# Dual eigenkets of the Susskind-Glogower phase operator

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By contour integration we show that the phase operator $\widehat{e^{-i\phi}}$ (one of the pair of Susskind-Glogower operators) also possesses eigenkets $|\gamma\rangle_*$, which are the dual vector of $\widehat{e^{i\phi}}$'s eigenkets. The properties of $|\gamma\rangle_*$ are studied and we see that $|\gamma\rangle_*$ and $\widehat{e^{i\phi}}$'s eigenkets can also make up a phase-state representation. [S1050-2947(96)01212-7]

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## I. INTRODUCTION

The phases of optical fields play the decisive role in many optical phenomena, particularly in diffraction and interference of light. Therefore much attention has been paid to the problem of defining and measuring an appropriate phase for radiation fields [1–3]. There are quite a few ways to propose phase operators. For example, Paul [4] defined the phase operator by the diagonal coherent state representation (called Glauber-Sudarshan $P$ representation) as

$$\int \frac{d^2z}{\pi} e^{i\theta}|z\rangle\langle z| = \widehat{e_P^{i\theta}}, \quad \text{with } z=|z|e^{i\theta}, \tag{1}$$

where $|z\rangle$ is the coherent state.

One can also use Weyl correspondence [5,6] to map a classical phase $e^{i\theta}$ onto a quantum-mechanical operator by

$$\int d^2\alpha \hat{\Delta}(\alpha,\alpha^*)e^{i\theta} = \widehat{e_W^{i\theta}}, \quad \alpha=|\alpha|e^{i\theta} \tag{2}$$

where $\hat{\Delta}(\alpha,\alpha^*)$ is the Wigner operator, usually expressed as an integration,

$$\hat{\Delta}(\alpha,\alpha^*) = \int \frac{d^2z}{2\pi^2} e^{-z^*(\alpha-\hat{a})+z(\alpha^*-\hat{a}^\dagger)}. \tag{3}$$

Using the technique of integration within an ordered product (IWOP) of operators [7], one can derive the explicit normally ordered form of $\hat{\Delta}(\alpha,\alpha^*)$ [see Eq. (2.1) in Ref. [8]],

$$\hat{\Delta}(\alpha,\alpha^*) = \frac{1}{\pi} :e^{-2(\hat{a}^\dagger-\alpha^*)(\hat{a}-\alpha)}: \tag{4}$$

and [7]

$$|z\rangle\langle z| = :e^{-(\hat{a}^\dagger-z^*)(\hat{a}-z)}:. \tag{5}$$

Substituting Eqs. (4) and (5) into Eqs. (1) and (2), respectively, one can obtain the explicit form of $\widehat{e_P^{i\theta}}$ and $\widehat{e_W^{i\theta}}$.

Another widely used and apparently easier phase operator is the Susskind-Glogower (SG) phase operator [2]. This phase operator comes from classical optics by introducing the phase as the "approximate" polar decomposition of the annihilation and creation operators, e.g.,

$$\hat{a}^\dagger = \widehat{e^{-i\phi}}\sqrt{\hat{a}\hat{a}^\dagger}, \quad \hat{a} = \sqrt{\hat{a}\hat{a}^\dagger}\ \widehat{e^{i\phi}}. \tag{6}$$

Although the SG phase operators are nonunitary, as

$$\widehat{e^{i\phi}}\,\widehat{e^{-i\phi}} = 1 \quad \text{and} \quad \widehat{e^{-i\phi}}\,\widehat{e^{i\phi}} = 1-|0\rangle\langle 0|, \tag{7}$$

where $|0\rangle$ is the ground state, and $|0\rangle\langle 0|$ effectively vanishes for those states with a negligible vacuum component, they are still widely used and studied in theoretical calculations in quantum optics [9]. Despite the fact that a more practical way is to operationally define a phase operator for a given experimental arrangement as was done by Mandel and coworkers [10], an appropriate phase operator can still be very useful in theoretical investigations of quantized radiation fields. Thus it will be helpful to investigate the SG phase operator in more detail. In Ref. [11], the relation between SG phase operators and the inverse operators $\hat{a}^{-1}$ and $(\hat{a}^\dagger)^{-1}$ are shown as

$$\widehat{e^{i\phi}} = (\hat{a}^\dagger)^{-1}\sqrt{\hat{N}} \quad \text{and} \quad \widehat{e^{-i\phi}} = \sqrt{\hat{N}}\hat{a}^{-1}, \tag{8}$$

which indicates that the nonunitarity of SG phase operators is intrinsically related to the noncommutative property of $\hat{a}$ and $\hat{a}^{-1}$ [$\hat{a}^\dagger$ and $(\hat{a}^\dagger)^{-1}$], e.g.,

$$[\hat{a},\hat{a}^{-1}] = [(\hat{a}^\dagger)^{-1},\ \hat{a}^\dagger] = |0\rangle\langle 0|, \tag{9}$$

which implies that $\hat{a}$ and $\hat{a}^\dagger$ cannot have regular polar decompositions due to their singularities. $\hat{N}\equiv\hat{a}^\dagger\hat{a}$ is the number operator.

The eigenstate of $\widehat{e^{i\phi}}$ is given by [12]

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$$|\gamma\rangle=\sum_{n=0}^{\infty}\gamma^n|n\rangle,\quad |\gamma|<1\quad \text{with}\quad \gamma=|\gamma|e^{i\varphi}, \tag{10}$$

where $|n\rangle$ is the number state. Equations (10) and (6) yield the expectation values

$$\langle\widehat{\sin\varphi}\rangle=|\gamma|\sin\varphi \quad\text{and}\quad \langle\widehat{\cos\varphi}\rangle=|\gamma|\cos\varphi, \tag{11}$$

with corresponding variances

$$\langle(\Delta\ \widehat{\sin\varphi})^2\rangle=\langle(\Delta\ \widehat{\cos\varphi})^2\rangle=\frac{1-|\gamma|^2}{4}. \tag{12}$$

When $|\gamma|\to 1$, one gets the (non-normalizable) SG phase state

$$|e^{i\phi}\rangle\equiv\sum_{n=0}^{\infty}e^{in\phi}|n\rangle,\quad -\pi<\phi\leqslant\pi, \tag{13}$$

which is an eigenket of $\widehat{e^{i\phi}}$ and a ''phase'' $\phi$ can be assigned. The SG ''phase state'' can lead to an approximate description of phase measurement via the relation

$$P_{\text{SG}}[\phi\in(\theta_0,\theta_0+2\pi)|\psi]=\langle F(\Delta)\rangle_\psi,\quad \Delta\text{:angle window}, \tag{14}$$

where

$$F(\Delta)=\int_\Delta \frac{d\phi}{2\pi}|e^{i\phi}\rangle\langle e^{i\phi}|. \tag{15}$$

When $\Delta=2\pi$,

$$F(2\pi)=\oint\frac{d\phi}{2\pi}|e^{i\phi}\rangle\langle e^{i\phi}|=1. \tag{16}$$

However, one can easily show that

$$\begin{aligned}\langle e^{i\phi'}|e^{i\phi}\rangle&=\sum_{n=0}^{\infty}e^{-in\phi'}\left\langle n\left|\sum_{n'=0}^{\infty}e^{in'\phi}\right|n'\right\rangle=\sum_{n=0}^{\infty}e^{in(\phi-\phi')}\\&=\frac{1}{2}+\pi\delta(\phi-\phi')+\frac{1}{2i}\cot\frac{1}{2}(\phi'-\phi),\end{aligned} \tag{17}$$

which means that the eigenstates of the SG phase operator $\widehat{e^{i\phi}}$ are non-orthogonal.

A question thus naturally arises. Can we find an orthogonal relation within the SG phase scheme? This question is closely related to whether $\widehat{e^{-i\phi}}$ possesses an eigenket or the creation operator $\hat{a}^\dagger$ has an eigenket, since $\widehat{e^{-i\phi}}=\hat{a}^\dagger(1/\sqrt{\hat{a}\hat{a}^\dagger})$. In Refs. [13–15], we have shown explicitly that $\hat{a}^\dagger$'s eigenket is not identically zero, instead, it can be composed in terms of complex $\delta$-function and Fock states and is useful in constructing the Drummond-Gardiner complex $P$ representation [15,16]. This encourages us to derive $\widehat{e^{-i\phi}}$'s eigenkets and then study their properties. In Sec. II we construct $\widehat{e^{-i\phi}}$'s eigenkets $|\gamma\rangle_*$. In Sec. III we discuss $|\gamma\rangle_*$'s properties. This work complements earlier work on phase-state representation and seems to be important in quantum optics.

## II. $\widehat{e^{-i\phi}}$'s EIGENKETS

We now discuss if $\widehat{e^{-i\phi}}$ possesses eigenkets. Operating $\widehat{e^{-i\phi}}$ on $|e^{i\phi}\rangle$, we have

$$\widehat{e^{-i\phi}}|e^{i\phi}\rangle=e^{-i\phi}\sum_{n=0}^{\infty}e^{i(n+1)\phi}|n+1\rangle=e^{-i\phi}(|e^{i\phi}\rangle-|0\rangle), \tag{18}$$

which means that $|e^{i\phi}\rangle$ is not the eigenket of $\widehat{e^{-i\phi}}$. This is due to the fact that

$$[\widehat{e^{i\phi}},\ \widehat{e^{-i\phi}}]=1-|0\rangle\langle 0|. \tag{19}$$

Thus we must reconsider this problem using another approach.

Supposing $\widehat{e^{-i\phi}}$ has eigenkets $|\gamma\rangle_*$, satisfying

$$\widehat{e^{-i\phi}}|\gamma\rangle_*=\gamma^*|\gamma\rangle_*. \tag{20}$$

Then, expanding $|\gamma\rangle_*$ in terms of $|n\rangle$ leads to

$$|\gamma\rangle_*=\sum_{n=0}^{\infty}C_n|n\rangle. \tag{21}$$

Using $\hat{a}^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$, we have

$$\begin{aligned}\widehat{e^{-i\phi}}|\gamma\rangle_*&=\sum_{n=0}^{\infty}|n+1\rangle C_n\\&=\gamma^*\sum_{n=0}^{\infty}|n\rangle C_n.\end{aligned} \tag{22}$$

The recursive relations then follow:

$$C_0\gamma^*=0,\quad C_0=\gamma^*C_1,\quad C_1=\gamma^*C_2,\ldots,\quad C_n=\gamma^*C_{n+1}. \tag{23}$$

The first equation, if one notices the relation $x\delta(x)=0$, has the solution of $\delta$-function form

$$C_0=\delta(\gamma^*), \tag{24}$$

where $\delta(\gamma^*)$ is a contour integration, defined by Heitler [17] as

$$\delta(\gamma^*)=\left.\frac{1}{2\pi i\gamma^*}\right|_C, \tag{25}$$

where the contour $C$ encircles the origin. Obviously $\gamma^*\delta(\gamma^*)=0$, which holds in the sense of contour integration. Thus

$$C_n=\frac{C_{n-1}}{\gamma^*}=\cdots=\frac{C_0}{\gamma^{*n}}=\frac{1}{2\pi i\gamma^{*n+1}}\bigg|_C. \tag{26}$$

Therefore the eigenket of $\widehat{e^{-i\phi}}$ is

$$|\gamma\rangle_*=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{\gamma^{*n+1}}|n\rangle|_C. \tag{27}$$

Now, one can easily check that

$$\begin{aligned}\widehat{e^{-i\phi}}|\gamma\rangle_*&=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{\gamma^{*n+1}}|n+1\rangle|_C\\&=\gamma^*\left[\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{\gamma^{*n+1}}|n\rangle|_C-\delta(\gamma^*)|0\rangle\right]\\&=\gamma^*|\gamma\rangle_*.\end{aligned} \tag{28}$$

When $|\gamma^*|=1$, $\gamma^*=e^{i\phi}$, Eq. (27) becomes

$$|e^{i\phi}\rangle_*=\frac{1}{2\pi i}\sum_{n=0}^{\infty}e^{-i(n+1)\phi}|n\rangle|_C, \tag{29}$$

and

$$\widehat{e^{-i\phi}}|\gamma\rangle_*=e^{i\phi}|\gamma\rangle_*=e^{i\phi}\sum_{n=0}^{\infty}e^{-i(n+1)\phi}|n\rangle|_C. \tag{30}$$

## III. THE PROPERTIES OF $|\gamma\rangle_*$

When $|\gamma^*|>1$, $|\gamma'|<1$, we calculate the overlap between $\langle\gamma'|$ [the conjugate of Eq. (10)] and $|\gamma\rangle_*$,

$$\begin{aligned}\langle\gamma'|\gamma\rangle_*&=\frac{1}{2\pi i}\sum_{n,n'=0}^{\infty}\gamma'^{*n}\langle n|n'\rangle\frac{1}{\gamma^{*n'+1}}\bigg|_C\\&=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\gamma'^{*n}}{\gamma^{*n+1}}\bigg|_C=\delta(\gamma^*-\gamma'^*).\end{aligned} \tag{31}$$

Operating $\widehat{e^{i\phi}}$ on $|\gamma\rangle_*$ yields

$$\begin{aligned}\widehat{e^{i\phi}}|\gamma\rangle_*&=\frac{1}{\sqrt{N+1}}\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\sqrt{n}}{\gamma^{*n+1}}|n-1\rangle|_C\\&=\frac{1}{\gamma^*}\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{\gamma^{*n+1}}|n\rangle|_C=\frac{1}{\gamma^*}|\gamma\rangle_*.\end{aligned} \tag{32}$$

Using $\langle\gamma|$ ($|\gamma\rangle$ is the eigenket of $\widehat{e^{i\phi}}$) and $|\gamma\rangle_*$, we make up the following contour integration:

$$\begin{aligned}\oint_C|\gamma\rangle_*\langle\gamma|d\gamma^*&=\sum_{n,n'=0}^{\infty}\oint_C d\gamma^*\frac{|n\rangle\langle n'|\gamma^{*n'}}{2\pi i\gamma^{*n+1}}\\&=\sum_{n=0}^{\infty}|n\rangle\langle n|=1,\end{aligned} \tag{33}$$

which represents a completeness relation in contour integration form.

Equation (33) is a generalization of Eq. (16), because when $\gamma^*=e^{i\phi}$, the left-hand side of Eq. (33) becomes (with $d\gamma^*=de^{i\phi}=ie^{i\phi}d\phi$)

$$\begin{aligned}\oint_C d\gamma^*|e^{i\phi}\rangle_*\langle e^{i\phi}|&=\frac{1}{2\pi}\int_0^{2\pi}d\phi\sum_{n,n'=0}^{\infty}e^{-in\phi}|n\rangle\\&\quad\times\langle n'|e^{in'\phi}=1.\end{aligned} \tag{34}$$

The phase distribution over the window $-\pi<\phi\leqslant\pi$ for any state is then defined either by

$$P(\phi)=\frac{1}{2\pi}|\langle e^{i\phi}|\psi\rangle|^2 \tag{35}$$

or by

$$P(\phi)=\frac{1}{2\pi}\langle e^{i\phi}|\psi\rangle\langle\psi|e^{i\phi}\rangle_*|_C, \tag{36}$$

with the normalization integration

$$\int_{-\pi}^{\pi}d\phi P(\phi)=1. \tag{37}$$

As an application of $\widehat{e^{-i\phi}}$'s eigenket, we calculate

$$\begin{aligned}&\exp(\lambda\widehat{e^{-i\phi}})\\&=\oint_C e^{\lambda\gamma^*}|\gamma\rangle_*\langle\gamma|d\gamma^*\\&=\oint_C d\gamma^*\sum_{n,n'=0}^{\infty}\frac{e^{\lambda\gamma^*}|n\rangle\langle n'|\gamma^{*n'}}{2\pi i\gamma^{*n+1}}\\&=\sum_{n,n'=0}^{\infty}\frac{1}{(n-n')!}\left(\frac{d}{d\gamma^*}\right)^{n-n'}e^{\lambda\gamma^*}|n\rangle\langle n'|\,|_{\gamma^*=0}\\&=\sum_{n,n'=0}^{\infty}\frac{1}{(n-n')!}\lambda^{n-n'}|n\rangle\langle n'|.\end{aligned} \tag{38}$$

In summary, we have derived the eigenket of the SG phase operator $\widehat{e^{-i\phi}}$, which can be considered as a dual vector (or a partner) of $\widehat{e^{i\phi}}$'s eigenket. The completeness relation (33) in contour integration form is an extension of the phase distribution completeness relation (16). That is to say that Eqs. (27), (31), and (33) should be included in the SG phase operator formalism. Although a lot of works have discussed the phase-state representation in quantum optics (for example, Ref. [18], [19]), so far as we know, the eigenkets of operator $\widehat{e^{-i\phi}}$, which form a complementary part for the SG phase operators, have not been given in the literature before.

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