Tomographic reconstruction of the density operator from its normally ordered moments

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The reconstruction of the density operator from the tomographic data (rotated quadrature components) via the normally ordered moments of the density operator is investigated. It is shown how arbitrary normally ordered moments of arbitrary order n can be obtained from the quadrature components for $n+1$ discrete angles that can be chosen arbitrarily. An integration over the angles of the rotated quadrature components multiplied by discrete phase factors is not necessary and uses more than the minimally necessary information about the rotated quadrature components. [S1050-2947(96)00312-5]

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Tomographic reconstruction is the reconstruction of a field function over an N-dimensional space from the integrated field functions over all possible $(N-1)$-dimensional hyperplanes, which means from their Radon transform. The importance of the paper of Vogel and Risken [1] for quantum optics is that it practically applies the inverse Radon transformation (e.g., [2,3]) to the reconstruction of the Wigner quasiprobability in two-dimensional phase space. However, to the author's knowledge, the Radon transformation is first mentioned in this connection in [4]. In quantum optics of a single mode the Radon transform corresponds to the rotated quadrature components dependent on the angle. Reference [1] gives the inversion of the Radon transform leading to the Wigner quasiprobability by three integrations that correspond to the transition from the quadrature components to the Fourier transform of the Wigner quasiprobability in the first step and to the inversion of the two-dimensional Fourier transform in the second step. The recent interest in this field is connected with the progress in experiments to determine the quadrature components dependent on the angle by homodyne detection [4–7], which makes it possible to determine the Wigner quasiprobability. Some theoretical papers are concerned with the problem of determining directly from measured data some other characteristics of the density operator such as, for example, the matrix elements of the density operator in the Fock-state representation [8–17].

Recently, Richter [18] has shown that the normally ordered moments of the density operator can be determined from the rotated quadrature components by twofold integration over pattern functions including one integration over the angle. However, this uses more than the necessary minimal information about the rotated quadrature components. I show in this paper that the integration over the angle is not necessary and that any normally ordered moment of order n (sum of the powers of $\hat{a}$ and $\hat{a}^\dagger$) can be obtained from the rotated quadrature components for $n+1$ discrete different angles. This is of great importance for the experiment. Usually, one does not intend to determine normally ordered moments of very high order. For example, in photon statistics one is often satisfied by the incomplete information about the state consisting of the mean value of the number operator and its variance that corresponds to normally ordered moments up to fourth order. For this purpose one needs minimally only the quadrature components for five arbitrary different angles. Any additional angle provides redundancies that can be used to check the consistency of the results by combining five different angles. This becomes impossible if we integrate from the beginning over the angle, which is equivalent to averaging over the angle.

Let us derive our basic result. The knowledge of all normally ordered moments $\langle \hat{a}^{\dagger k}\hat{a}^l\rangle$ is equivalent to the complete information contained in the density operator $\hat{\varrho}$ via the reconstruction relation

$$\hat{\varrho}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\hat{a}_{k,l}\langle \hat{a}^{\dagger k}\hat{a}^l\rangle,\quad \hat{a}_{k,l}\equiv\sum_{j=0}^{\{k,l\}}\frac{(-1)^j|l-j\rangle\langle k-j|}{j!\sqrt{(k-j)!(l-j)!}}, \tag{1}$$

which was derived in [19], later in [20], and recently in a different way in [21]. The operators for the reconstruction $\hat{a}_{k,l}(k,l=0,1,2,\dots)$ are given here in the Fock-state representation. The measurable quadrature components $\langle q;\varphi|\hat{\varrho}|q;\varphi\rangle$, together with their connection to the Radon transform $\breve{W}(u,v;c)$ of the Wigner quasiprobability $W(q,p)$, are defined by

$$\breve{W}(\cos\varphi,\sin\varphi;q)\equiv\langle q;\varphi|\hat{\varrho}|q;\varphi\rangle$$
$$\equiv\langle q|(\hat{R}(\varphi))^\dagger\hat{\varrho}\hat{R}(\varphi)|q\rangle,$$
$$\hat{R}(\varphi)\equiv\exp(i\varphi\hat{a}^\dagger\hat{a}),$$
$$\breve{W}(u,v;c)\equiv\int dq\wedge dp\,\delta(c-(uq+vp))W(q,p)$$
$$=|\mu|\breve{W}(\mu u,\mu v;\mu c),\quad \mu\neq 0, \tag{2}$$

with arbitrary μ. The two-dimensional vector $(\cos\varphi,\sin\varphi)$ is the normal unit vector to the hyperplanes (here lines) in the Radon transform. In (2) the unitary rotation operator $\hat{R}(\varphi)$ has been introduced. Its most important properties for our purpose are

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$$\hat{R}(\varphi)\hat{Q}(\hat{R}(\varphi))^{\dagger}=\hat{Q}\cos\varphi+\hat{P}\sin\varphi = \sqrt{\frac{\hbar}{2}}(\hat{a}e^{-i\varphi}+\hat{a}^{\dagger}e^{i\varphi})\equiv\hat{Q}(\varphi),$$

$$\hat{R}(\varphi)\hat{P}(\hat{R}(\varphi))^{\dagger}=-\hat{Q}\sin\varphi+\hat{P}\cos\varphi = -i\sqrt{\frac{\hbar}{2}}(\hat{a}e^{-i\varphi}-\hat{a}^{\dagger}e^{i\varphi}), \tag{3}$$

and the relation of the eigenvalue problem of $\hat{Q}(\varphi)$ to the eigenvalue problem of $\hat{Q}$

$$\hat{Q}|q\rangle=q|q\rangle,\quad |q;\varphi\rangle\equiv\hat{R}(\varphi)|q\rangle,$$

$$\to\hat{Q}(\varphi)|q;\varphi\rangle=\hat{R}(\varphi)\hat{Q}|q\rangle=q|q;\varphi\rangle. \tag{4}$$

To obtain the connection of $\langle q;\varphi|\hat{\varrho}|q;\varphi\rangle$ with the normally ordered moments $\langle\hat{a}^{\dagger k}\hat{a}^{l}\rangle$ we have to use Eq. (1),

$$\langle q;\varphi|\hat{\varrho}|q;\varphi\rangle=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle q;\varphi|\hat{a}_{k,l}|q;\varphi\rangle\langle\hat{a}^{\dagger k}\hat{a}^{l}\rangle. \tag{5}$$

By using the explicit representation of $\hat{a}_{k,l}$ given in (1) and the action of the rotation operator $\hat{R}(\varphi)$ onto Fock states $|n\rangle$ resulting in a multiplication by the phase factor $\exp(in\varphi)$ and, furthermore, the well-known position representation $\langle q|n\rangle$ of the Fock states by Hermite functions, we find [22]

$$\langle q;\varphi|\hat{a}_{k,l}|q;\varphi\rangle=\frac{1}{\sqrt{\pi\hbar}}\exp\left(-\frac{q^2}{\hbar}\right)\frac{e^{i(k-l)\varphi}}{\sqrt{2^{k+l}}k!l!}\times\sum_{j=0}^{\{k,l\}}\frac{(-2)^j k!l!}{j!(k-j)!(l-j)!}H_{k-j}\left(\frac{q}{\sqrt{\hbar}}\right)\times H_{l-j}\left(\frac{q}{\sqrt{\hbar}}\right)=\frac{1}{\sqrt{\pi\hbar}}\exp\left(-\frac{q^2}{\hbar}\right)\frac{e^{i(k-l)\varphi}}{\sqrt{2^{k+l}}k!l!}H_{k+l}\left(\frac{q}{\sqrt{\hbar}}\right), \tag{6}$$

where a known identity for finite sums over Hermite polynomials is used [formula (36), Chap. 10.13 in [23]]. It can be proved by complete induction. The Hermite polynomial in this relation is not specific for k and l separately, but only for the sum $k+l=n$. Therefore by multiplication of (5) with $H_n(q/\sqrt{\hbar})$, integration over the variable q with the derived form of $\langle q;\varphi|\hat{a}_{k,l}|q;\varphi\rangle$ in (6), and using the well-known orthonormality relations for the Hermite functions one obtains the following linear combination of normally ordered moments $\langle\hat{a}^{\dagger k}\hat{a}^{l}\rangle$ with $n=k+l$:

$$\frac{1}{\sqrt{2^n}}\int_{-\infty}^{+\infty}dq\langle q;\varphi|\hat{\varrho}|q;\varphi\rangle H_n\left(\frac{q}{\sqrt{\hbar}}\right)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}e^{i(2k-n)\varphi}\langle\hat{a}^{\dagger k}\hat{a}^{n-k}\rangle=\langle\mathcal{N}\{(\hat{a}e^{-i\varphi}+\hat{a}^{\dagger}e^{i\varphi})^n\}\rangle=\left(\sqrt{\frac{2}{\hbar}}\right)^n\langle\mathcal{N}\{(\hat{Q}(\varphi))^n\}\rangle, \tag{7}$$

where $\mathcal{N}\{\}$ denotes normal ordering of the content in curly braces.

The moment of zeroth order is simply the trace of the density operator $\hat{\varrho}$ or according to Eq. (7),

$$\int_{-\infty}^{+\infty}dq\langle q;\varphi|\hat{\varrho}|q;\varphi\rangle=1 \tag{8}$$

for arbitrary angles φ, and obviously no integration over φ is necessary. To determine the moments of first order one can choose two arbitrary different angles $\varphi_1\neq\varphi_2$ (pairs of angles φ and $\varphi+\pi$ are equivalent and cannot be used as ''different'' angles) and after solution of the corresponding two equations of (7) with $H_1(z)=2z$, one obtains

$$\langle\hat{a}\rangle=\frac{1}{i2\sin(\varphi_2-\varphi_1)}\sqrt{\frac{2}{\hbar}}\int_{-\infty}^{+\infty}dq(e^{i\varphi_2}\langle q;\varphi_1|\hat{\varrho}|q;\varphi_1\rangle - e^{i\varphi_1}\langle q;\varphi_2|\hat{\varrho}|q;\varphi_2\rangle)q,$$

$$\langle\hat{a}^{\dagger}\rangle=\frac{1}{i2\sin(\varphi_1-\varphi_2)}\sqrt{\frac{2}{\hbar}}\int_{-\infty}^{+\infty}dq(e^{-i\varphi_2}\langle q;\varphi_1|\hat{\varrho}|q;\varphi_1\rangle - e^{-i\varphi_1}\langle q;\varphi_2|\hat{\varrho}|q;\varphi_2\rangle)q. \tag{9}$$

In particular, for $\varphi_1=0$ and $\varphi_2=\pi/2$,

$$\langle\hat{Q}\rangle=\int_{-\infty}^{+\infty}dq\langle q;0|\hat{\varrho}|q;0\rangle q=\sqrt{\frac{\hbar}{2}}(\langle\hat{a}\rangle+\langle\hat{a}^{\dagger}\rangle),$$

$$\langle\hat{P}\rangle=\int_{-\infty}^{+\infty}dq\left\langle q;\frac{\pi}{2}\right|\hat{\varrho}\left|q;\frac{\pi}{2}\right\rangle q=-i\sqrt{\frac{\hbar}{2}}(\langle\hat{a}\rangle-\langle\hat{a}^{\dagger}\rangle). \tag{10}$$

The normally ordered moments of second order $\langle\hat{a}^2\rangle,\langle\hat{a}^{\dagger}\hat{a}\rangle$, and $\langle\hat{a}^{\dagger 2}\rangle$ can be found from the special case $n=2$ in (7) that is dependent on the quadrature components for three arbitrary different angles. Because of the length of the formulas I will not write them down. However, there is a general simple form of the solution of Eq. (7) with respect to the normally ordered moments of arbitrary order n if one chooses $n+1$ equally spaced angles from $2(n+1)$ angles that solve the circle division problem. This solution has the form

$$\langle \hat{a}^{\dagger k}\hat{a}^{n-k}\rangle = \frac{k!(n-k)!}{(n+1)!}\sum_{m=0}^{n} \exp\left\{ -i(2k-n)\left(\varphi_0 + \frac{m\pi}{n+1}\right)\right\} \times \frac{1}{\sqrt{2^n}}\int_{-\infty}^{+\infty} dq \left\langle q;\varphi_0 + \frac{m\pi}{n+1}\right| \times \hat{\varrho}\left| q;\varphi_0+\frac{m\pi}{n+1}\right\rangle H_n\left(\frac{q}{\sqrt{\hbar}}\right), \tag{11}$$

where φ_0 is an initial angle that can be chosen arbitrarily. The proof can be given by considering the solutions of the circle division problem. The complex solutions z for the division of the unit circle into $n+1$ equal parts satisfy the equation

$$0 = z^{n+1} - 1 = (z-1)(z^n + z^{n-1} + \cdots + z + 1) = (z-1)\sum_{m=0}^{n} z^m, \quad z = \exp\left(is\frac{2\pi}{n+1}\right). \tag{12}$$

From this relation it follows that each solution $z=\exp\{is\,2\pi/(n+1)\}$ of the circle division problem with exception of $z=1$ corresponding to $s=0$ satisfies the equation $\Sigma_{m=0}^{n} z^m = 0$. By inserting (7) into (11) and by using the discussed properties of the circle division problem one proves the solution (11). The solution given by Richter [18] corresponds to an additional averaging over all possible initial angles φ_0 in (11) and is true but needs more than the minimally necessary information about the quadrature components.

Let us consider squeezed coherent states as an important example with nontrivial angular dependence. The Wigner quasiprobability of such states has the form [24]

$$W(q,p) = \frac{1}{\hbar\pi}\exp\left\{ -\frac{[(1+\zeta)(q-\overline{Q}) + i(1-\zeta)(p-\overline{P})][(1+\zeta^*)(q-\overline{Q}) - i(1-\zeta^*)(p-\overline{P})]}{(1-\zeta\zeta^*)\hbar}\right\}, \tag{13}$$

with $\overline{Q}\equiv\langle\hat{Q}\rangle$ and $\overline{P}\equiv\langle\hat{P}\rangle$ as the real displacement parameters with respect to canonical coordinates (q,p) and with ζ as the squeezing parameter in the nonunitary approach [25,26]. The Radon transform $\breve{W}(u,v;c)$ according to the definition in (2) is explicitly given by

$$\breve{W}(u,v;c) = \sqrt{\frac{1-\zeta\zeta^*}{[(1-\zeta)(1-\zeta^*)u^2 + (1+\zeta)(1+\zeta^*)v^2 + i(\zeta-\zeta^*)2uv]\hbar\pi}} \times \exp\left\{ -\frac{(1-\zeta\zeta^*)(c-u\overline{Q}-v\overline{P})^2}{[(1-\zeta)(1-\zeta^*)u^2+(1+\zeta)(1+\zeta^*)v^2+i(\zeta-\zeta^*)2uv]\hbar}\right\}. \tag{14}$$

Now, by forming $\breve{W}(\cos\varphi,\sin\varphi;q)$ from (14) and after its multiplication with the Hermite polynomials and integration one finds

$$\frac{1}{\sqrt{2^n}}\int_{-\infty}^{+\infty} dq\,\breve{W}(\cos\varphi,\sin\varphi;q) H_n\left(\frac{q}{\sqrt{\hbar}}\right) = \sum_{j=0}^{[n/2]} \frac{(-1)^j n!}{j!(n-2j)!}\left(\frac{\zeta e^{-i2\varphi} - 2\zeta\zeta^* + \zeta^* e^{i2\varphi}}{2(1-\zeta\zeta^*)}\right)^j (\overline{a}e^{-i\varphi} + \overline{a^*}e^{i\varphi})^{n-2j}$$

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} e^{i(2k-n)\varphi}\left(\sqrt{\frac{\zeta^*}{2(1-\zeta\zeta^*)}}\right)^k \left(\sqrt{\frac{\zeta}{2(1-\zeta\zeta^*)}}\right)^{n-k} \times \sum_{j=0}^{\{k,n-k\}} \frac{k!(n-k)!}{j!(k-j)!(n-k-j)!}(2\sqrt{\zeta\zeta^*})^j H_{k-j}\left(\sqrt{\frac{1-\zeta\zeta^*}{2\zeta^*}}\overline{a^*}\right) \times H_{n-k-j}\left(\sqrt{\frac{1-\zeta\zeta^*}{2\zeta}}\overline{a}\right), \tag{15}$$

with $\overline{a}\equiv\langle\hat{a}\rangle$ and $\overline{a^*}\equiv\langle\hat{a}^\dagger\rangle$. In (15) I have omitted some, in principle, simple intermediate steps of the transition from the result of the integration to its representation similar to (7). The result of integration [the first line in (15)] can also be written by means of the Hermite polynomial $H_n(z)$, however, without an advantage for our purpose. By comparison with (7) one finds that the normally ordered moments for squeezed coherent states are given by

$$\langle \hat{a}^{\dagger k}\hat{a}^{l}\rangle = \left(\sqrt{\frac{\zeta^*}{2(1-\zeta\zeta^*)}}\right)^k \left(\sqrt{\frac{\zeta}{2(1-\zeta\zeta^*)}}\right)^l \times \sum_{j=0}^{\{k,l\}} \frac{k!l!}{j!(k-j)!(l-j)!}(2\sqrt{\zeta\zeta^*})^j \times H_{k-j}\left(\sqrt{\frac{1-\zeta\zeta^*}{2\zeta^*}}\overline{a^*}\right) H_{l-j}\left(\sqrt{\frac{1-\zeta\zeta^*}{2\zeta}}\overline{a}\right), \quad (16)$$

which coincides with special moments up to fourth order calculated in another way in [25,26]. Therefore, sometimes it is possible to use the relation (7) for the calculation of the normally ordered moments themselves, apart from its importance for analyzing experimental results.

In conclusion, by using formula (1) for the reconstruction of the density operator from its normally ordered moments I have derived in a simple way the basic equations (7) and (11) which relate the measurable quadrature components to the normally ordered moments, where no integration over the angle is necessary. The approach is illustrated for squeezed coherent states. The determination of the normally ordered moments has some advantages in comparison to the determination of the matrix elements of the density operator in the Fock-state representation and is more simple. In the Fock-state representation one often has to determine matrix elements $\langle m|\hat{\varrho}|n\rangle$ for relatively high numbers of m and n, which is difficult. On the other hand, according to the usual belief, a small number of low-order normally ordered moments $\langle \hat{a}^{\dagger k}\hat{a}^{l}\rangle$ gives sufficient information about the state of the system. This last statement is not fully clear when we look to the reconstruction formula (1) of the density operator and its background should be investigated in the future.

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