\leqslant 1$ is the probability that an injected atom indeed interacts with the field, the operator $M_{\rm tr}$ introduced in Eq. (2.3) takes the form

$$M_{\rm tr}=pM+(1-p)\underline{1}=\underline{1}+p(M-\underline{1}), \tag{2.14}$$

where the operator $M$ describes the atom-field interaction according to the Jaynes-Cummings model and $\underline{1}$ is the unit operator. With this splitting of the operator $M_{\rm tr}$, the basic equations (2.7)–(2.13) can still be applied when the probability $p$ is taken into account. In particular, the steady-state density operator $\overline{\rho}$ then obeys the equation

$$p(M-1)\frac{L}{1-\left[\int_0^\infty dt\ f(t)e^{Lt}\right]^{-1}}\,\overline{\rho}+L\,\overline{\rho}=0, \tag{2.15}$$

where $f(t)$ is the waiting-time distribution that refers to all incoming atoms including the nonresonant, i.e., noninteracting, ones.

Interestingly, the statistics of the interacting atoms, considered separately, is still described by a renewal process. To show this, we calculate their exclusive probability densities $Q_m^{\rm int}(t_0,t_1,\ldots,t_{m-1})$ $(m=2,3,\ldots)$. These refer to the presence of exactly $m$ interacting atoms that occur with probability $p$ each time, at the instants $t_0,t_1,\ldots,t_{m-1}$, and of an arbitrary number of randomly distributed noninteracting atoms in the intervals in between, which occur with probability $1-p$. Starting from the nonselective exclusive probability densities of all atoms, which obey the renewal condition (2.5), we obtain

$$Q_m^{\rm int}(t_0,t_1,\ldots,t_{k-1})=p^m r\prod_{i=1}^{m-1}F(t_i,t_{i-1}), \tag{2.16}$$

where

$$\begin{aligned}F(t_i,t_{i-1})=f(t_i-t_{i-1})+\sum_{k=1}^{\infty}(1-p)^k\int_{t_{i-1}}^{t_i}d\tau_k\\ \times\int_{t_{i-1}}^{\tau_k}d\tau_{k-1}\cdots\int_{t_{i-1}}^{\tau_2}d\tau_1 f(t_i-\tau_k)\\ \times\left[\prod_{l=2}^{k}f(\tau_l-\tau_{l-1})\right]f(\tau_1-t_{i-1})\ .\end{aligned} \tag{2.17}$$

The multiplication of all terms $F(t_i,t_{i-1})$ $(i=1,\ldots,m)$ in Eq. (2.16) just takes into account all possible distributions of the noninteracting atoms. Obviously, the functions

$F(t_i,t_{i-1})$ depend only on the difference $t_i-t_{i-1}$, which can be easily seen by making the substitutions $t_i-t_{i-1}=\tau$ and $\tau_l-t_{i-1}=\xi_l$ $(l=1,\ldots,k)$ in Eq. (2.17). Therefore we can write, instead of Eq. (2.16),

$$Q_m^{\rm int}(t_0,t_1,\ldots,t_{k-1})=pr\prod_{i=1}^{m-1} f^{\rm int}(t_i-t_{i-1}), \quad (2.18)$$

where

$$f^{\rm int}(\tau)=pf(\tau)+p\sum_{k=1}^{\infty}(1-p)^k\int_0^{\tau}d\xi_k f(\tau-\xi_k) \times\int_0^{\xi_k}d\xi_{k-1}f(\xi_k-\xi_{k-1})\cdots\int_0^{\xi_2}d\xi_1 f(\xi_2-\xi_1)f(\xi_1) \quad (2.19)$$

is the waiting-time distribution of the interacting atoms. Equation (2.18) indeed expresses the property of being a renewal process with rate $rp$ for the statistics of the interacting atoms. Hence, instead of Eq. (2.15) we could use the equation

$$(M-1)\frac{L}{1-\left[\int_0^{\infty}dt\ f^{\rm int}(t)e^{Lt}\right]^{-1}}\overline{\rho}+L\rho=0\,. \quad (2.20)$$

In order to show explicitly the equivalence of Eqs. (2.15) and (2.20) we consider the Laplace transforms

$$\tilde{f}^{\rm int}(z)=\int_0^{\infty}dt\ e^{-zt}f^{\rm int}(t) \quad (2.21)$$

and

$$\tilde{f}(z)=\int_0^{\infty}dt\ e^{-zt}f(t)\,. \quad (2.22)$$

Applying the convolution theorem, we find from Eq. (2.19)

$$\tilde{f}^{\rm int}(z)=p\tilde{f}(z)+p\sum_{k=1}^{\infty}(1-p)^k[\tilde{f}(z)]^{k+1}=\frac{p\tilde{f}(z)}{1-(1-p)\tilde{f}(z)}\,. \quad (2.23)$$

Therefore, we get

$$\left[\int_0^{\infty}dt\ e^{Lt}f^{\rm int}(t)\right]^{-1}=[\tilde{f}^{\rm int}(-L)]^{-1} =\frac{1-(1-p)\int_0^{\infty}dt\ e^{Lt}f(t)}{p\int_0^{\infty}dt\ e^{Lt}f(t)}\,. \quad (2.24)$$

When we substitute the expression (2.24) into Eq. (2.20) we immediately arrive at Eq. (2.15). We remark that for practical calculations it is much more convenient to use the operator $M_{\rm tr}=1+p(M-1)$ and the original waiting-time distribution $f(t)$ than to start from the operator $M$ and the waiting-time distribution $f^{\rm int}(t)$.

Finally, it is interesting to calculate the waiting-time distribution for the renewal process that results from randomly choosing interacting atoms out of regularly distributed injected atoms, i.e., for the discrete pumping process we will be interested in. With the help of Eqs. (2.13), (2.22), and (2.23) we obtain

$$\tilde{f}^{\rm int}(z)=p\left[e^{-zT}+\sum_{k=1}^{\infty}(1-p)^k e^{-z(k+1)T}\right]=\frac{p}{e^{zT}-(1-p)}, \quad (2.25)$$

which yields

$$f^{\rm int}(t)=p\left\{\delta(t-T)+\sum_{k=1}^{\infty}(1-p)^k\delta[t-(k+1)T]\right\}\,. \quad (2.26)$$

From Eqs. (2.6) and (2.26) the pumping rate can be calculated with a little algebra to be

$$r_{\rm int}=\left[\int_0^{\infty}tf^{\rm int}(t)dt\right]^{-1}=\frac{p}{T}\,. \quad (2.27)$$

In the double limit $T\to 0$, $p\to 0$, Eq. (2.25) can be written as $\tilde{f}^{\rm int}(z)=r_{\rm int}(z+r_{\rm int})^{-1}$, which is the Laplace transform of the exponential waiting-time distribution $f(t)=r_{\rm int}\exp(-r_{\rm int}t)$ characteristic for a Poissonian pumping process, as should be expected in this limit [3].

### C. Manifestation of non-Markovian dynamics

In classical physics the Markov approximation consists in the assumption that the conditional probability of a process is entirely determined by the knowledge of the most recent condition [13]. In quantum mechanics, this is equivalent to the evolution-operator equation

$$V_s(\tau,t')=V_s(\tau,t)V_s(t,t') \quad (2.28)$$

for $\tau>t>t'$, where $V_s(t,t')$ describes the evolution of the system from the initial time $t'$ to the final time $t$. Obviously, the above equation implies that $V_s$ can be written as

$$V_s(t,t')=e^{L_s(t-t')}\equiv V(t-t'), \quad (2.29)$$

where $L_s$ is the Liouvillian of the total system. Because of Eq. (2.1) the density operator of the system then evolves according to $\rho(t)=e^{L_s(t-t_0)}\rho(t_0)$, which yields the Markovian master equation

$$\dot{\rho}=L_s\rho\,. \quad (2.30)$$

From inspection of Eq. (2.2) we conclude that the Markov condition (2.28) or (2.29), respectively, can be fulfilled only when the injected atoms are distributed according to a renewal process obeying the factorization condition (2.5) and when, in addition, the waiting-time distribution $f(t)$ decays exponentially in time, thus describing Poissonian injection. From Eqs. (2.5) and (2.6) we find, for Poissonian injection,

$$Q_{k+2}(t_0,t_1,\ldots,t_{k+1})=r^{k+2}e^{-r(t_{k+1}-t_0)}, \quad (2.31)$$

where $k=0,1,\ldots$. Indeed, it has been shown [10] that in this case Eqs. (2.1) and (2.2) are equivalent to the Markovian master equation $\dot{\rho}=r(M_{\rm tr}-1)\rho+L\rho$, which can be written as

$$\dot{\rho}=rp(M-1)\rho+L\rho \tag{2.32}$$

when Eq. (2.14) is taken into account. We remark at this point that the assumption of a negligibly short atom-field interaction time [see Eq. (2.3)] has been essential for the manifestation of Markovian dynamics for Poissonian pumping. Without this assumption, the operator products $M_{\rm tr}e^{L(t_m-t_{m-1})}$ occurring in Eq. (2.2) would have to be replaced by the expressions $V_{\rm tr}(t_m,t_m-t_{\rm int})e^{L(t_m-t_{\rm int}-t_{m-1})}$, where $V_{\rm tr}(t_m,t_m-t_{\rm int})$ describes the evolution of the reduced cavity-field density matrix from the initial time $t_m-t_{\rm int}$ to the final time $t_m$ under the influence of the atom-field interaction. Hence, despite Poissonian injection the Markovian condition (2.28) would not be fulfilled on a time scale that takes into account the finite duration of the atom-field interaction.

For any kind of non-Poissonian injection statistics, where Eq. (2.31) does not hold, the field evolution can never be a Markovian one. In particular, for injection according to a non-Poissonian renewal process, the field-density-operator evolution equation has been transformed into an integro-differential equation, the kernel of which is responsible for the non-vanishing memory time being characteristic for the non-Markovian evolution [9]. We stress once again that the non-Markovian character of Eq. (2.1) for non-Poissonian injection does not originate from the quantum-mechanical interaction between the field and the individual atoms, the duration of which has been assumed to be negligibly short. Rather, the non-Markovian dynamics is caused by the statistics of the arrival times of the atoms that can be described by a classical stochastic process.

Since the exact solution of the non-Markovian evolution equation (2.1) is rather difficult even with the assumption (2.5), one may ask the question as to how the evolution equation can be approximated by a Markovian one, which could then be written in the simple form of Eq. (2.30). In view of the steady-state solution given by Eqs. (2.11) and (2.14) it seems reasonable to use the approximation

$$\dot{\rho}\approx p(M-1)\frac{L}{1-\left[\int_0^\infty dt\, f(t)e^{Lt}\right]^{-1}}\rho+L\rho\ . \tag{2.33}$$

We remark that for Poissonian injection the equivalence of the above equation to the exact result (2.32) is easily verified by inserting the Poissonian waiting-time distribution $f(t)=re^{-rt}$ into Eq. (2.33). For regular injection of atoms, where $f(t)=\delta(t-T)$, Eq. (2.33) takes the form

$$\dot{\rho}\approx\left(p(M-1)\frac{L}{1-e^{-LT}}+L\right)\rho, \tag{2.34}$$

which has been derived in [8] by a time-averaging procedure and was called there a macroscopic master equation. For all kinds of non-Poissonian injection statistics Eq. (2.33) is only an approximation. In these cases the non-Markovian character of the field dynamics is revealed by the deviation of the actual dynamical behavior [which is described by Eqs. (2.1) and (2.2)] from the predictions made by the Markovian approximation (2.33). In Sec. III we shall discuss this for discrete pumping by investigating two-time expectation values of the steady-state field.

Whereas for Markovian processes the evaluation of all multitime expectation values can be easily performed with the help of the evolution equation for the reduced density operator of the system under consideration, this no longer holds true, in general, for non-Markovian processes. However, for the calculation of two-time expectation values $\langle A_s(t_0+\tau)B_s(t_0)\rangle$ with $A_s$ and $B_s$ being arbitrary system operators, the non-Markovian reduced-density-operator evolution equation is sufficient provided the expectation values refer to a time $t_0$, where the states $\rho_s(t_0)$ and $\rho_r(t_0)$ of the system and of the reservoir are known separately, i.e., where the density operator $\rho_{s+r}(t_0)$ of the combined system formed by the reservoir and by the system under consideration factorizes according to $\rho_{s+r}(t_0)=\rho_s(t_0)\rho_r(t_0)$. This is true when the interaction starts at time $t_0$ or when at $t_0$ the system is in a steady state that is independent of the state of the reservoir. Setting, for simplicity, $t_0=0$, we then find, in analogy to the standard treatment for Markovian processes [13],

$$\begin{aligned}\langle A_s(\tau)B_s(0)\rangle&=\mathrm{Tr}_{s+r}[e^{iH\tau/\hbar}A_s(0)e^{-iH\tau/\hbar}B_s(0)\rho_s(0)\rho_r(0)]\\&=\mathrm{Tr}_s\{A_s(0)\mathrm{Tr}_r[e^{-iH\tau/\hbar}B_s(0)\rho_s(0)\rho_r(0)e^{iH\tau/\hbar}]\}\\&=\mathrm{Tr}_s[A_s(0)V_s(\tau,0)B_s(0)\rho_s(0)],\end{aligned} \tag{2.35}$$

where $H$ is the Hamilton of the combined system and $V_s$ is the evolution operator for the reduced density operator of the system [see Eq. (2.1), where the index $s$ has been suppressed at the density operator].

When the injection statistics of the atoms corresponds to a renewal process, it can be explicitly shown that the system under consideration, i.e., the cavity field, indeed reaches a steady state, described by the density operator $\overline{\rho}$, which is known separately [see Eqs. (2.7) and (2.9) or Eq. (2.11), respectively]. Therefore, we may apply Eq. (2.35) in the steady state and obtain, e.g.,

$$\langle a^\dagger(\tau)a(0)\rangle_{\rm SS}=\mathrm{Tr}[a^\dagger V_s(\tau,0)a\overline{\rho}], \tag{2.36}$$

where $a$ and $a^\dagger$ are the photon annihilation and creation operators of the cavity mode, respectively, and the evolution operator $V_s(\tau,0)$ is given by Eqs. (2.2) and (2.5). Similarly, if we were interested in the steady-state intensity correlation function of the cavity field, we could write, in analogy to the treatment of Markovian processes [13],

$$\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle_{\rm SS}=\mathrm{Tr}[CV_s(\tau,0)C\overline{\rho}], \tag{2.37}$$

where $[C\overline{\rho}]_{nn}=(n+1)\rho_{n+1,n+1}$.

## III. APPLICATION TO DISCRETE NON-POISSONIAN PUMPING

### A. Statistics of the pump atoms

We now use the general results of the preceding section in order to investigate the non-Markovian dynamics of the cavity field for the case that the interacting atoms are chosen with probability $p$ out of a regular atomic beam. To begin with, let us calculate the probabilities $W_k(\tau)$ that exactly $k$ atoms are present in the atomic beam in an interval of length $\tau$, which is located arbitrarily with respect to the regularly spaced atomic arrival times being separated by the time distance $T$. For convenience, we express the length of this interval by $\tau=mT+xT$, where $m=0,1,2,\ldots$ and $0\leqslant x<1$. When we substitute Eqs. (2.5) and (2.13) into Eq. (2.4) and perform all integrations we find after some algebra

$$W_k(mT+xT)=0 \quad \text{for} \quad k\neq m,m+1 \tag{3.1}$$

$$W_{m+1}(mT+xT)=x, \tag{3.2}$$

$$W_m(mT+xT)=1-x\ . \tag{3.3}$$

Equations (3.1)–(3.3) could have been found immediately by noticing that the probability to have one atom injected in an interval of length $xT$ is equal to $rxT=x$, with $r=T^{-1}$ being the injection rate following from Eqs. (2.6) and (2.13). The mean number of atoms injected in an interval of length $\tau=mT+xT$ is equal to

$$\langle k\rangle_{mT+xT}=\sum_k kW_k(mT+xT)=m+x \tag{3.4}$$

and the variance is given by the expression

$$\langle\Delta k^2\rangle_{mT+xT}=\sum_k k^2W_k(mT+xT)-\langle k\rangle^2_{mT+xT}=x(1-x), \tag{3.5}$$

which takes its maximum value for $x=\frac{1}{2}$. Since

$$\frac{\langle\Delta k^2\rangle_{mT+xT}}{\langle k\rangle_{mT+xT}}=\frac{x(1-x)}{m+x}<1 \tag{3.6}$$

the injection statistics is always sub-Poissonian, but depends on the length of the interval $\tau$ when $m$ is not large compared to 1. For the forthcoming considerations it is important to keep in mind that, except for the singular case $x=0$, at any given length of the time interval $\tau$ two different numbers of atoms, which differ just by 1, may travel, with appropriate probability, through the cavity. When we intend to identify time averages with quantum-mechanical expectation values referring to the whole ensemble, it is necessary that the time interval over which the averaging is performed is located arbitrarily with respect to the arrival times of the incoming atoms as has been assumed above.

For completeness we still calculate the statistics of the atoms that actually interact with the field in the model of random choice of pump atoms. For this purpose we suppose that before the injection into the cavity each of the regularly spaced incoming atoms is brought into resonance with the field with probability $p$ $(0<p\leqslant 1)$, whereby this probability does not depend on the state of the other atoms so that no additional correlations between the resonant atoms can arise due to the resonance-producing physical process. The probability $W_k^{\rm int}(mT+xT)$ for finding $k$ resonant, i.e., interacting incoming atoms in an interval of length $\tau=mT+xT$, can be obtained with the help of Eqs. (3.1)–(3.3). We arrive at

$$W_k^{\rm int}(mT+xT)=0 \quad \text{for} \quad k>m+1\ , \tag{3.7}$$

$$W_{m+1}^{\rm int}(mT+xT)=xp^{m+1}\ , \tag{3.8}$$

$$W_k^{\rm int}(mT+xT)=(1-x)\binom{m}{k}p^k(1-p)^{m-k}+x\binom{m+1}{k}p^k(1-p)^{m+1-k} \quad \text{for} \quad 0\leqslant k\leqslant m\ . \tag{3.9}$$

The mean number of interacting atoms and its relative variance follow to be

$$\langle k\rangle^{\rm int}_{mT+xT}=p(m+x) \tag{3.10}$$

and

$$\frac{\langle\Delta k^2\rangle^{\rm int}_{mT+xT}}{\langle k\rangle^{\rm int}_{mT+xT}}=1-p\frac{m+x^2}{m+x}<1, \tag{3.11}$$

which again depend on the length of the interval $\tau$ provided that $x$ cannot be neglected in comparison to $m$.

### B. Evolution equation for the density operator of the cavity field

To derive the evolution equation of the cavity-field density operator for regular injection of atoms and random choice of the interacting pump atoms we start from the general equations (2.1) and (2.2), where the operator $M_{\rm tr}$ is given by Eq. (2.14). Taking into account Eqs. (2.5), (2.6), and (2.13) and performing the integrations (see [9]) we arrive at

$$\begin{aligned}\rho(mT+xT)=&\frac{1}{T}\int_0^{xT}dt'e^{L(xT-t')}[1+p(M-1)]\\&\times\{e^{LT}[1+p(M-1)]\}^me^{Lt'}\rho(0)\\&+\frac{1}{T}\int_{xT}^{T}dt'e^{L(xT-t')}\\&\times\{e^{LT}[1+p(M-1)]\}^me^{Lt'}\rho(0),\end{aligned} \tag{3.12}$$

where again the decomposition $\tau=mT+xT$ $(m=0,1,\ldots,\ 0\leqslant x<1)$ has been used. The physical interpretation of Eq. (3.12) is quite obvious in view of the statistics of the injected atoms considered in Sec. III A [see Eqs. (3.1)–(3.3)]. The first integral on the right-hand side corresponds to the possibility that $m+1$ atoms arrive over the interval $\tau=mT+xT$, which is located arbitrarily with respect to the atomic arrival times, whereas the second integral

corresponds to the arrival of only $m$ atoms. Both possibilities are taken into account with proper probability, which can be checked by setting $M=\underline{1}$ and $e^{Lt}=\underline{1}$ for all $t$. In order to determine the steady-state solution $\overline{\rho}=\lim_{\tau\to\infty}\rho(\tau)$ we make use of the general equations (2.7) and (2.8). With the help of Eqs. (2.13) and (2.14) we find

$$\frac{p}{T}(M-1)\overline{\rho^c}+L\overline{\rho}=0, \tag{3.13}$$

where $\overline{\rho^c}$ is the injection-time conditioned density operator referring to the time instant immediately before the transit of an atom. According to Eqs. (2.7) and (2.14) it obeys the mapping condition

$$\overline{\rho^c}=e^{LT}[1+p(M-1)]\overline{\rho^c}. \tag{3.14}$$

Combining Eqs. (3.13) and (3.14), as it has been done to derive the general equation (2.11), we finally obtain the formal steady-state equation

$$p(M-1)\frac{L}{1-e^{-LT}}\overline{\rho}+L\overline{\rho}=0\,. \tag{3.15}$$

### C. Equivalence of ensemble averaging and time averaging

When deriving Eq. (3.15) we had in mind the whole quantum-mechanical ensemble of cavities with regular injection of atoms where the arrival time of the first atom varies statistically in the members of the ensemble. The latter assumption is equivalent to the fact that the time evolution is not injection-time conditioned. In the steady state the quantum-mechanical expectation value of any field variable is equal to its time averaged value. Therefore, Eq. (3.15) can be also obtained by performing the time-average in a single subensemble that belongs to a fixed value of the arrival time of the first atom, which, for simplicity, we denote by $\tau=0$. In this subensemble one can define a steady-state density operator $\overline{\rho^{(x)}}$ that refers to the state of the field at the time instants $\tau=mT+xT$ with $m=0,1,\ldots$ and $0\leqslant x<1$ [5]. This operator obeys the mapping equation [8]

$$\overline{\rho^{(x)}}=e^{LxT}[1+p(M-1)]e^{L(1-x)T}\overline{\rho^{(x)}}, \tag{3.16}$$

which for $x=1$ is identical to Eq. (3.14), i.e., $\overline{\rho^{(1)}}$ corresponds to the injection-time conditioned density operator $\overline{\rho^c}$ introduced earlier. Since Eq. (3.16) can be transformed into the mapping condition $e^{-LxT}\overline{\rho^{(x)}}=[1+p(M-1)]e^{LT}e^{-LxT}\rho^{(x)}$ we conclude that the operator $e^{-LxT}\overline{\rho^{(x)}}$ does not depend on $x$. In particular, we have $e^{-LxT}\overline{\rho^{(x)}}=e^{-LT}\overline{\rho^{(1)}}=e^{-LT}\overline{\rho^c}$. Substituting this result into the right-hand side of Eq. (3.16) we find the alternative expression

$$\overline{\rho^{(x)}}=e^{LxT}[1+p(M-1)]\overline{\rho^c}\,. \tag{3.17}$$

When we define the unconditioned steady-state operator $\overline{\rho}$ as the time-averaged value of $\overline{\rho^{(x)}}$ [5] we can write [8]

$$\overline{\rho}=\int_0^1 dx\;\overline{\rho^{(x)}}=\frac{e^{LT}-1}{LT}[1+p(M-1)]\overline{\rho^c}=\frac{1-e^{-LT}}{LT}\overline{\rho^c}, \tag{3.18}$$

where Eq. (3.17) and the mapping condition (3.14) have been used. With the help of the above equation $\overline{\rho^c}$ can be expressed by $\overline{\rho}$ in Eq. (3.14) and after a short transformation we arrive at the desired Eq. (3.15). Thus, due to the periodicity of the underlying regular injection process used in our model, it has been possible to show explicitly that for a micromaser with discrete pumping time averaging and ensemble averaging are equivalent in the stationary regime, i.e., to demonstrate the validity of the ergodic hypothesis.

### D. Non-Markovian dynamics in the transients and the spectrum

#### *1. General equations*

In the following we want to use the density-operator evolution equation (3.12) in order to calculate the transients and the spectrum of a micromaser with discrete non-Poissonian pumping. Whereas the previous considerations have been quite general, from now on we restrict ourselves to the usual situation that before the interaction the resonant atoms are in the upper level of the microwave transition. In the photon-number representation the effect of the Jaynes-Cummings-interaction on the field then reads [14]

$$\begin{aligned}[M\rho]_{n,n+q}=&\cos(gt_{\rm int}\sqrt{n+q+1})\cos(gt_{\rm int}\sqrt{n+1})\rho_{n,n+q}\\&+\sin(gt_{\rm int}\sqrt{n+q})\sin(gt_{\rm int}\sqrt{n})\rho_{n-1,n+q-1}\end{aligned} \tag{3.19}$$

$(n,q=0,1,\ldots)$. Here $t_{\rm int}$ and $g$ denote the atom-field interaction time and coupling constant, respectively. For simplicity, we assume that thermal photons can be neglected. Therefore the damping process can be described by [14]

$$\begin{aligned}[L\rho]_{n,n+q}=\gamma\Big[&\sqrt{(n+q+1)(n+1)}\rho_{n+1,n+q+1}\\&-\left(n+\frac{q}{2}\right)\rho_{n,n+q}\Big]\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}[e^{Lt}\rho]_{n,n+q}=&e^{-\gamma t(n+q/2)}\sum_{j=0}^{\infty}(1-e^{-\gamma t})^j\\&\times\sqrt{\binom{n+q+j}{j}\binom{n+j}{j}}\rho_{n+j,n+q+j},\end{aligned} \tag{3.21}$$

with $\gamma$ being the cavity damping constant. Since only density-matrix elements $\rho_{nn'}$ with the same value of $n-n'$ are coupled by the interaction and damping process, it is advantageous to introduce the column vector

$$\vec{p}^{(q)} \equiv \begin{pmatrix} \rho_{0,q} \\ \rho_{1,1+q} \\ \vdots \\ \rho_{n_{\max}-q,n_{\max}} \end{pmatrix}, \tag{3.22}$$

where we have assumed that the photon-number distribution is truncated at a certain maximum photon number $n_{\max}$. Equation (3.12) can be decomposed into separate evolution equations for the vectors $\vec{p}^{(q)}$ $(q=0,1,\ldots)$, where the operators $e^{Lt}$ and $1+p(M-1)$ are replaced by the corresponding matrices. Making use of Eq. (3.21), the integration with respect to $t'$ in Eq. (3.12) is easily performed analytically. Finally, the evolution equation (3.12) can be replaced by the set of equations

$$\vec{p}^{(q)}(mT+xT) = \boldsymbol{V}^{(q)}(mT+xT)\vec{p}^{(q)}(0) \tag{3.23}$$

$(q=0,1,\ldots)$, where $\boldsymbol{V}^{(q)}$ is a matrix, the elements of which are explicitly known. With the help of Eq. (3.23) the transients of all one-time expectation values of the cavity-field variables can be directly calculated. In particular, for $q=0$, Eq. (3.23) yields the evolution of the photon statistics. When we are interested in two-time expectation values we have to use Eq. (3.23) with modified initial conditions. In particular, taking into account Eqs. (2.36), (3.22), and (3.23) we obtain the steady-state correlation function

$$\begin{aligned} \langle a^\dagger(\tau)a(0)\rangle_{\rm SS} &= \sum_{n=0}^{n_{\max}-1} \sqrt{n+1}\,[V_s(\tau,0)a\overline{\rho}]_{n,n+1} \\ &= \sum_{n=0}^{n_{\max}-1} \sqrt{n+1} \sum_{m=0}^{n_{\max}-1} V^{(1)}_{nm}(\tau)\sqrt{m+1} \\ &\quad \times \overline{\rho}_{m+1,m+1}, \end{aligned} \tag{3.24}$$

where $\tau = mT + xT$.

In order to discuss the non-Markovian properties of the field dynamics, the results obtained with the help of Eq. (3.23) have to be compared with those that would follow from the Markovian approximation (2.33). In the following, instead of the usual correlation function (3.24) we consider the more general steady-state two-time expectation values

$$\begin{aligned} \langle a^{\dagger k}(\tau)a^k(0)\rangle_{\rm SS} &= \langle V_s(\tau,0)a^k\overline{\rho}a^{\dagger k}\rangle \\ &= \sum_{n=0}^{n_{\max}-k} \sqrt{\frac{(n+k)!}{n!}}\,[V_s(\tau,0)a^k\overline{\rho}]_{n,n+k} \\ &= \sum_{n=0}^{n_{\max}-k} \sqrt{\frac{(n+k)!}{n!}} \sum_{m=0}^{n_{\max}-k} V^{(k)}_{nm} \\ &\quad \times \sqrt{\frac{(m+k)!}{m!}}\,\overline{\rho}_{m+k,m+k} \end{aligned} \tag{3.25}$$

$(k=1,2,\ldots)$. By Fourier transforming we find the generalized spectrum

$$S_k(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty d\tau \frac{\langle a^{\dagger k}(\tau)a^k(0)\rangle_{\rm SS}}{\langle a^{\dagger k}a^k\rangle_{\rm SS}} e^{-i(\omega-k\nu)\tau}, \tag{3.26}$$

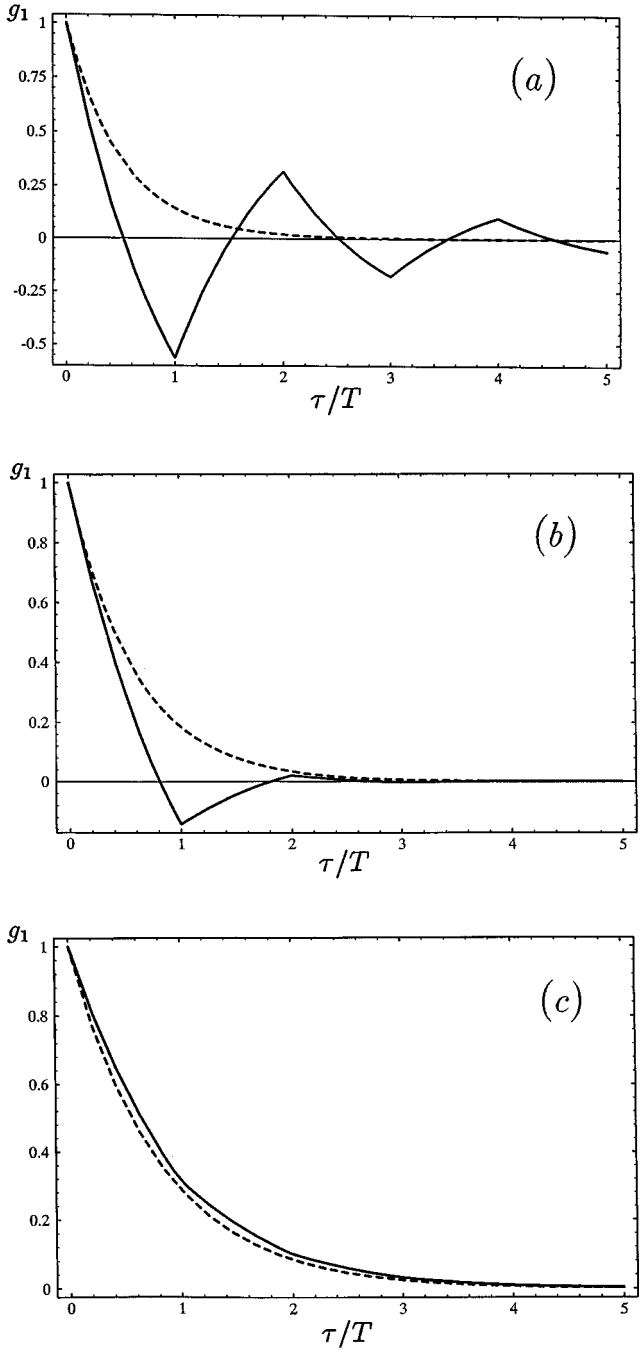


FIG. 1. Decay of the normalized field correlation function $g_1(\tau) = \langle a^\dagger(\tau)a(0)\rangle_{\rm SS}/\langle a^\dagger a\rangle_{\rm SS}$ in a micromaser operating under the one-photon trapping condition $gt_{\rm int} = 3\pi/\sqrt{2}$ at discrete non-Poissonian pumping with $T=\gamma^{-1}$ and (a) $p=1$, (b) $p=0.8$, and (c) $p=0.5$. The dashed line corresponds to the result of the Markovian approximation. The thermal photon number is equal to zero.

which for $k=1$ corresponds to the usual power spectrum of the cavity mode of frequency $\nu$ and which we denote as $k$-photon spectrum. For $k=2$ it has been shown that this spectrum could be determined by examining the two-photon absorption in a weak beam of two-photon resonant atoms probing the cavity field [15].

### 2. Analytical results for a micromaser in the k-photon trapped state

We want to calculate the field correlation function and the spectrum given by Eqs. (3.25) and (3.26) for the stationary regime of a micromaser that is operated under the $k$-photon trapping condition [16]

$$gt_{\rm int}=l\frac{\pi}{\sqrt{k+1}}\quad (l=1,2,\ldots)\ . \tag{3.27}$$

In this case the maximum photon number $n_{\max}$ in the cavity is equal to $k$ provided that thermal photons can be neglected. For $q=k$ and $n_{\max}=k$, Eq. (3.23) reduces to a $c$-number equation for the density-matrix element $\rho_{0k}$ so that all results can be easily obtained analytically. From Eqs. (3.19)–(3.21) we obtain

$$[L\rho]_{0k}=-k\frac{\gamma}{2}\rho_{0k}, \tag{3.28}$$

$$[e^{Lt}\rho]_{0k}=e^{-k(\gamma/2)t}\rho_{0k}, \tag{3.29}$$

and

$$[M\rho]_{0k}=b_k(l)\rho_{0k}, \tag{3.30}$$

where in Eq. (3.30) the quantity

$$b_k(l)=\cos(l\pi)\cos\left(l\frac{\pi}{\sqrt{k+1}}\right), \tag{3.31}$$

following from the $k$-photon trapping condition (3.27), has been introduced. When we apply Eq. (3.25) for $n_{\max}=k$ and take into account that in this case $\langle a^{\dagger k}a^k\rangle_{\rm SS}=k!\,\overline{\rho_{kk}}$, we arrive at

$$\frac{\langle a^{\dagger k}(\tau)a^k(0)\rangle_{\rm SS}}{\langle a^{\dagger k}a^k\rangle_{\rm SS}}=V_{00}^{(k)}(\tau)=\frac{\rho_{0k}(\tau)}{\rho_{0k}(0)}, \tag{3.32}$$

where use has been made of Eqs. (3.22) and (3.23). Substitution of Eqs. (3.29) and (3.30) into the evolution equation (3.12) finally yields, for $mT\leq\tau\leq(m+1)T$ $(m=0,1,2,\ldots)$,

$$V_{00}^{(k)}(\tau)=e^{-(k/2)\gamma\tau}[1+p(b_k-1)]^m\left[1+p(b_k-1)\left(\frac{\tau}{T}-m\right)\right]. \tag{3.33}$$

To obtain the above equation we used the fact that, because of the $c$-number structure of Eq. (3.23) resulting for $n_{\max}=k$, the operators $L$ and $M$ may be commuted when Eq. (3.12) is applied to calculate $\rho_{0k}(mT+xT)$. By introducing the notation $[\tau/T]$ for the maximum integer that does not exceed the value of $\tau/T$, we may write

$$\frac{\langle a^{\dagger k}(\tau)a^k(0)\rangle_{\rm SS}}{\langle a^{\dagger k}a^k)\rangle_{\rm SS}}=e^{-(k/2)\gamma\tau}[1+p(b_k-1)]^{[\tau/T]}\times\left\{1+p(b_k-1)\left(\frac{\tau}{T}-\left[\frac{\tau}{T}\right]\right)\right\}\ . \tag{3.34}$$

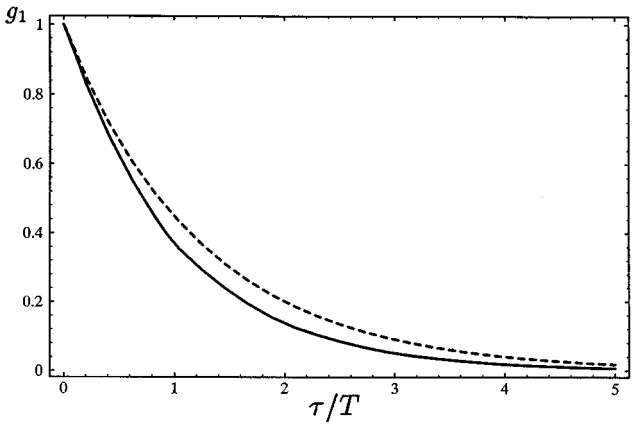


FIG. 2. Same as Fig. 1, but for $gt_{\rm int}=\pi/\sqrt{2}$ and $p=1$.

The $k$-photon spectrum, defined by Eq. (3.26), of a micromaser operated under the $k$-photon trapping condition is found to be

$$S_k(\omega)=\frac{1}{\pi}{\rm Re}\left[\frac{1}{\frac{k}{2}\gamma+i(\omega-k\nu)}\left\{1+\frac{p(b_k-1)}{\left[k\frac{\gamma}{2}+i(\omega-k\nu)\right]T}\times\frac{1-e^{-[k(\gamma/2)+i(\omega-k\nu)]T}}{1-[1+p(b_k-1)]e^{-[k(\gamma/2)+i(\omega-k\nu)T]}}\right\}\right]\ . \tag{3.35}$$

Obviously, the properties of the decay curve of the field-field correlation function described by Eq. (3.34) and of the $k$-photon spectrum crucially depend on the value of the parameter $b_k$ given by Eq. (3.31). Depending on the number $l$ $(l=1,2,\ldots)$, the parameters $b_k(l)$ take on values between $-1$ and 1 for any $k$. Therefore it is sufficient to discuss Eqs. (3.34) and (3.35) for $k=1$ and different values of $b_1(l)$; the behavior for $k\neq1$ is principally the same. In the following we investigate the normalized correlation function

$$g_1(\tau)=\frac{\langle a^\dagger(\tau)a(0)\rangle_{\rm SS}}{\langle a^\dagger a\rangle_{\rm SS}} \tag{3.36}$$

for the one-photon trapped state and the corresponding one-photon spectrum $S_1(\omega)$, i.e., the usual power spectrum. We stress here that Eqs. (3.34) and (3.35) result from the non-Markovian density-operator evolution equation (3.12), which can be interpreted as the exact solution of a non-Markovian evolution equation. When we would instead use the Markovian approximation (2.34) for the master equation and again apply Eqs. (3.28)–(3.30) we would find the approximate result

$$g_1(\tau)\approx\exp\left[-\frac{1-[1+p(b_1-1)]e^{-(\gamma/2)T}}{1-e^{-(\gamma/2)T}}\frac{\gamma}{2}\tau\right]. \tag{3.37}$$

The decay of the normalized field correlation function $g_1(\tau)$ is depicted in Figs. 1 and 2. As it has become obvious from Eq. (3.34), the function $g_1(\tau)$ reveals a sawtoothlike behavior when the quantity $1+p(b_k-1)$ takes on negative values [see Figs. 1(a) and 1(b), where $b_1=-0.9282$]. Since

$b_k \geqslant -1$ [see Eq. (3.31)] the requirement $1+p(b_k-1)<0$ can be only fulfilled when $p>\frac{1}{2}$. Physically, the origin for the sawtoothlike behavior lies in the fact that the interaction with a single atom can cause a phase reversal of the complex quantity $\langle a^{\dagger k}\rangle$. In fact, for the $k$-photon trapped state with $n_{\max}=k$ we find with the help of Eq. (3.30)

$$\langle a^{\dagger k}(t+t_{\rm int})\rangle = {\rm Tr}[a^{\dagger k} M \rho(t)] = b_k \sqrt{k!}\,\rho_{0k}(t) = b_k \langle a^{\dagger k}(t)\rangle \ . \tag{3.38}$$

Since we assumed that the micromaser is pumped by atoms being in a definite energy state, the off-diagonal elements of the field density matrix vanish in the stationary state and the same is true for the expectation values of the operators $a^{\dagger k}$. To be more precise, instead of Eq. (3.38) we therefore consider the field correlation

$$\begin{aligned}\langle a^{\dagger k}(t+t_{\rm int}) a^{\dagger k}(t)\rangle_{\rm SS} &= {\rm Tr}[M a^k \overline{\rho a^k}] \\ &= b_k k!\,\overline{\rho_{kk}} = b_k \langle a^{\dagger k} a^k\rangle_{\rm SS} \ .\end{aligned} \tag{3.39}$$

Again, for $b_k<0$ the phase of this complex field correlation is reversed due to the interaction with a single atom. Now, in our model the probability that an injected atom indeed interacts with the field is given by the parameter $p$. To estimate the average change of the field due to the transit of one atom one has to take into account both the possibilities that the atom traversing the cavity does not interact with the field (which occurs with probability $1-p$) and that the atom is in resonance with the field (probability $p$). In the first case the field remains unchanged, whereas in the second case Eqs. (3.38) or (3.39) can be applied. The average effect of the transit of a single atom in the $k$-photon trapped state hence can be described by

$$\begin{aligned}\frac{\langle a^{\dagger k}(t+t_{\rm int})\rangle}{\langle a^{\dagger k}(t)\rangle} &= \frac{\langle a^{\dagger k}(t+t_{\rm int}) a^k(t)\rangle_{\rm SS}}{\langle a^{\dagger k} a^k\rangle_{\rm SS}} \\ &= 1-p+p b_k \ .\end{aligned} \tag{3.40}$$

The requirement $1+p(b_k-1)<0$ for the sawtoothlike behavior to occur therefore becomes obvious from physical considerations.

In Figs. 1(c) and 2 this requirement is not fulfilled; the field correlation function $g_1(\tau)$ decreases monotonically. Nevertheless, the deviation from the behavior expected from the Markovian approximation (3.37) is apparent. In these cases the so-called macroscopic [8] master equation (2.34) hence does not correctly describe the macroscopic behavior where changes on a time scale of the order $1/T$ are averaged out.

As a consequence of the sawtoothlike decay of $g_1(\tau)$ the spectrum is split into several equidistant peaks, separated by $\Delta\omega = 2\pi/T$ (see Figs. 3 and 4). This behavior has already previously been found [6,7] for certain parameter regions under general operating conditions. The previous treatments have been based on a stroboscopic approach with subsequent time averaging. In contrast to this, we investigated the density matrix of the whole quantum-mechanical ensemble. Thus it was possible to explain the fact that line splitting can occur only for $p>\frac{1}{2}$.

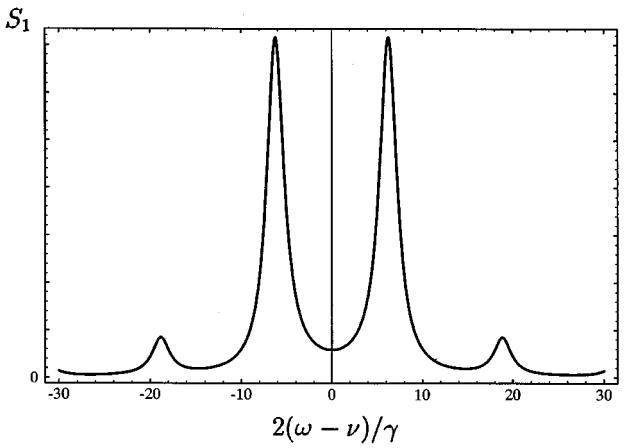


FIG. 3. Steady-state spectrum $S_1(\omega)$ of a micromaser operating under the one-photon trapping condition $gt_{\rm int}=3\pi/\sqrt{2}$ with zero thermal photon number for discrete non-Poissonian pumping with time distance $T=\gamma^{-1}$ and interaction probability $p=1$.

## IV. CONCLUSION

We have presented a unified treatment of discrete and continuous non-Poissonian micromaser pumping, which rests on an ensemble-averaging approach. The resulting dynamics of the micromaser field proved to be a non-Markovian one. The equivalence of ensemble averaging and time averaging could be explicitly shown for a micromaser with discrete non-Poissonian pumping. Moreover, for this case an exact analytical solution could be found for the time-dependent field-field correlation function and the generalized $k$-photon spectrum ($k=1,2,\ldots$) of the cavity field provided that the micromaser is operated under the $k$-photon trapping condition. It turned out that for certain values of the interaction parameter the $k$-photon spectrum is split into several equidistant peaks. We showed that this effect is explained by

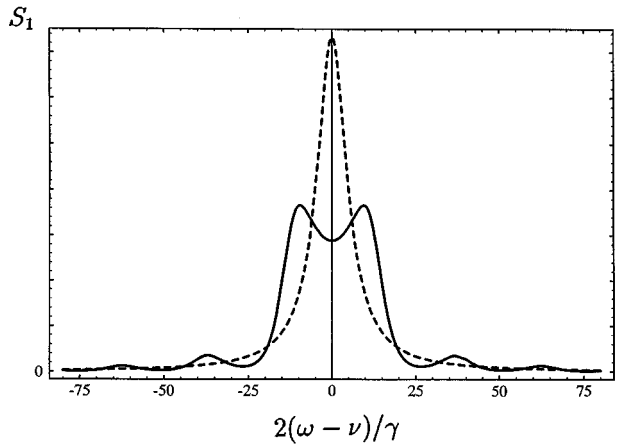


FIG. 4. Same as Fig. 3, but for $gt_{\rm int}=2\pi/\sqrt{2}$ and $T=0.5\gamma^{-1}$. The dashed line corresponds to the result of the Markovian approximation.

the phase reversal of the entire cavity field due to the transit of a single atom.

We note here that our analytic approach can also be applied to more general operating conditions of a micromaser with discrete pumping, provided that the resulting photon-number distribution is single peaked at a sufficiently large mean photon number. This is planned to be the topic of a future paper [17], where we plan to show that the spectral splitting into equidistant peaks that results in certain conditions or, equivalently, the phase reversal of the field due to the transit of a single atom, is caused by the occurrence of Rabi oscillations with discrete frequencies. Hence the effect is due to field quantization and is closely related to the Jaynes-Cummings revival. It has nothing to do with the line splitting that can occur for a micromaser with Poissonian pumping when the dynamics is governed by several exponentially decaying distributions having different time constants [18]. Finally, we remark that, though microwave photons cannot be directly detected, our investigations nevertheless might be of practical interest since they can be applied also to a microlaser that has been successfully operated in the optical region [19].

## ACKNOWLEDGMENTS

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