Corrections to $O(\alpha^7 mc^2)$ fine-structure splittings in helium

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The times-order external-potential Bethe-Salpeter formalism is reformulated in a Schrödinger-like equation and in a form suitable for calculation of contributions arising from the relativistic momentum region. $O(\alpha^7 mc^2)$ corrections to the fine structure of helium arising from the relativistic momentum region due to exchange diagrams are derived and presented. They are expressed in the form of expectation values of nonrelativistic operators. These off-leading-order contributions arising from the relativistic momentum region are not sensitive to any experiment in hydrogen, positronium, or muonium but they are larger than experimental errors in the measurements of fine structure in helium. Therefore, they provide a test of corrections of this kind, which has not been carried out in any other bound-state system. [S1050-2947(96)08512-5]

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I. INTRODUCTION

Since the development of QED theory, helium has played an active role in testing the bound-state QED theory for two-electron systems. In fact, helium can provide some interesting tests that cannot be carried out in other bound-state systems such as hydrogen, positronium, and muonium, due to either unique features of helium or the fact that the tests are not sensitive to the measurements in those systems currently. For example, a unique higher-order cancellation of nonperturbative Coulomb binding is found [1] to occur only in the multielectron atoms. Another interesting test is on the off-leading-order corrections arising from the relativistic momentum region, which will be the main focus of this paper. One potential test is the test of three-body terms. This is particularly interesting because it is not clear whether any test so far in QED or QCD bound systems has tested true three-body terms. Most many-body calculations are done on the interaction of any two particles out of many particles. In other cases, the true three-body terms are not explicitly sensitive to the experiments. However, the lowest-order three-body terms in helium are found to be of order $\alpha^6 m^2 c^2/M$ and might give a correction of a few kHz to the helium fine structure while the current experimental error in measuring the fine structure is about 3 kHz [7]. The calculation of these three-body corrections is completed within a three-body formalism recently developed. This development will be reported in a separate paper.

The first systematic calculation of helium energy levels of order $\alpha^5 mc^2$ was accomplished by Araki [2] and by Sucher [3] in the 1950s, which was considered a milestone. All the corrections to the $O(\alpha^5 mc^2)$ energy levels come from two-photon diagrams and arise from both relativistic and nonrelativistic momentum regions. Both the relativistic and the nonrelativistic contributions are of leading order. In comparison with hydrogen, positronium, and muonium, the major difference in this calculation is the separation of ultraviolet logarithmic singularity from the finite corrections at the operator level rather than the numerical level, and the derivation of the correct nonrelativistic operator for the corrections since the nonrelativistic wave function of helium is unknown. The second monumental endeavor was the calculation of QED and relativistic corrections to the helium fine structure splittings of order $\alpha^6 mc^2$. These corrections were derived by Douglas and Kroll [4] in the 1970s, and expressed in terms of expectation values of nonrelativistic operators. This work puts helium in a leading position as a candidate for higher-order QED tests since a similar calculation for positronium was done 20 years later [5]. Furthermore, the calculation of the $O(\alpha^6 mc^2)$ positronium fine structure requires only the evaluation of expectation values of Douglas and Kroll's nonrelativistic operators using the nonrelativistic wave function for positronium as demonstrated in a previous paper [1]. For the $O(\alpha^6 mc^2)$ fine structure, all corrections come from one- and two-photon diagrams and arise only from the nonrelativistic momentum region. Therefore, these corrections are of off-leading order. The absence of the relativistic contributions simplifies the calculation significantly. In Douglas and Kroll's work, a beautiful generalization of the Foldy-Wouthysen transformation was introduced. This generalization turned out to be a very helpful tool for higher-order QED analysis, including the calculation presented in this paper.

The development of a highly accurate nonrelativistic wave function for helium [6] and a recent high-precision measurement of the helium fine structure [7] made it possible to test even higher-order QED and relativistic effects in helium. In previous papers [1,8], we derived nonrelativistic operators of order $\alpha^7 mc^2$ contributing to the fine structure splittings of helium. They arise from exchange and radiative diagrams, and are obtained in a nonrelativistic approximation. The approximation is accurate for radiative corrections since no QED correction is found from the relativistic momentum region. This is confirmed by the absence of $\ln\alpha$ terms due to ultraviolet origin (detailed presentation of infrared and ultraviolet logarithmic corrections is given in Ref. [1] and all logarithmic terms of the two-electron type arise from ultraviolet origin). However, there are exchange corrections arising from the relativistic momentum region. This is signaled by the appearance of logarithmic terms due to ultraviolet singularity. Furthermore, these corrections are of off-leading order and appear only in bound states. Tests of these corrections are interesting since to our knowledge there has been no similar calculation done in any one- or two-body bound-state system. In a bound system such as hydrogen under one-body approximation, no net correction of order $\alpha^7 mc^2$ arises from pure Coulomb exchange. The only correction of this order comes from radiative QED effects such as self-energy modification, which is of nominal order $\alpha^5 mc^2$ and gives nonrelativistic contribution. The leading order of corrections from the relativistic momentum region is $\alpha^6 mc^2$. These corrections were obtained by Karplus *et al.* [9] and by Baranger *et al.* [10]. The next-to-leading order of the relativistic corrections is $\alpha^8 mc^2$, which would be nonrelativistic corrections of relative order α^2 to the terms obtained by Karplus *et al.* [9] and by Baranger *et al.* [10]. Of course, the three-potential contributions due to self-energy corrections from the relativistic momentum region are of leading order $\alpha^7 mc^2$. The relativistic contributions of the next-to-leading order are of order $\alpha^7 mc^2$ in positronium or order $\alpha^7 m^2 c^2/M$ in hydrogen and muonium. In any case, the corrections are not sensitive to the current experiments in hydrogen, positronium, or muonium. Therefore, no correction of off-leading order from the relativistic momentum region has ever been tested in any bound-state system. However, they are sensitive to the measurement of fine structure in helium. The magnitude of order of these corrections is about 10 kHz and larger than the current experimental error 3 kHz [7]. A test of these corrections is important in view of the fact that the correction of off-leading order is the characteristic of bound-state systems. In this paper, we will present our analysis on these corrections to the fine structure splittings of helium. Together with the corrections presented previously [1,8], they consist of most corrections of order $\alpha^7 mc^2$ to the helium fine structure splittings. Corrections that have not been derived may partially arise from vertex modification. The vertex correction was calculated in Ref. [8] phenomenologically. A rigorous analysis may be required for a complete evaluation. The additional correction could be a few kHz.

Although the contributions presented in Ref. [1] arise from the nonrelativistic momentum region, fully relativistic kernels due to single and double transverse photon exchange were derived based on the times-order Bethe-Salpeter formalism developed by Sucher [3]. As we shall show, calculation of contributions arising from the relativistic momentum region is very complicated for no-pair Coulomb and no-pair single transverse photon diagrams within the times-order formalism developed by Sucher [3]. The main difficulty is due to the application of the Brillouin-Wigner perturbation theory, which is more suitable for calculation of contributions coming from the nonrelativistic momentum region. In this paper, we will reformulate the times-order formalism and rederive these kernels in a form that is more suitable for calculation of contributions arising from the relativistic momentum region. Most of the derived kernels can also be obtained using the Salpeter [11] perturbation theory, which is more convenient for the relativistic calculation in comparison with the Brillouin-Wigner perturbation theory used in Sucher's times-order formalism [3,4,1]. The difference between our formulation and the Salpeter theory [11] for the current calculation is that the no-pair ladder kernels of Coulomb exchange and single transverse photon exchange are derived using the Salpeter perturbation theory in the Salpeter formalism, while ours are obtained from scattering amplitudes directly. In the Salpeter perturbation theory, the no-pair ladder kernels are obtained by the wave-function perturbation relativistically or nonrelativistically depending on whether the corrections arise from the relativistic or nonrelativistic momentum region. The times-order ladder kernels do not show up in nonrelativistic approximation. In our formalism, the relativistic no-pair ladder kernels are derived from the scattering theory and the nonrelativistic kernels are obtained by perturbation of the nonrelativistic wave function using the Schrödinger nonrelativistic perturbation theory. The times-order no-pair ladder kernels in nonrelativistic approximation are shown to cancel out in Ref. [1]. For the fine structure of order $\alpha^7 mc^2$, the main difference between our relativistic kernels and those derived in the Salpeter perturbation theory is the no-pair single transverse photon ladder kernel. We found that ours is more convenient and instructive for the calculation of the off-leading-order relativistic contribution. Our reformulation of the times-order formalism is similar to the S-matrix theory. The idea is simply to sandwich a modified scattering amplitude between four-dimensional bound-state wave functions. In this reformulation of the times-order Bethe-Salpeter formalism, all kernels may be written in an explicitly covariant form at first. Different methods may be used for different calculations, depending on relativistic contributions or nonrelativistic contributions. For the one-photon kernel, all contributions are nonrelativistic and of orders $\alpha^2 mc^2$, $\alpha^4 mc^2$, $\alpha^6 mc^2$, and so on. For the two-photon kernel, contributions come from both relativistic and nonrelativistic momentum regions, and are of orders $\alpha^5 mc^2$, $\alpha^6 mc^2$, $\alpha^7 mc^2$, and so on. In nonrelativistic approximation, all the no-pair ladder kernels give contributions by perturbation of the nonrelativistic wave function. The only exception is the no-pair double transverse photon corrections, which come in part from two diagrams that are not times-order ladder graphs, although they are parts of the covariant ladder Feynman diagrams. There is a difference between the times-order ladder diagrams and covariant ladder diagrams. With our reformulation, the two-photon relativistic contributions can be calculated using a simple formula as given by the following:

$$\begin{aligned}\Delta E=&\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int d^4k d^4k' D_{\mu\nu}(k)D_{\alpha\beta}(k')\\&\times\langle\psi(p_{1\mu}p_{2\mu})|\gamma_1^0\gamma_1^\mu S_1(p_1-k)\gamma_1^0\gamma_1^\alpha\\&\times[\gamma_2^0\gamma_2^\nu S_2(p_2+k)\gamma_2^0\gamma_2^\beta+\gamma_2^0\gamma_2^\beta S_2(p_2+k')\gamma_2^0\gamma_2^\nu]\\&\times|\psi(p'_{1\mu}p'_{2\mu})\rangle,\end{aligned}\tag{1}$$

where $p'_{1\mu}=p_{1\mu}-k_\mu-k'_\mu$ $p'_{2\mu}=p_{2\mu}+k_\mu+k'_\mu$, and S_1 and S_2 are the external-potential single-particle propagators defined by

$$S_1(p_1-k)=\frac{1}{\mu_1 E+\epsilon-\omega-H(\mathbf{p}_1-\mathbf{k})}\tag{2}$$

and

$$S_2(p_2+k)=\frac{1}{\mu_2 E-\epsilon+\omega-H(\mathbf{p}_2+\mathbf{k})}\tag{3}$$

with

$$\mu_1 = \frac{m_1}{m_1 + m_2} \tag{4}$$

and

$$H(\mathbf{p}_1 - \mathbf{k}) = \boldsymbol{\alpha}_1 \cdot (\mathbf{p}_1 - \mathbf{k}) + m_1 \gamma_1^0 + V_1, \tag{5}$$

and similarly for μ_2 and $H(\mathbf{p}_2 + \mathbf{k})$. $D_{\mu\nu}$ and $D_{\alpha\beta}$ are covariant propagators of photons. ψ is the four-dimensional bound-state wave function [11]. Seven-dimensional integrations over $p_{1\mu} = (\epsilon, \mathbf{p}_1)$ and $p_{2\mu} = (\epsilon, \mathbf{p}_2)$ are implied. The first term in Eq. (1) corresponds to a covariant ladder diagram and the second to a covariant crossed-ladder diagram. Similarly, three- or four-photon relativistic contributions can be calculated by replacing the above amplitude by the modified three- or four-photon amplitude, respectively. The revised times-order formalism is obtained by expressing the photons' propagators in terms of Coulomb and transverse photon propagators corresponding to Coulomb-Coulomb, Coulomb-transverse, and transverse-transverse photon diagrams, and the single-particle propagators in terms of positive- and negative-energy projection operators connected with the no-pair, one-pair, and two-pair times-order diagrams. Although Eq. (1) is developed for calculation of relativistic contributions, it also can be used to calculate the two-photon contributions arising from the nonrelativistic momentum region, which were calculated in Ref. [1]. The exception is the no-pair times-order ladder corrections, which arise from perturbation of nonrelativistic wave functions. This is conceivable since the only difference between a bound state and a free particle system is that the former is a nonrelativistic system and the latter is not. It is the nonrelativistic wave function of a bound state that causes corrections of infinite orders arising from a given diagram. In contrast, the plane wave function of a free particle system leads to corrections of one order due to a given diagram. The two-photon energy formula (1) can be generalized to a formula for infinite photon exchange diagrams. Summing all exchanged photons, we get

$$\begin{aligned} E = {} & \langle \phi_c(\mathbf{p}_1\mathbf{p}_2) | H(\mathbf{p}_1) + H(\mathbf{p}_2) | \phi_c(\mathbf{p}_1\mathbf{p}_2) \rangle \\ & + \sum_{n=1}^{\infty} \left(\frac{\alpha}{2\pi^2} \right)^n \left(\frac{1}{-2\pi i} \right)^{n-1} (-1)^n \int \prod_{i=1}^{n} d^4k_i D_{\mu_i\nu_i}(k_i) \\ & \times \left\langle \psi(p_{1\mu}p_{2\mu}) \right| \left[\prod_{i=1}^{n-1} \gamma_1^0 \gamma_1^{\mu_i} S_1(p_1 - k_i) \right] \gamma_1^0 \gamma_1^{\mu_n} \\ & \times \left\{ \sum_{l_1=\nu_1}^{\nu_n} \cdots \sum_{l_n=\nu_1}^{\nu_n} \epsilon_{l_1,\ldots,l_n} \left[\prod_{i=1}^{n-1} \gamma_2^0 \gamma_2^{l_i} S_2(p_2 + k^{l_i}) \right] \right. \\ & \left. \times \gamma_2^0 \gamma_2^{l_n} \right\} \left| \psi(p'_{1\mu}p'_{2\mu}) \right\rangle, \end{aligned} \tag{6}$$

where $\epsilon_{l_1,\ldots,l_n} = 0$ if $l_i = l_j$ or 1 otherwise. ϕ_c is the Coulomb ladder wave function [3,4,1]. This is a Schrödinger-like equation and reduces to the Schrödinger equation in the nonrelativistic limit. For n photons, the number of terms is n. For example, there are one, two, and six terms for one, two, and three photons, respectively. For radiative diagrams, the fermion and photon propagators are replaced by the modified self-energy and vacuum polarization propagators. The Dirac gamma matrix is replaced by the modified vertex. For more complicated radiative diagrams, the amplitudes are written according to the Feynman rules for the scattering problem. The above formula is obtained as a combined extension of the formalisms in Refs. [1] and [16]. No overcounting occurs since the ladder kernels in nonrelativistic approximation cancel out as demonstrated in Ref. [1].

The general idea for calculation of relativistic contributions is to let the momenta of exchanged photons be relativistic. To be relativistic, at least two exchanged photons are required since the external observable or the variables in nonrelativistic wave functions must be nonrelativistic. That is, the linear combination of photon momenta is nonrelativistic. On the other hand, only two-photon exchange diagrams contribute to the helium fine-structure splittings of order α^7mc^2 from the relativistic momentum region. The contribution is a nonrelativistic expansion of order α^2 relative to the lowest-order relativistic energy corrections obtained by Araki [2] and by Sucher [3]. To the order of interest, a fully relativistic kernel for a given diagram must be derived. Such a kernel contains both relativistic variables and nonrelativistic variables. To lowest order α^5mc^2, only relativistic variables are retained while nonrelativistic variables are neglected. To order α^7mc^2, all nonrelativistic variables are expanded to order α^2 relative to the lowest order. Three-photon diagrams contribute to relativistic energy levels of leading order α^6mc^2. Their corrections to the fine structure are of order α^8mc^2.

II. COULOMB PHOTON EXCHANGE

Although no Coulomb exchange correction to $O(\alpha^7mc^2)$ fine structure in helium is found in nonrelativistic approximation, contributions arise from the relativistic momentum region. These contributions come from the no-pair Coulomb ladder corrections of first and second order in the Brillouin-Wigner perturbation theory, and from one-pair and two-pair diagrams. To lowest order, they contribute to the energy corrections of order α^5mc^2, which were obtained by Araki [2] and by Sucher [3]. Let us start with the no-pair Coulomb ladder equation. The no-pair Coulomb ladder Hamiltonian that contributes to the energy levels of order α^5mc^2 from a relativistic momentum region was obtained by Sucher, and is given by

$$\begin{aligned} H_c = mI_c \Bigg[& -\frac{1}{4m^2} + \frac{1}{E(\mathbf{p}_1)[E(\mathbf{p}_1)+m]} - \frac{1}{[E(\mathbf{p}_1)+m]^2} \\ & - \frac{1}{p_1^2} \left(\frac{E(\mathbf{p}_1)-m}{2E(\mathbf{p}_1)} \right)^2 \Bigg] I_c, \end{aligned} \tag{7}$$

where terms of order α^4mc^2 are subtracted. Here, I_c is the interelectron Coulomb interaction operator, and

$$E(\mathbf{p}_1) = \sqrt{m^2 + p_1^2}.$$

In addition, a correction of second order due to wave function perturbation of first order was derived by Sucher, and is given by

$$H_2 = I_c \frac{E(\mathbf{p}_1)-m}{2[E(\mathbf{p}_1)+m]}\left[\frac{1}{E(\mathbf{p}_1)} - \frac{1}{E(\mathbf{p}_1)+m} + \frac{m-E(\mathbf{p}_1)}{4E^2(\mathbf{p}_1)}\right] I_c, \tag{8}$$

where the lower-order nonrelativistic corrections are subtracted. The total energy correction of order $\alpha^5 mc^2$ arising from the no-pair Coulomb ladder equation is

$$\Delta E_{cle} = \langle \phi_0 | H_c + H_2 | \phi_0 \rangle = -\alpha^5 mc^2 \left(\frac{\pi}{2} + \frac{5}{3}\right) \langle \phi_0 | \delta(\mathbf{r}) | \phi_0 \rangle \tag{9}$$

obtained by Sucher [3]. Another correction due to a two-pair diagram is given by

$$\begin{aligned}\Delta E_{--}^{C\cdot C} &= \langle \phi_c | I_c (-D_c)^{-1} \mathcal{L}_{--} I_c | \phi_c \rangle \\ &= -\left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\mathbf{k}}{k^2} \frac{d\mathbf{k}'}{k'^2} \\ &\quad \times \left\langle \phi_c(\mathbf{p}_1, \mathbf{p}_2) \middle| \frac{\Lambda_{1-}(\mathbf{p}_1 - \mathbf{k}) \Lambda_{2-}(\mathbf{p}_2 + \mathbf{k})}{E + E(\mathbf{p}_1 - \mathbf{k}) + E(\mathbf{p}_2 + \mathbf{k})} \right. \\ &\quad \times \left| \phi_c(\mathbf{p}_1 - \mathbf{k} - \mathbf{k}', \mathbf{p}_2 + \mathbf{k} + \mathbf{k}') \right\rangle. \end{aligned} \tag{10}$$

This two-pair Coulomb correction can also be derived from Eq. (1). To lowest order, the correction becomes

$$\Delta E_{--}^{C\cdot C} = \alpha^5 mc^2 \left(\frac{\pi}{2} - \frac{5}{3}\right) \langle \phi_0 | \delta(\mathbf{r}) | \phi_0 \rangle, \tag{11}$$

which was obtained by Sucher [3].

If we consider two distinct particles with masses m_1 and m_2, and $Z_1 = -Z_2 = 1$, the corresponding Hamiltonian of first order due to the no-pair Coulomb ladder interaction becomes

$$\begin{aligned} H_c = \mu I_c \Bigg\{ & 2\frac{\mu}{m_1}\left[\frac{1}{4m_1^2} - \frac{1}{(E_{p_1}+m_1)^2}\right] \\ & + 2\frac{\mu}{m_2}\left[\frac{1}{4m_2^2} - \frac{1}{(E_{p_2}+m_2)^2}\right] - \frac{1}{2m_1} \\ & + \frac{1}{E_{p_1}(E_{p_1}+m_1)} - \frac{1}{2m_2} + \frac{1}{E_{p_2}(E_{p_2}+m_2)} \\ & - \frac{p_1^2}{2} \frac{1}{E_{p_1}(E_{p_1}+m_1)} \frac{1}{E_{p_2}(E_{p_2}+m_2)} \Bigg\} I_c, \end{aligned} \tag{12}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass. The corresponding Hamiltonian of second order reads

$$\begin{aligned} H_2 = 2\mu^2 I_c & \frac{m_2(E_{p_2}+m_2)(E_{p_1}-m_1) + m_1(E_{p_1}+m_1)(E_{p_2}-m_2)}{m_1 m_2 (E_{p_1}+m_1)(E_{p_2}+m_2) - \mu m_2 (E_{p_2}+m_2)(E_{p_1}-m_1) - \mu m_1 (E_{p_1}+m_1)(E_{p_2}-m_2)} \\ & \times \left[-\frac{\mu}{m_1} \frac{1}{(E_{p_1}+m_1)^2} + \frac{1}{2E_{p_1}(E_{p_1}+m_1)} - \frac{\mu}{m_2} \frac{1}{(E_{p_2}+m_2)^2} + \frac{1}{2E_{p_2}(E_{p_2}+m_2)} \right. \\ & \left. - \frac{p_1^2}{2E_{p_1}(E_{p_1}+m_1)} \frac{1}{2E_{p_2}(E_{p_2}+m_2)} \right] I_c. \end{aligned} \tag{13}$$

The no-pair Coulomb ladder correction is given by the expectation value of the above two Hamiltonians. To lowest order, all nonrelativistic variables are dropped. The energy correction due to the two-pair diagram is found to be

$$\Delta E_{--}^{C\cdot C} = \alpha^5 \mu^3 c^2 I \langle \phi_0 | \delta(\mathbf{r}) | \phi_0 \rangle, \tag{14}$$

where

$$I = -2 \int_0^\infty \frac{dk}{k^2 E_1 E_2} \frac{(E_1 - m_1)(E_2 - m_2)}{E_1 + E_2 + m_1 + m_2}. \tag{15}$$

Here

$$E_1 = \sqrt{m_1^2 + k^2}, \quad E_2 = \sqrt{m_2^2 + k^2}. \tag{16}$$

Combining the above two corrections and performing the integration, we find

$$\Delta E_{cle} + \Delta E_{--}^{C\cdot C} = -\frac{2}{3} \alpha^5 \mu^3 c^2 \left[\frac{2}{m_1^2} + \frac{1}{m_1 m_2} + \frac{2}{m_2^2}\right] \times \langle \phi_0 | \delta(\mathbf{r}) | \phi_0 \rangle \tag{17}$$

which agrees with ΔE_{Ca} of Fulton and Martin [12]. As observed, the above calculation is quite complicated due to the Brillouin-Wigner perturbation theory. The higher-order terms are even more difficult to calculate. However, the calculation is simplified significantly using either the Salpeter

[11] perturbation theory or our reformulation equation (1). Since all the $O(\alpha^7 mc)$ relativistic contributions come from pure electron-electron interactions, external potentials are neglected. The Bethe-Salpeter Coulomb ladder (including pair ladder diagrams) equation is given by

$$[E_c-H(\mathbf{p}_1)-H(\mathbf{p}_2)]\phi_c(\mathbf{p}_1,\mathbf{p}_2)=[\Lambda_{++}(\mathbf{p}_1,\mathbf{p}_2)-\Lambda_{--}(\mathbf{p}_1,\mathbf{p}_2)]\frac{-\alpha}{2\pi^2}\times\int\frac{d\mathbf{k}}{k^2}\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k}) \quad (18)$$

for two electrons. Subtracting the Breit equation for Coulomb exchange leads to

$$\Delta E=\frac{\alpha}{2\pi^2}\int\frac{d\mathbf{k}}{k^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\Lambda_{+-}(\mathbf{p}_1,\mathbf{p}_2)+\Lambda_{-+}(\mathbf{p}_1,\mathbf{p}_2)+2\Lambda_{--}(\mathbf{p}_1,\mathbf{p}_2)|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k}\rangle. \quad (19)$$

Applying the Salpeter perturbation on the left wave function (perturbation of the right wave function gives zero relativistic contribution to the order of interest), we obtain

$$\Delta E=-\left(\frac{\alpha}{2\pi}\right)^2\int\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\left\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)\left|\frac{\Lambda_{+-}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})}{E-E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_2+\mathbf{k})}+\frac{\Lambda_{-+}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})}{E+E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2+\mathbf{k})}+\frac{2\Lambda_{--}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})}{E+E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_2+\mathbf{k})}\right|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\right\rangle. \quad (20)$$

To lowest order $\alpha^5\mu c^2$, the nonrelativistic variables $\mathbf{p}_1$ and $\mathbf{p}_2$ may be dropped. Thus we get

$$\Delta E=\frac{\alpha^5\mu^3c^2}{2\pi}\langle\phi_0|\delta(\mathbf{r})|\phi_0\rangle\int\frac{d\mathbf{k}}{k^4}\frac{1}{E_1E_2}\left[\frac{(E_1+m_1)(E_2-m_2)}{E_1-E_2-m_1-m_2}+\frac{(E_1-m_1)(E_2+m_2)}{E_2-E_1-m_1-m_2}-\frac{2(E_1-m_1)(E_2-m_2)}{E_1+E_2+m_1+m_2}\right]. \quad (21)$$

Performing integration over $\mathbf{k}$ gives

$$\Delta E=-\frac{2}{3}\alpha^5\mu^3c^2\left(\frac{2}{m_1^2}+\frac{1}{m_1m_2}+\frac{2}{m_2^2}\right)\langle\phi_0|\delta(\mathbf{r})|\phi_0\rangle. \quad (22)$$

This calculation is much simpler than that derived from the Brillouin-Wigner perturbation theory since the Salpeter perturbation is more suitable to relativistic calculation. This advantage of the Salpeter perturbation theory becomes more obvious in the case of single transverse photon exchange, which will be discussed later.

For the helium fine structure of order $\alpha^7 mc^2$, we need to calculate relativistic contributions of the next-to-leading order. The relativistic momenta $\mathbf{k}$ and $\mathbf{k}'$ must be treated more accurately:

$$\mathbf{k}+\mathbf{k}'=\mathbf{k}'', \quad (23)$$

where $\mathbf{k}''$ has to be nonrelativistic momentum as an external observable in the nonrelativistic wave function. $\mathbf{k}'$ is expanded nonrelativistically in terms of $\mathbf{k}''$, which makes the calculation more complicated and very singular.

In order to compare the Brillouin-Wigner perturbation theory with the Salpeter perturbation theory, we calculate the no-pair Coulomb ladder corrections in two different ways. First, we employ the Brillouin-Wigner perturbation theory. The spin dependent Hamiltonian of first order in the no-pair Coulomb ladder equation is

$$\begin{aligned}H_c^7=&\frac{1}{2}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{m}I_c\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\left[\frac{1}{E(\mathbf{p}_1)+m}-\frac{1}{2m}\left(1-\frac{p_1^2}{4m^2}\right)\right]\frac{-m}{p_1^2}I_c+\frac{1}{2}I_c\frac{-m}{p_1^2}\left[\frac{E(\mathbf{p}_2)+m}{2E(\mathbf{p}_2)}\frac{1}{2E(\mathbf{p}_1)}-\frac{1}{2m}\left(1-\frac{3p_1^2}{4m^2}\right)\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{m}\\&+\frac{1}{2}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{m}I_c\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\left[\frac{1}{E(\mathbf{p}_2)+m}-\frac{1}{2m}\left(1-\frac{p_2^2}{4m^2}\right)\right]\frac{-m}{p_2^2}I_c+\frac{1}{2}I_c\frac{-m}{p_2^2}\left[\frac{E(\mathbf{p}_1)+m}{2E(\mathbf{p}_1)}\frac{1}{2E(\mathbf{p}_2)}-\frac{1}{2m}\left(1-\frac{3p_2^2}{4m^2}\right)\right]\\&\times\boldsymbol{\sigma}_2\cdot\mathbf{p}_2I_c\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{m}+\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2m}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{2m}I_c\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\left[\frac{1}{(E(\mathbf{p}_1)+m)^2}-\frac{1}{4m^2}\right]\frac{-m}{p_1^2}I_c\\&+I_c\frac{-m}{p_1^2}\left[\frac{1}{4E^2(\mathbf{p}_1)}-\frac{1}{4m^2}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{p}_2I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2m}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{2m}.\end{aligned} \quad (24)$$

Dropping off the spin-independent terms and those that give zero contribution to the fine structure, the energy correction due to the above Hamiltonian becomes

$$\Delta E_c^{(1)}=\langle\phi_0|H_c^7|\phi_0\rangle=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\left\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)\right|\frac{-1}{|\mathbf{p}_1-\mathbf{k}|^2}\left[\frac{1}{2m}\left(1-\frac{|\mathbf{p}_1-\mathbf{k}|^2}{4m^2}\right)-\frac{1}{E(\mathbf{p}_1-\mathbf{k})+m}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}$$
$$-\frac{1}{2|\mathbf{p}_1-\mathbf{k}|^2}\left[\frac{1}{m}\left(1-\frac{3|\mathbf{p}_1-\mathbf{k}|^2}{4m^2}\right)-\frac{E(\mathbf{p}_2+\mathbf{k})+m}{2E(\mathbf{p}_1-\mathbf{k})E(\mathbf{p}_2+\mathbf{k})}\right][\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}''+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')]$$
$$\times\left|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\right\rangle, \tag{25}$$

where some higher-order corrections are dropped. After nonrelativistic expansion, we arrive at

$$\Delta E_c^7=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')$$
$$\times|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{26}$$

where

$$I_{so}=\frac{1}{6}\int_0^\infty\frac{dk}{k^4}\left[\frac{4}{E_k+m}-\frac{1}{m}-\frac{1}{2E_k}\left(1+\frac{m}{E_k}\right)+\frac{5k^2}{4m^3}\right.$$
$$\left.-\frac{k^2}{2E_k^3}\left(1+\frac{2m}{E_k}\right)\right]. \tag{27}$$

This is the first-order correction to the $O(\alpha^7 mc^2)$ fine structure obtained in the Sucher's formulation [3].

The Coulomb ladder Hamiltonian of second order is given by

$$H_2=\Delta H\frac{-2m}{p_1^2+p_2^2}\left[1-g\frac{-2m}{p_1^2+p_2^2}\right]^{-1}\Delta H$$
$$=\Delta H_c^L\frac{-(m+E_{p_1})}{2p_1^2}\Delta H_c^R, \tag{28}$$

where

$$g=\frac{m-E_{p_1}}{m+E_{p_1}}\frac{p_1^2}{m}, \tag{29}$$

$$\Delta H_c^L=g+\Delta H_{SD}, \tag{30}$$

and

$$\Delta H_c^R=g+\frac{p_1^2}{m}\frac{E_{p_1}-m}{E_{p_1}}+\left(\frac{E_{p_1}-m}{2E_{p_1}}\right)^2 I_c+\Delta H_{SD}. \tag{31}$$

The spin-dependent part is given by

$$\Delta H_{SD}=2\frac{E_{p_2}+m}{2E_{p_2}}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2E_{p_1}}I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{E_{p_1}+m}$$
$$+\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2E_{p_1}}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{2E_{p_2}}I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{E_{p_1}+m}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{E_{p_2}+m}. \tag{32}$$

This Hamiltonian is of nominal order $\alpha^4 mc^2$ nonrelativistically. The lowest-order nonrelativistic correction is not subtracted from the Hamiltonian until the final calculation is done. This is different from the subtraction in the leading-order relativistic contributions. The energy correction of second order is found to be

$$\Delta E_c^{(2)}=\langle\phi_0|H_2|\phi_0\rangle$$
$$=\left\langle\phi_0\right|\frac{E_{p_1}-m}{2m}\left[2\frac{E_{p_2}+m}{2E_{p_2}}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2E_{p_1}}I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{E_{p_1}+m}\right.$$
$$\left.+\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2E_{p_1}}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{2E_{p_2}}I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{E_{p_1}+m}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{E_{p_2}+m}\right]$$
$$+\left[2\frac{E_{p_2}+m}{2E_{p_2}}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2E_{p_1}}I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{E_{p_1}+m}\right.$$
$$\left.+\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2E_{p_1}}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{2E_{p_2}}I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{E_{p_1}+m}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{E_{p_2}+m}\right]$$
$$\times\frac{-(E_{p_1}+m)}{2p_1^2}\left\{\frac{p_1^2}{E_{p_1}}\frac{E_{p_1}-m}{E_{p_1}+m}+\left(\frac{E_{p_1}-m}{2E_{p_1}}\right)^2 I_c\right.$$
$$+2\frac{E_{p_2}+m}{2E_{p_2}}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2E_{p_1}}I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{E_{p_1}+m}$$
$$\left.+\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{2E_{p_1}}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{2E_{p_2}}I_c\frac{\boldsymbol{\sigma}_1\cdot\mathbf{p}_1}{E_{p_1}+m}\frac{\boldsymbol{\sigma}_2\cdot\mathbf{p}_2}{E_{p_2}+m}\right\}\left|\phi_0\right\rangle. \tag{33}$$

The relativistic contribution comes from two Coulomb potentials while each term in the above equation contains one Coulomb potential. The second Coulomb potential is obtained by repeatedly making use of the Schrödinger equation on both the left and the right wave functions and keeping the terms of up to order $\alpha^7 mc^2$. The relativistic variables $\mathbf{k}'$, $E(\mathbf{p}_1-\mathbf{k})$, and $E(\mathbf{p}_2+\mathbf{k})$ are expanded nonrelativistically. After some manipulation, we get

$$\Delta E_c^{(2)}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')$$
$$+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{34}$$

where

$$I_{so}=\int_0^\infty dk\Bigg\{-\frac{1}{24mE_k^2}\left(\frac{1}{m^2}+\frac{2}{E_k^2}\right)+\frac{1}{3mk^2}\Bigg[\frac{1}{E_k+m}\left(\frac{1}{E_k+m}-\frac{1}{E_k}+\frac{E_k-m}{4E_k^2}\right)+\frac{5(E_k+m)}{16mE_k^2}\Bigg]-\frac{1}{8m^3k^2}\Bigg\} \tag{35}$$

and

$$I_{ss}=\int_0^\infty\left[-\frac{E_k+m}{48m^2k^2E_k^2}+\frac{1}{24m^3k^2}\right]dk. \tag{36}$$

The total contribution to the $O(\alpha^7mc^2)$ fine-structure splittings of helium arising from the Coulomb ladder equation is given by Eqs. (26) and (34). As observed, it is quite complicated to calculate the above corrections in the original times-order formalism because of difficult application of the Brillouin-Wigner perturbation to relativistic calculation. Using either our formula (1) or the Salpeter perturbation theory, the calculation is simplified significantly.

In the Salpeter perturbation theory, the no-pair Coulomb effects may be calculated by using Eq. (20), in which the third term should be divided by 2 since half of the third term comes from the no-pair diagram and the other half is due to the two-pair diagram. However, the result obtained in this way disagrees with that in the Brillouin-Wigner perturbation. The subtraction of the Breit correction of lower order needs to be treated more carefully to higher order. Only exact terms of lower order may be subtracted. The correct starting point should be from the following formula:

$$\Delta E_{++}^{C\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\times\left\langle\phi_c\left|\frac{\Lambda_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})}{E-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2+\mathbf{k})}\right|\phi_c\right\rangle, \tag{37}$$

which can be obtained from either the Salpeter perturbation theory or our reformulation equation (1). The kernel in the above equation is singular even to order α^5mc^2 due to the Breit corrections, in contrast to the finite kernel in Eq. (20). After subtraction of singular terms $-8m/k^4+2/(mk^2)$, the previously obtained energy correction of order α^5mc^2 due to the no-pair Coulomb diagram is reproduced. For the $O(\alpha^7mc^2)$ fine structure of helium, the Coulomb ladder contribution including the two-pair effects from Eq. (10) becomes

$$\Delta E^{C\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{38}$$

where

$$I_{so}=\int_0^\infty\frac{dk}{8m^2k^2}\Bigg\{(E_k+m)^2\Bigg[\frac{1}{2k^2E_k}\left(1+\frac{m}{E_k}-\frac{2mk^2}{3E_k^3}+\frac{E_k-m}{3E_k^4}-\frac{m(E_k+m)}{3k^2E_k^3}\right)\Bigg]-\frac{1}{6mk^2}+\frac{1}{24m^3}+\frac{1}{E_k+m}\left(1-\frac{m}{E_k}\right)\times\Bigg[1-\frac{m}{E_k}+\frac{k^2}{3E_k^2}\left(2+\frac{4m}{E_k}+\frac{m}{E_k+m}\right)\Bigg]\Bigg\} \tag{39}$$

and

$$I_{ss}=\int_0^\infty\frac{dk}{24m^2E_k^2}\left[\frac{1}{m}-\frac{1}{E_k+m}\right]. \tag{40}$$

The first line in the above spin-orbit correction corresponds to the no-pair diagram and agrees with the spin-orbit terms in Eqs. (26) and (34) obtained by using the Brillouin-Wigner perturbation. The second line is due to two-pair diagrams. The first term and half of the second term in the above spin-spin correction come from the no-pair diagram and agree with that in Eq. (34) obtained in Sucher's formulation of the times-order formalism. The remaining part arises from two-pair diagrams. Agreement between two quite different calculations of the no-pair Coulomb correction provides a good check. As demonstrated above, the calculation of the no-pair Coulomb effects using the Brillouin-Wigner perturbation theory is very difficult. On the other hand, the Salpeter perturbation method is more convenient and instructive for such calculation. Another interesting check is to calculate the Dirac energy of order α^7mc^2 perturbatively. It is well known that the Dirac energy for a particle bound by an external Coulomb field is a function of even powers of fine-structure constant α. In one-body approximation, the Coulomb ladder correction is

$$\Delta E^{C\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{si}\mathbf{p}_1\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{41}$$

where

$$I_{so}=\int_0^\infty\frac{dk}{8m^2E_k(E_k+m)}\left[\frac{1}{E_k+m}+\frac{2m}{3E_k}\left(\frac{1}{E_k}+\frac{1}{E_k+m}\right)\right] \tag{42}$$

and

$$I_{si}=-\int_0^\infty\frac{dk}{2k^2(E_k+m)}\Bigg[\frac{2}{3E_k(E_k+m)}\left(1-\frac{m}{E_k}\right)-\frac{1}{4m^2}\left(1-\frac{m}{E_k}\right)-\frac{2m}{3E_k^3}-\frac{k^2}{6mE_k^2(E_k+m)}-\frac{k^2}{6mE_k^3}\Bigg], \tag{43}$$

where we have retained the spin-independent terms for the Dirac energy. The above Coulomb ladder correction is found to cancel a correction arising from one-pair crossed-ladder Coulomb diagrams to be presented in the following.

The main difference between the reformulation equation (1) and the times-order Bethe-Salpeter formalism developed by Sucher [3] is the calculation of the no-pair Coulomb-Coulomb and Coulomb-transverse photon diagrams. The kernels due to pair diagrams derived from both formulations are exactly the same. In the case of Coulomb exchange, energy correction from one-pair diagrams is

$$\Delta E^{C\times C}_{-+}+\Delta E^{C\times C}_{+-}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}\frac{d\mathbf{k}'}{k'^2}\left\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)\right|-\frac{\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})}{E-E(\mathbf{p}_1)-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})}$$
$$-\frac{\Lambda_{1+}(\mathbf{p}_1-\mathbf{k}')\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})}{E-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_2)-E(\mathbf{p}_2+\mathbf{k})-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')}\left|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\right\rangle \tag{44}$$

derived by using our equation (1). The same equation was also derived in Ref. [1] using Sucher's formulation [3].

From the relativistic momentum region, the energy correction of lowest order is

$$\Delta E^{C\times C}=\alpha^5\mu^3c^2\langle\phi_0|I\delta(\mathbf{r})|\phi_0\rangle, \tag{45}$$

where

$$I=2\int_0^\infty\frac{dk}{k^2E_1E_2}\left[\frac{(E_1-m_1)(E_2+m_2)}{E_1+E_2+m_1-m_2}+\frac{(E_1+m_1)(E_2-m_2)}{E_1+E_2-m_1+m_2}\right]. \tag{46}$$

On computation, we obtain

$$\Delta E^{C\times C}=\frac{2}{3}\alpha^5\mu^3c^2\left[\frac{2}{m_1^2}-\frac{1}{m_1m_2}+\frac{2}{m_2^2}\right]\langle\phi_0|\delta(\mathbf{r})|\phi_0\rangle, \tag{47}$$

which agrees with ΔE_{Cb} of Fulton and Martin [12] for two distinct particles and that of Sucher [3] for helium.

For the helium fine structure of order α^7mc^2, using the generalized Foldy-Wouthuysen (FW) transformation [1]

$$\langle\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\rangle$$
$$=\langle\Lambda_{1+}(\mathbf{p}_1-\mathbf{k}')\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\rangle$$
$$=\frac{1}{4}\left\{-\frac{1}{2m^2}\left[1-\frac{m^2}{E_k^2}+\frac{2m^2k^2}{3E_k^4}\right]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\frac{k^2}{3m^2E_k^2}\left(1-\frac{m^2}{E_k^2}\right)i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+\frac{k^2}{6m^2E_k^2}\times\boldsymbol{\sigma}_1\cdot\mathbf{p}_1i\boldsymbol{\sigma}_2\cdot\mathbf{k}''+\frac{1}{mE_k}i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')\right\} \tag{48}$$

and expanding the denominators nonrelativistically, we obtain

$$\Delta E^{C\times C}(\alpha^7)=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{49}$$

where

$$I_{so}=-\frac{1}{8m^2}\int_0^\infty\frac{dk}{k^2E_k}\left[1-\frac{m^2}{E_k^2}+\frac{2k^2}{3E_k^2}\right] \tag{50}$$

and

$$I_{ss}=\frac{1}{24m^2}\int_0^\infty\frac{dk}{E_k^3}. \tag{51}$$

For hydrogen, the above crossed-ladder Coulomb correction becomes

$$\Delta E^{C\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{si}\mathbf{p}_1\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{52}$$

where

$$I_{so}=-\int_0^\infty\frac{dk}{8m^2E_k(E_k+m)}\left[\frac{1}{E_k+m}+\frac{2m}{3E_k}\left(\frac{1}{E_k}+\frac{1}{E_k+m}\right)\right] \tag{53}$$

and

$$I_{si}=\int_0^\infty\frac{dk}{2k^2(E_k+m)}\left[\frac{2}{3E_k(E_k+m)}\left(1-\frac{m}{E_k}\right)-\frac{1}{4m^2}\left(1-\frac{m}{E_k}\right)-\frac{2m}{3E_k^3}-\frac{k^2}{6mE_k^2(E_k+m)}-\frac{k^2}{6mE_k^3}\right], \tag{54}$$

which is seen to cancel exactly the Coulomb ladder correction in Eq. (41) term by term. Therefore, the zero Dirac energy of order α^7mc^2 for non-S states is reproduced in our perturbative calculation. The S-state correction due to relativistic origin arises from four-photon diagrams. The corresponding nonrelativistic operator is just the δ function. The reproduction of the Dirac zero energy provides a good check for our Coulomb calculation.

The total contribution to the $O(\alpha^7 mc^2)$ fine structure splittings, arising from Coulomb photon exchange, is obtained by combining corrections in Eqs. (38) and (49). On computation, we get

$$\Delta E_c = \left(\frac{\alpha}{2\pi^2}\right)^2 4\pi \int d\mathbf{k}'' \langle \phi_0(\mathbf{p}_1,\mathbf{p}_2)| I_{so} i \boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 \times \mathbf{k}'') + I_{ss} \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \boldsymbol{\sigma}_1 \cdot \mathbf{k}'' | \phi_0(\mathbf{p}_1 - \mathbf{k}'', \mathbf{p}_2 + \mathbf{k}'')\rangle, \tag{55}$$

where

$$I_{so} = -\frac{\pi}{32} + \frac{1}{8} \tag{56}$$

and

$$I_{ss} = \tfrac{1}{12}. \tag{57}$$

The above finite result is consistent with the fact that no contribution arises from the nonrelativistic momentum region since the appearance of logarithmic cutoff terms indicates that contributions come from both relativistic and nonrelativistic momentum regions. Upon Fourier transformation described in our previous paper [1], we obtain

$$\Delta E_c = \alpha^7 mc^2 \left\langle \phi_0 \middle| I_{so} \delta(\mathbf{r}) \boldsymbol{\sigma}_1 \cdot \left(\frac{\mathbf{r}}{r^2} \times \mathbf{p}_1\right) + I_{ss} \delta(\mathbf{r}) \frac{1}{r^2} \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} \middle| \phi_0 \right\rangle, \tag{58}$$

where

$$I_{so} = -\frac{3\pi}{4} + 3 \tag{59}$$

and

$$I_{ss} = -5. \tag{60}$$

Here we have used the following formulas:

$$\int d\mathbf{k} i \boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 \times \mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r}) = 4\pi \boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 \times \mathbf{r}) ks(3,1) = 24\pi^3 \delta(\mathbf{r}) \frac{1}{r^2} \boldsymbol{\sigma}_1 \cdot (\mathbf{r} \times \mathbf{p}_1) \tag{61}$$

and

$$\begin{aligned} \int d\mathbf{k} \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r}) &= 2\int d\mathbf{k} \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \boldsymbol{\sigma}_2 \cdot \mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r}) \\ &= 4\pi \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} [ks(3,1) - ks(4,2)] \\ &= -120\pi^3 \delta(\mathbf{r}) \frac{1}{r^2} \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}. \end{aligned} \tag{62}$$

In this section, we presented calculation of the no-pair Coulomb effects in three different methods, namely, Sucher's times-order approach, the Salpeter perturbation method, and a direct application of the S-matrix method. The last two are the same for Coulomb exchange. The first one is quite complicated in relativistic approximation. Calculation of pair effects is not different from one to another.

III. SINGLE TRANSVERSE PHOTON EXCHANGE

Relativistic energy corrections due to the single transverse photon exchange arise from no-pair, one-pair, and two-pair diagrams. Formulas in closed form were derived on the basis of the times-order Bethe-Salpeter formalism and presented in our previous paper [1]. Here, we rederive them starting from our Eq. (1). They all come from a transverse photon exchange plus a Coulomb photon.

A. No pair

In order to compare the Brillouin-Wigner perturbation theory with the Salpeter perturbation method, we recalculate the $O(\alpha^5 mc^2)$ relativistic energy levels of helium using both methods. In the Brillouin-Wigner perturbation method, energy corrections due to the no-pair single transverse photon exchange arising from the relativistic momentum region come from the Breit corrections of first order and second order as well as recoil corrections. For two distinct particles, the correction arising from the relativistic Breit operators is given by

$$\begin{aligned} \Delta E_B^5 = 2\left\langle \phi_0 \middle| B \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \boldsymbol{\sigma}_2 \cdot \mathbf{p}_2 \left[\frac{1}{\sqrt{2E_{p_1}(E_{p_1}+m_1)}} \times \frac{1}{\sqrt{2E_{p_1}(E_{p_1}+m_1)}} - \frac{1}{4m_1 m_2} \right] \middle| \phi_0 \right\rangle \\ + 2\left\langle \phi_0 \middle| B \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \boldsymbol{\sigma}_2 \cdot \mathbf{p}_2 \frac{1}{\sqrt{2E_{p_1}(E_{p_1}+m_1)}} \times \frac{1}{\sqrt{2E_{p_1}(E_{p_1}+m_1)}} \middle| \phi_1 \right\rangle, \end{aligned} \tag{63}$$

where the perturbed wave function of first order is

$$\phi_1=2\mu\left[1-\frac{E_{p_1}-m_1}{E_{p_1}+m_1}\frac{\mu}{m_1}-\frac{E_{p_2}-m_2}{E_{p_2}+m_2}\frac{\mu}{m_2}\right]^{-1}\times\left[-\frac{\mu}{m_1}\frac{1}{(E_{p_1}+m_1)^2}+\frac{1}{2E_{p_1}(E_{p_1}+m_1)}-\frac{\mu}{m_2}\frac{1}{(E_{p_2}+m_2)^2}+\frac{1}{2E_{p_2}(E_{p_2}+m_2)}-\frac{p_1^2}{2E_{p_1}(E_{p_1}+m_1)}\frac{1}{2E_{p_2}(E_{p_2}+m_2)}\right]I_c\phi_0 \quad (64)$$

and the transverse integral operator B is defined as

$$B\phi(\mathbf{p}_1,\mathbf{p}_2)=\frac{\alpha}{2\pi^2}\int\frac{d\mathbf{k}}{-k^2}\sigma_1^i\sigma_2^i\phi(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k}). \quad (65)$$

After nonrelativistic expansion of the denominators and some manipulation, we obtain

$$\Delta E_B^5=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\times|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \quad (66)$$

where

$$I=\int_0^\infty\frac{dk}{k^2}\left\{\frac{8\mu}{3}\left[\frac{1}{E_1+m_1}+\frac{1}{E_2+m_2}-\frac{1}{4m_1m_2}\right]-\frac{4k^2}{3}\frac{1}{E_1+m_1}\frac{1}{E_2+m_2}f\right\}. \quad (67)$$

Here

$$f=2\mu\left[1-\frac{E_1-m_1}{E_1+m_1}\frac{\mu}{m_1}-\frac{E_2-m_2}{E_2+m_2}\frac{\mu}{m_2}\right]^{-1}\left[-\frac{\mu}{m_1}\frac{1}{(E_1+m_1)^2}+\frac{1}{2E_1(E_1+m_1)}-\frac{\mu}{m_2}\frac{1}{(E_2+m_2)^2}+\frac{1}{2E_2(E_2+m_2)}-\frac{k^2}{2E_1(E_1+m_1)}\frac{1}{2E_2(E_2+m_2)}\right]. \quad (68)$$

For $m_1=m_2=m$, we get

$$\Delta E_B^5=-\frac{2\alpha^5mc^2}{3}(\pi+2)\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle, \quad (69)$$

which reproduces Sucher's result.

Recoil corrections are separated from the Breit corrections explicitly in terms of the numbers of photons. The Breit corrections correspond to pure single transverse photon exchanged. In order to extract relativistic contributions, two photons are required. A Coulomb photon comes in either from repeatedly applying the Schrödinger equation on both the left and the right wave functions or from the wave function perturbation. The relativistic recoil corrections arise explicitly from two-photon exchange. Therefore, no relativistic recoil corrections of second order contribute to the order of interest due to the perturbation of the wave function. One of the corrections due to recoil effects arises from ladder diagrams and is given by

$$\Delta E_{RI}^0=\frac{\alpha}{2\pi^2}\int\frac{d\mathbf{k}}{2k^2}\left\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)\left|\left[\alpha_1^i\frac{D_c^{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)}{D_c^{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)-k}\alpha_2^i+\alpha_2^i\frac{D_c^{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})}{D_c^{++}(\mathbf{p}_1,\mathbf{p}_2-\mathbf{k})-k}\alpha_1^i\right]\right.\right.$$
$$\times\frac{1}{D_c^{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})}\mathcal{L}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})I_c\mathcal{L}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})+\mathcal{L}_{++}(\mathbf{p}_1,\mathbf{p}_2)I_c\mathcal{L}_{++}(\mathbf{p}_1,\mathbf{p}_2)\frac{1}{D_c^{++}(\mathbf{p}_1,\mathbf{p}_2)}$$
$$\left.\left.\times\left[\alpha_1^i\frac{D_c^{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)}{D_c^{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)-k}\alpha_2^i+\alpha_2^i\frac{D_c^{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})}{D_c^{++}(\mathbf{p}_1,\mathbf{p}_2-\mathbf{k})-k}\alpha_1^i\right]\right|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\right\rangle. \quad (70)$$

After nonrelativistic reduction to lowest order, it becomes

$$\Delta E_{RI}^0=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \quad (71)$$

where

$$I=\int_0^\infty\frac{dk}{6E_1E_2}\frac{1}{m_1+m_2-E_1-E_2}\left[\frac{m_1-E_1}{m_1-E_1-k}+\frac{m_2-E_2}{m_2-E_2-k}\right]. \quad (72)$$

Another recoil correction due to crossed-ladder diagrams is given by

$$\Delta E_{RI}^{1}=\frac{\alpha}{2\pi^2}\int\frac{d\mathbf{k}}{2k}\left\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)\right|\alpha_1^i\frac{1}{E-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2)-k}\mathcal{L}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)I_c\mathcal{L}_{++}(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2)\frac{1}{E-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2)-k}\alpha_2^i$$
$$+\alpha_2^i\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}\mathcal{L}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})I_c\mathcal{L}_{++}(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k})\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}\alpha_1^i\left|\phi_c(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2+\mathbf{k})\right\rangle. \tag{73}$$

Reduction yields

$$\Delta E_{RI}^{1}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{74}$$

where

$$I=\int_0^\infty\frac{kdk}{6E_1E_2}\frac{1}{E_1+k-m_1}\frac{1}{E_2+k-m_2}. \tag{75}$$

Combining the two recoil corrections, we get

$$\Delta E_{RI}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{76}$$

where

$$I=\int_0^\infty\frac{dk}{6E_1E_2}\left[\frac{1}{m_1+m_2-E_1-E_2}\left(\frac{E_1-m_1}{E_1+k-m_1}+\frac{E_2-m_2}{E_2+k-m_2}\right)+\frac{k}{E_1+k-m_1}\frac{1}{E_2+k-m_2}\right]. \tag{77}$$

For $m_1=m_2$, the correction becomes

$$\Delta E_{RI}=-\frac{2\alpha^5mc^2}{3}\ln 2\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle, \tag{78}$$

which reproduces Sucher's result. In the above calculation, it is quite difficult to track all the relevant terms in the Brillouin-Wigner perturbation expansion. This makes it much more complicated to calculate the $O(\alpha^7mc^2)$ fine structure. The detailed calculation of the fine structure using this method will not be presented here. In the following we use our formulation of the times-order theory.

Similar to Coulomb ladder corrections, the ladder correction due to no-pair single transverse photon exchange is calculated more conveniently using the formulation equation (1). The corresponding formula becomes

$$\Delta E_{++}^{T\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i\frac{\mathcal{L}_{1+}(\mathbf{p}_1-\mathbf{k})}{\mu_1E-\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega+i\delta}\alpha_2^i$$
$$\times\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k})}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k})-\epsilon+\omega+i\delta}\psi(p'_{1\mu}p'_{2\mu})+\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})$$
$$\times\frac{\mathcal{L}_{1+}(\mathbf{p}_1-\mathbf{k}')}{\mu_1E-\varepsilon_1(\mathbf{p}_1-\mathbf{k}')+\epsilon-\omega'+i\delta}\alpha_1^i\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k}')}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k}')-\epsilon+\omega'+i\delta}\alpha_2^i\psi(p'_{1\mu}p'_{2\mu}). \tag{79}$$

Performing integration over the energy variables and dropping the external potentials, it becomes

$$\Delta E_{++}^{T\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{k'^2}\left\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)\right|\frac{1}{E-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2+\mathbf{k})}\left[\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}\right.$$
$$\left.+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2)-k}\right]\alpha_1^i\Lambda_{1+}(\mathbf{p}_1-\mathbf{k})\alpha_2^i\Lambda_{1+}(\mathbf{p}_2+\mathbf{k})$$
$$+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k}')}\left[\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k}+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k}')-k}\right]$$
$$\times\Lambda_{1+}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\Lambda_{1+}(\mathbf{p}_2+\mathbf{k}')\alpha_2^i\left|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\right\rangle. \tag{80}$$

To lowest order $\alpha^5 mc^2$, the above contribution to helium energy levels is

$$\Delta E_{++}^{T\cdot C} = -\frac{4\alpha^5 m^3 c^2}{3}\int_0^\infty \frac{dk[2E_k^2 + k(2E_k+m)]}{mE_k^2(E_k+m)(E_k+k-m)}\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle, \tag{81}$$

which is the sum of ΔE_B^5 and ΔE_{RI}^0 in Sucher's calculation [3]. This calculation is much simpler and more instructive, especially for the $O(\alpha^7 mc^2)$ fine structure.

The recoil correction ΔE_{RI}^1 in Sucher's times-order formalism corresponds to crossed-ladder diagrams and can be derived from Eq. (1). The corresponding times-order formula becomes

$$\Delta E_{++}^{T\times C} = \left(\frac{\alpha}{2\pi^2}\right)^2 \frac{1}{-2\pi i}\int \frac{d^4p_1 d\mathbf{p}_2 d^4k d^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i \frac{\mathcal{L}_{1+}(\mathbf{p}_1-\mathbf{k})}{\mu_1 E-\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega+i\delta}2$$
$$\times \frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k}')}{\mu_2 E-\varepsilon_2(\mathbf{p}_2+\mathbf{k}')-\epsilon+\omega'+i\delta}\alpha_2^i\psi(p'_{1\mu}p'_{2\mu}) + \left(\frac{\alpha}{2\pi^2}\right)^2 \frac{1}{-2\pi i}\int \frac{d^4p_1 d\mathbf{p}_2 d^4k d^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})$$
$$\times \frac{\mathcal{L}_{1+}(\mathbf{p}_1-\mathbf{k}')}{\mu_1 E-\varepsilon_1(\mathbf{p}_1-\mathbf{k}')+\epsilon-\omega'+i\delta}\alpha_1^i\alpha_2^i \frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k})}{\mu_2 E-\varepsilon_2(\mathbf{p}_2+\mathbf{k})-\epsilon+\omega+i\delta}\psi(p'_{1\mu}p'_{2\mu}). \tag{82}$$

Performing integration over the energy variables and neglecting the external potentials lead to

$$\Delta E_{++}^{T\times C} = \left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{k'^2}\left\langle \phi_c(\mathbf{p}_1,\mathbf{p}_2)\right| \frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}\,\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k}$$
$$\times \Lambda_{1+}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k}) + \frac{1}{E-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2)-k}\,\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k}')-k}$$
$$\times \alpha_1^i\Lambda_{1+}(\mathbf{p}_1-\mathbf{k})\Lambda_{2+}(\mathbf{p}_2+\mathbf{k}')\alpha_2^i\left|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\right\rangle. \tag{83}$$

To lowest order $\alpha^5 mc^2$, it becomes

$$\Delta E_{++}^{T\times C} = \frac{4\alpha^5 m^3 c^2}{3}\int_0^\infty \frac{kdk}{E_k^2(E_k+k-m)^2}\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle, \tag{84}$$

which reproduces ΔE_{RI}^1 The total correction due to no-pair single transverse photon exchange is given by Eqs. (81) and (84) and becomes

$$\Delta E_{++}^{T\times C} + \Delta E_{++}^{T\cdot C} = \frac{4\alpha^5 mc^2}{3}\left(-1-\frac{1}{2}\ln 2-\frac{\pi}{2}\right)\times\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle = \Delta E_B^5 + \Delta E_{RI}, \tag{85}$$

which reproduces that of Sucher [3] arising from the no-pair single transverse photon diagrams.

For the helium fine structure of order $\alpha^7 mc^2$, the relativistic contribution due to the no-pair crossed-ladder single transverse photon exchange is derived from Eq. (83) by expanding the denominators nonrelativistically and using the FW transformation,

$$\langle\Lambda_{1+}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\rangle$$
$$=\frac{1}{16}\left\{2\left(\frac{1}{m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k}')}\right)(p_1^i-k''^i)+\left(\frac{1}{m}-\frac{1}{E(\mathbf{p}_1-\mathbf{k}')}\right)\boldsymbol{\sigma}_1\cdot\mathbf{k}''\sigma_1^i+\frac{2}{E(\mathbf{p}_1-\mathbf{k}')}\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\right.$$
$$\left.+\frac{1}{2m^2E(\mathbf{p}_1-\mathbf{k}')}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')\right\}\left\{2\left(\frac{1}{m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})}\right)p_2^i+\left(\frac{1}{m}-\frac{1}{E(\mathbf{p}_2+\mathbf{k})}\right)\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}''\right.$$
$$\left.+\frac{2}{E(\mathbf{p}_2+\mathbf{k})}\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}+\frac{1}{2m^2E(\mathbf{p}_2+\mathbf{k})}\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')\right\}$$
$$=J_2^0+J_1^1+J_3^1+J_0^2+J_2^2+J_4^2, \tag{86}$$

where

$$J_2^0=-\frac{1}{4E_k^2}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k},$$

$$J_1^1=\frac{1}{4E_k}\left(\frac{1}{m}+\frac{1}{E_k}\right)i\boldsymbol{\sigma}_1\cdot[(2\mathbf{p}_1-\mathbf{k}'')\times\mathbf{k}]+\frac{1}{4E_k}\left(\frac{1}{E_k}-\frac{1}{m}\right)\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}'',$$

$$J_3^1=-\frac{1}{4E_k^4}\mathbf{k}''\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k},$$

$$J_0^2=\frac{1}{6}\left(\frac{1}{m^2}-\frac{1}{E_k^2}+\frac{k^2}{2m^2E_k^2}\right)i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\frac{1}{12}\left[\left(\frac{1}{m}-\frac{1}{E_k}\right)^2+\frac{2k^2}{m^2E_k^2}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'',$$

$$J_2^2=\frac{k^2}{12E_k^3}\left[\left(\frac{1}{m}+\frac{2}{E_k}\right)i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\left(\frac{1}{m}-\frac{2}{E_k}\right)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''\right],$$

$$J_4^2=-\frac{k^4}{30E_k^6}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''. \tag{87}$$

In deriving the above result, one needs to keep all terms containing up to four **k**'s and care is required. After some manipulation, we obtain

$$\Delta E_{++}^{T\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int_0^\infty dk\times\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{88}$$

where

$$I_{so}=\frac{1}{6k(E_k+k-m)^2}\left\{\left[\frac{1}{m^2}-\frac{1}{E_k^2}+\frac{k^2}{2m^2E_k^2}\right]+\frac{k^2}{2E_k^3}\left(\frac{1}{m}+\frac{2}{E_k}\right)+\frac{1}{E_k}\left(\frac{1}{E_k}+\frac{1}{m}\right)\times\left[2+\frac{k^2}{E_k(E_k+k-m)}\right]\right\} \tag{89}$$

and

$$I_{ss}=\frac{1}{6k(E_k+k-m)^2}\left\{-\frac{1}{2}\left(\frac{1}{E_k}-\frac{1}{m}\right)^2-\frac{k^2}{m^2E_k^2}+\frac{1}{E_k}\left[2+\frac{k^2}{E_k(E_k+k-m)}\right]\left[\frac{1}{E_k}-\frac{1}{m}-\frac{2k^2}{5E_k^3}\right]-\frac{k^2}{2E_k^3}\left(\frac{1}{m}-\frac{2}{E_k}\right)-\frac{k^4}{5E_k^6}-\frac{1}{5E_k^3}\left[8+\frac{4k^2}{E_k(E_k+k-m)}+\frac{k^4}{E_k^2(E_k+k-m)^2}\right]\right\}. \tag{90}$$

The correction due to no-pair ladder single transverse photon exchange is derived from Eq. (80) and given by

$$\Delta E_{++}^{T\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int_0^\infty dk\times\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{91}$$

where

$$I_{so}=\frac{1}{6k(E_k-m)(E_k+k-m)}\left\{\left[\frac{1}{m^2}-\frac{1}{E_k^2}+\frac{k^2}{2m^2E_k^2}\right]+\frac{k^2}{2E_k^3}\left(\frac{1}{m}-\frac{2}{E_k}\right)+\frac{2}{E_k}\left(\frac{1}{E_k}+\frac{1}{m}\right)+\frac{k^2}{2E_k^2}\left(\frac{1}{m}-\frac{1}{E_k}\right)\times\left(\frac{1}{E_k-m}+\frac{1}{E_k+k-m}\right)\right\} \tag{92}$$

and

$$I_{ss}=\frac{1}{6k(E_k-m)(E_k+k-m)}\Bigg\{-\frac{1}{2}\left(\frac{1}{E_k}-\frac{1}{m}\right)^2-\frac{k^2}{m^2E_k^2}-\frac{k^2}{2E_k^3}\left(\frac{1}{m}-\frac{2}{E_k}\right)-\frac{2}{E_k}\left(\frac{1}{m}-\frac{1}{E_k}\right)-\frac{k^2}{2E_k^2}\left(\frac{1}{m}-\frac{1}{E_k}\right)\times\left(\frac{1}{E_k-m}+\frac{1}{E_k+k-m}\right)-\frac{4k^2}{5E_k^4}-\frac{1}{15E_k^2}\left[4+\frac{k^2}{E_k}\left(\frac{1}{E_k-m}+\frac{1}{E_k+k-m}\right)\right]\Bigg\}. \quad (93)$$

The total contribution due to the no-pair single transverse photon exchange is obtained by adding the corrections in Eqs. (88) and (91). Upon calculation, we get

$$\Delta E^T_{++++}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \quad (94)$$

where

$$I_{so}=\frac{5}{72}-\frac{1}{6}\ln B+\frac{1}{24}\ln 2+\frac{1}{48}\pi \quad (95)$$

and

$$I_{ss}=\frac{1}{40}\ln 2+\frac{69}{960}\pi+\frac{17}{240}+\frac{1}{30}\ln B, \quad (96)$$

where B is the cutoff. The logarithmic cutoff is supposed to cancel that which arises from nonrelativistic contributions. Taking Fourier transform yields

$$\Delta E^T_{++++}=\alpha^7mc^2\left\langle\phi_0\middle|I_{so}\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\left(\frac{\mathbf{r}}{r^2}\times\mathbf{p}_1\right)+I_{ss}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\middle|\phi_0\right\rangle, \quad (97)$$

where

$$I_{so}=\tfrac{5}{3}-4\ln B+\ln 2+\tfrac{1}{2}\pi \quad (98)$$

and

$$I_{ss}=-\tfrac{3}{2}\ln 2-\tfrac{69}{16}\pi-\tfrac{17}{4}-2\ln B. \quad (99)$$

At first glance, the logarithmic cutoff terms do not seem to cancel those in Eqs. (307)–(309) in Ref. [1], arising from no-pair single transverse photon exchange in nonrelativistic approximation. A careful examination shows that they cancel the logarithmic singular terms due to the no-pair single transverse photon exchange plus a Coulomb photon or the divergent terms in Eqs. (78) and (93) in Ref. [1]. This is understood because the cancellation takes place between relativistic and nonrelativistic contributions arising from the same no-pair diagrams of one transverse and one Coulomb photons. The other part of no-pair single transverse photon exchange in Eq. (94) in Ref. [1] comes from the correction of another transverse photon and is a pure transverse photon correction. We will show that the singular terms in this correction cancel those in ladder double transverse photon exchange in relativistic approximation.

B. One pair

Relativistic energy corrections arising from one-pair diagrams are due to a transverse photon plus a Coulomb photon exchanged. The formula derived from our reformulation equation (1) for calculation of one-pair single transverse photon contribution is

$$\Delta E^{T\times C}_{-+}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k})}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega-i\delta}\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k}')}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k}')-\epsilon+\omega'+i\delta}\times\alpha_2^i\psi(p'_{1\mu}p'_{2\mu})+\left(\frac{\alpha}{2\pi^2}\right)\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k}')}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k}')+\epsilon-\omega'-i\delta}\times\alpha_1^i\alpha_2^i\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k})}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k})-\epsilon+\omega+i\delta}\psi(p'_{1\mu}p'_{2\mu}). \quad (100)$$

Performing integration over the energy variables and neglecting the external potentials lead to

$$\Delta E^{T\times C}_{-+}=\left(\frac{\alpha^2}{2\pi}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})I+\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\Lambda_{2+}(\mathbf{p}_2+\mathbf{k}')\alpha_2^iI'\times|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle, \quad (101)$$

where

$$I=\frac{-1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})}\left[\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}+\frac{-1}{E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k})+k}\right] \tag{102}$$

and

$$I'=\frac{-1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2+\mathbf{k}')}\left[\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k}')-k}+\frac{-1}{E(\mathbf{p}_1)+E(\mathbf{p}_2+\mathbf{k}')+k}\right]. \tag{103}$$

The above correction agrees with that derived in Ref. [1] using Sucher's formulation. To lowest order, the crossed-ladder correction becomes

$$\Delta E_{-+}^{T\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{104}$$

where

$$I=\frac{1}{6}\int_0^\infty\frac{kdk}{E_1E_2}\,\frac{1}{E_1+E_2+m_1-m_2}\left[\frac{1}{E_2+k-m_2}+\frac{1}{E_1+k+m_1}\right]. \tag{105}$$

The energy correction due to ladder diagrams is derived from the reformulation equation (1). The relevant times-order formula is

$$\Delta E_{-+}^{T\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k})}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega-i\delta}$$

$$\times\alpha_2^i\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k})}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k})-\epsilon+\omega+i\delta}\psi(p'_{1\mu}p'_{2\mu})+\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})$$

$$\times\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k}')}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k}')+\epsilon-\omega'-i\delta}\alpha_1^i\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k}')}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k}')-\epsilon+\omega'+i\delta}\alpha_2^i\psi(p'_{1\mu}p'_{2\mu}). \tag{106}$$

Performing integration over the energy variables and dropping the external potentials give

$$\Delta E_{-+}^{T\cdot C}=\left(\frac{\alpha^2}{2\pi}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})I$$
$$+\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k}')\alpha_2^iI'|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle, \tag{107}$$

where

$$I=\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k})+k}\,\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k} \tag{108}$$

and

$$I'=\frac{1}{E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+E(\mathbf{p}_1-\mathbf{k}')+k}\times\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})-k}. \tag{109}$$

To lowest order, it becomes

$$\Delta E_{-+}^{T\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)\times|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{110}$$

where

$$I=\frac{1}{6}\int_0^\infty\frac{kdk}{E_1E_2}\,\frac{1}{E_1-E_2+m_1+m_2}\times\left[\frac{1}{E_2+k-m_2}-\frac{1}{E_1+k+m_1}\right]. \tag{111}$$

Including the correction arising from the single pair diagrams in which a pair is on the second fermion line, we get

$$\Delta E^{TC}_{-+}+\Delta E^{TC}_{+-}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \times|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{112}$$

where

$$I=\frac{1}{6}\int_0^\infty\frac{kdk}{E_1E_2}\left\{\frac{1}{E_1-E_2+m_1+m_2}\times\left[\frac{1}{E_2+k-m_2}-\frac{1}{E_1+k+m_1}\right]+\frac{1}{E_1+E_2+m_1-m_2}\times\left[\frac{1}{E_2+k-m_2}+\frac{1}{E_1+k+m_1}\right]+\frac{1}{E_2-E_1+m_1+m_2}\times\left[\frac{1}{E_1+k-m_1}-\frac{1}{E_2+k+m_2}\right]+\frac{1}{E_1+E_2+m_2-m_1}\times\left[\frac{1}{E_1+k-m_1}+\frac{1}{E_2+k+m_2}\right]\right\}. \tag{113}$$

For helium $m_1=m_2=m$, we obtain

$$\Delta E^{TC}_{-+}+\Delta E^{TC}_{+-}=\frac{4\alpha^5mc^2}{3}(1+\ln 2)\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle, \tag{114}$$

which reproduces Sucher's result.

For the helium fine structure of order α^7mc^2, we expand nonrelativistically the spin dependent numerators and get

$$\begin{aligned}\langle\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2\pm}(\mathbf{p}_2+\mathbf{k})\rangle=&\frac{1}{16}\left\{2\left(\frac{1}{m}-\frac{1}{E(\mathbf{p}_1-\mathbf{k}')}\right)(p_1^i-k''^i)+\left(\frac{1}{m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k}')}\right)\boldsymbol{\sigma}_1\cdot\mathbf{k}''\sigma_1^i-\frac{2}{E(\mathbf{p}_1-\mathbf{k}')}\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\right.\\&\left.-\frac{1}{2m^2E(\mathbf{p}_1-\mathbf{k}')}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')\right\}\\&\times\left\{2\left(\frac{1}{m}\pm\frac{1}{E(\mathbf{p}_2+\mathbf{k})}\right)p_2^i+\left(\frac{1}{m}\mp\frac{1}{E(\mathbf{p}_2+\mathbf{k})}\right)\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}''\pm\frac{2}{E(\mathbf{p}_2+\mathbf{k})}\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}\right.\\&\left.\pm\frac{1}{2m^2E(\mathbf{p}_2+\mathbf{k})}\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')\right\}\\=&J_2^0+J_1^1+J_3^1+J_0^2+J_2^2+J_4^2,\end{aligned} \tag{115}$$

where

$$J_2^0=\pm\frac{1}{4E_k^2}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k},$$

$$J_1^1=\frac{1}{8E_k}\left\{\pm2\left(\frac{1}{m}-\frac{1}{E_k}\right)i\boldsymbol{\sigma}_1\cdot[(\mathbf{p}_1-\mathbf{k}'')\times\mathbf{k}]+2\left(\frac{1}{m}\pm\frac{1}{E_k}\right)i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_2\times\mathbf{k})\mp\frac{2}{E_k}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}''+\frac{1}{m}(1\pm1)i\boldsymbol{\sigma}_1\cdot(\mathbf{k}''\times\mathbf{k})+\frac{1}{m}(1\mp1)\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}''\right\},$$

$$J_3^1=\mp\frac{1}{4E_k^4}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k}'')\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k},$$

$$J_0^2=\frac{1}{12}\left[\left(\frac{1}{m}-\frac{1}{E_k}\right)\left(\frac{1}{m}\mp\frac{1}{E_k}\right)+\left(\frac{1}{m}+\frac{1}{E_k}\right)\left(\frac{1}{m}\pm\frac{1}{E_k}\right)\mp\frac{k^2}{m^2E_k^2}\right]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+\frac{1}{12}\left[-\left(\frac{1}{m}+\frac{1}{E_k}\right)\left(\frac{1}{m}\mp\frac{1}{E_k}\right)\pm\frac{2k^2}{m^2E_k^2}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'',$$

$$J_2^2=\frac{k^2}{24E_k^3}\left[-\frac{1}{m}(1\mp1)\mp\frac{4}{E_k}\right]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+\frac{k^2}{24E_k^3}\left[\frac{1}{m}(1\mp1)\mp\frac{4}{E_k}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'',$$

$$J_4^2=\pm\frac{k^4}{30E_k^6}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''. \tag{116}$$

Similarly, the FW transformation for the other one- and two-pair crossed-ladder numerators is given by

$$\langle\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\Lambda_{2\pm}(\mathbf{p}_2+\mathbf{k}')\alpha_2^i\rangle=\frac{1}{16}\left\{2\left(\frac{1}{m}-\frac{1}{E(\mathbf{p}_1-\mathbf{k})}\right)p_1^i-\left(\frac{1}{m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k})}\right)\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}''+\frac{2}{E(\mathbf{p}_1-\mathbf{k})}\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}\right.$$
$$\left.+\frac{1}{2m^2E(\mathbf{p}_1-\mathbf{k})}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')\right\}\left\{2\left(\frac{1}{m}\pm\frac{1}{E(\mathbf{p}_2+\mathbf{k}')}\right)(p_2^i+k''^i)\right.$$
$$-\left(\frac{1}{m}\mp\frac{1}{E(\mathbf{p}_2+\mathbf{k}')}\right)\boldsymbol{\sigma}_2\cdot\mathbf{k}''\sigma_2^i\mp\frac{2}{E(\mathbf{p}_2+\mathbf{k}')}\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}\mp\frac{1}{2m^2E(\mathbf{p}_2+\mathbf{k}')}$$
$$\left.\times\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')\right\}$$
$$=L_2^0+L_1^1+L_3^1+L_0^2+L_2^2+L_4^2, \tag{117}$$

where

$$L_2^0=\pm\frac{1}{4E_k^2}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k},$$

$$L_1^1=\frac{1}{8E_k}\left\{-\left[\frac{1}{m}(1\mp1)\pm\frac{2}{E_k}\right]i\boldsymbol{\sigma}_1\cdot[(2\mathbf{p}_1-\mathbf{k}'')\times\mathbf{k}]\right.$$
$$\left.+\left[\frac{1}{m}(1\mp1)\mp\frac{2}{E_k}\right]\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}''\right\},$$

$$L_3^1=\pm\frac{1}{4E_k^4}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{k}'')\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}, \tag{118}$$

$$L_0^2=\frac{1}{12}\left[\left(\frac{1}{m}-\frac{1}{E_k}\right)\left(\frac{1}{m}\mp\frac{1}{E_k}\right)\right.$$
$$\left.+\left(\frac{1}{m}+\frac{1}{E_k}\right)\left(\frac{1}{m}\pm\frac{1}{E_k}\right)\mp\frac{k^2}{m^2E_k^2}\right]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')$$
$$+\frac{1}{12}\left[-\left(\frac{1}{m}+\frac{1}{E_k}\right)\left(\frac{1}{m}\mp\frac{1}{E_k}\right)\pm\frac{2k^2}{m^2E_k^2}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'',$$

$$L_2^2=\frac{k^2}{24E_k^3}\left[-\frac{1}{m}(1\mp1)\mp\frac{4}{E_k}\right]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')$$
$$+\frac{k^2}{24E_k^3}\left[\frac{1}{m}(1\mp1)\mp\frac{4}{E_k}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'',$$

$$L_4^2=\pm\frac{k^4}{30E_k^6}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''.$$

Using the above results and expanding the denominators nonrelativistically, we derive

$$\Delta E_{-+}^{T\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi\int_0^\infty dk\int d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|$$
$$\times I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''$$
$$\times|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{119}$$

where

$$I_{so}=-\frac{1}{12kE_k}\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]$$
$$\times\left[\frac{1}{m^2}+\frac{1}{E_k^2}-\frac{k^2}{2m^2E_k^2}-\frac{3k^2}{2E_k^4}\right]$$
$$+\frac{k}{24E_k^4}\left[\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right]$$
$$+\frac{1}{6kE_k^3}\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right] \tag{120}$$

and

$$I_{ss}=\frac{1}{12kE_k}\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]\left[\frac{1}{2}\left(\frac{1}{m^2}-\frac{1}{E_k^2}\right)\right.$$
$$\left.-\frac{k^2}{E_k^2}\left(\frac{1}{m^2}-\frac{3}{2E_k^2}+\frac{k^2}{2E_k^4}\right)\right]+\frac{k}{24E_k^4}\left[\frac{1}{(E_k+k+m)^2}\right.$$
$$\left.+\frac{1}{(E_k+k-m)^2}\right]\left[1-\frac{3k^2}{5E_k^2}\right]-\frac{1}{30k^2m^3}$$
$$-\frac{k}{30E_k^4}\left[\frac{1}{(E_k+k+m)^2}+\frac{1}{(E_k+k-m)^2}\right.$$
$$\left.+\frac{1}{E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\right]$$
$$+\frac{1}{6kE_k^3}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\left[1-\frac{2k^2}{5E_k^2}\right]$$
$$-\frac{2}{15kE_k^3}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right). \tag{121}$$

The $O(\alpha^7mc^2)$ correction due to ladder diagrams is obtained by expanding the numerators and denominators in Eq. (107) nonrelativistically. Upon taking the FW transformation, the numerators become

$$\langle \Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i \Lambda_{2\pm}(\mathbf{p}_2+\mathbf{k}')\alpha_2^i\rangle = \frac{1}{16}\left\{2\left(\frac{1}{m}-\frac{1}{E(\mathbf{p}_1-\mathbf{k}')}\right)(p_1^i-k''^i)+\left(\frac{1}{m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k}')}\right)\boldsymbol{\sigma}_1\cdot\mathbf{k}''\sigma_1^i-\frac{2}{E(\mathbf{p}_1-\mathbf{k}')}\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\right.$$
$$\left.-\frac{1}{2m^2E(\mathbf{p}_1-\mathbf{k}')}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')\right\}\left\{2\left(\frac{1}{m}\pm\frac{1}{E(\mathbf{p}_2+\mathbf{k}')}\right)(p_2^i+k''^i)\right.$$
$$-\left(\frac{1}{m}\mp\frac{1}{E(\mathbf{p}_2+\mathbf{k}')}\right)\boldsymbol{\sigma}_2\cdot\mathbf{k}''\sigma_2^i\mp\frac{2}{E(\mathbf{p}_2+\mathbf{k}')}\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i$$
$$\left.\mp\frac{1}{2m^2E(\mathbf{p}_2+\mathbf{k}')}\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')\right\}$$
$$=J_2^0+J_1^1+J_3^1+J_0^2+J_2^2+J_4^2, \tag{122}$$

where

$$J_2^0=\pm\frac{1}{4E_k^2}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k},$$

$$J_1^1=\frac{1}{8E_k}\left\{\pm2\left(\frac{1}{m}-\frac{1}{E_k}\right)i\boldsymbol{\sigma}_1\cdot[(\mathbf{p}_1-\mathbf{k}'')\times\mathbf{k}]-2\left(\frac{1}{m}\pm\frac{1}{E_k}\right)i\boldsymbol{\sigma}_1\cdot[(\mathbf{p}_1-\mathbf{k}'')\times\mathbf{k}]+\left[\frac{1}{m}(1\mp1)\mp\frac{2}{E_k}\right]\right.$$
$$\left.\times[i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}'']\right\},$$

$$J_3^1=\mp\frac{1}{4E_k^4}(\mathbf{p}_1-\mathbf{p}_2-2\mathbf{k}'')\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k},$$

$$J_0^2=\frac{1}{12}\left[\left(\frac{1}{m}-\frac{1}{E_k}\right)\left(\frac{1}{m}\mp\frac{1}{E_k}\right)+\left(\frac{1}{m}+\frac{1}{E_k}\right)\left(\frac{1}{m}\pm\frac{1}{E_k}\right)\mp\frac{k^2}{m^2E_k^2}\right]i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')$$
$$-\frac{1}{12}\left[\left(\frac{1}{m}+\frac{1}{E_k}\right)\left(\frac{1}{m}\mp\frac{1}{E_k}\right)\mp\frac{2k^2}{m^2E_k^2}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'',$$

$$J_2^2=\frac{k^2}{24E_k^3}\left[-\frac{1}{m}(1\mp1)\pm\frac{4}{E_k}\right][i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''],$$

$$J_4^2=0. \tag{123}$$

Similarly,

$$\langle\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\alpha_2^i\Lambda_{2\pm}(\mathbf{p}_2+\mathbf{k})\rangle=\frac{1}{16}\left\{2\left(\frac{1}{m}-\frac{1}{E(\mathbf{p}_1-\mathbf{k})}\right)p_1^i-\left(\frac{1}{m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k})}\right)\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}''+\frac{2}{E(\mathbf{p}_1-\mathbf{k})}\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}\right.$$
$$\left.+\frac{1}{2m^2E(\mathbf{p}_1-\mathbf{k})}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')\right\}\left\{2\left(\frac{1}{m}\pm\frac{1}{E(\mathbf{p}_2+\mathbf{k})}\right)p_2^i\right.$$
$$+\left(\frac{1}{m}\mp\frac{1}{E(\mathbf{p}_2+\mathbf{k})}\right)\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}''\pm\frac{2}{E(\mathbf{p}_2+\mathbf{k})}\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}\pm\frac{1}{2m^2E(\mathbf{p}_2+\mathbf{k})}$$
$$\left.\times\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\sigma_2^i\boldsymbol{\sigma}_2\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')\right\}$$
$$=L_2^0+L_1^1+L_3^1+L_0^2+L_2^2+L_4^2, \tag{124}$$

where

$$L_2^0 = \pm \frac{1}{4E_k^2} \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k},$$

$$L_1^1 = \frac{1}{8E_k} \left\{ \left[\pm 2\left(\frac{1}{m} - \frac{1}{E_k}\right) - 2\left(\frac{1}{m} \pm \frac{1}{E_k}\right) \right] i\boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 \times \mathbf{k}) + \left[\pm \left(\frac{1}{m} + \frac{1}{E_k}\right) - \left(\frac{1}{m} \mp \frac{1}{E_k}\right) \right] \times [i\boldsymbol{\sigma}_1 \cdot (\mathbf{k} \times \mathbf{k}'') - \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k}''] \right\},$$

$$L_3^1 = \pm \frac{1}{4E_k^4} (\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{k} \boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k},$$

$$L_0^2 = \frac{1}{12} \left[\left(\frac{1}{m} - \frac{1}{E_k}\right)\left(\frac{1}{m} \mp \frac{1}{E_k}\right) + \left(\frac{1}{m} + \frac{1}{E_k}\right)\left(\frac{1}{m} \pm \frac{1}{E_k}\right) \mp \frac{k^2}{m^2 E_k^2} \right] i\boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 \times \mathbf{k}'') - \frac{1}{12} \left[\left(\frac{1}{m} + \frac{1}{E_k}\right)\left(\frac{1}{m} \mp \frac{1}{E_k}\right) \mp \frac{2k^2}{m^2 E_k^2} \right] \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \boldsymbol{\sigma}_2 \cdot \mathbf{k}'',$$

$$L_2^2 = \frac{k^2}{24E_k^3} \left[-\frac{1}{m}(1 \mp 1) \pm \frac{4}{E_k} \right] \times [i\boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 \times \mathbf{k}'') - \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \boldsymbol{\sigma}_2 \cdot \mathbf{k}''],$$

$$L_4^2 = 0. \tag{125}$$

The above results also apply to the two-pair calculation to be presented in the next section. Using the above results and expanding the denominators nonrelativistically, we obtain

$$\Delta E_{-+}^{T\cdot C} = \left(\frac{\alpha}{2\pi^2}\right)^2 4\pi \int_0^\infty dk \times \int d\mathbf{k}'' \langle \phi_0(\mathbf{p}_1, \mathbf{p}_2) | I_{so} i\boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 \times \mathbf{k}'') + I_{ss} \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \boldsymbol{\sigma}_2 \cdot \mathbf{k}'' | \phi_0(\mathbf{p}_1 - \mathbf{k}'', \mathbf{p}_2 + \mathbf{k}'') \rangle, \tag{126}$$

where

$$I_{so} = -\frac{1}{6k} \frac{1}{(E_k+k)^2 - m^2} \left[\frac{1}{m^2} + \frac{1}{E_k^2} - \frac{k^2}{2m^2 E_k^2} + \frac{k^2}{E_k^4} + \frac{k^2}{2E_k^3}\left(\frac{1}{E_k+k-m} + \frac{1}{E_k+k+m}\right) \right] + \frac{1}{3kE_k^2} \frac{1}{(E_k+k)^2 - m^2} \tag{127}$$

and

$$I_{ss} = \frac{1}{12k} \frac{1}{(E_k+k)^2 - m^2} \left[\frac{1}{m^2} - \frac{1}{E_k^2} - \frac{2k^2}{m^2 E_k^2} + \frac{2k^2}{E_k^4} + \frac{k^2}{E_k^3}\left(\frac{1}{E_k+k-m} + \frac{1}{E_k+k+m}\right) \right] + \frac{1}{3kE_k^2} \frac{1}{(E_k+k)^2 - m^2}\left(1 - \frac{2k^2}{5E_k^2}\right) - \frac{k}{15E_k^2} \frac{1}{(E_k+k)^2 - m^2} \left[\frac{4}{k^2} + \frac{1}{E_k}\left(\frac{1}{E_k+k-m} + \frac{1}{E_k+k+m}\right) \right] - \frac{1}{30k^2 m^3}. \tag{128}$$

The pure singular spin-spin terms in Eqs. (120) and (127) correspond to the subtraction of nonrelativistic contributions of lower order $\alpha^6 mc^2$ or of the last term in Eq. (5.15) in Ref. [4].

The total contribution due to the one-pair single transverse photon exchange is obtained by computing the two corrections in Eqs. (119) and (126), and is given by

$$2\Delta E_{-+}^{TC} = 2\left(\frac{\alpha}{2\pi^2}\right)^2 4\pi \int_0^\infty dk \times \int d\mathbf{k}'' \langle \phi_0(\mathbf{p}_1, \mathbf{p}_2) | I_{so} i\boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 \times \mathbf{k}'') + I_{ss} \boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \boldsymbol{\sigma}_2 \cdot \mathbf{k}'' | \phi_0(\mathbf{p}_1 - \mathbf{k}'', \mathbf{p}_2 + \mathbf{k}'') \rangle, \tag{129}$$

where

$$I_{so} = -\tfrac{1}{12} \ln 2 \tag{130}$$

and

$$I_{ss} = -\tfrac{103}{360} - \tfrac{1}{12} \ln 2 + \tfrac{1}{30} \ln B. \tag{131}$$

Taking the Fourier transform, we get

$$\Delta E_{-+}^{TC} + \Delta E_{+-}^{TC} = \alpha^7 mc^2 \left\langle \phi_0 \left| I_{so} \delta(\mathbf{r}) \frac{1}{r^2} \boldsymbol{\sigma}_1 \cdot (\mathbf{r} \times \mathbf{p}_1) + I_{ss} \delta(\mathbf{r}) \frac{1}{r^2} \boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}} \boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}} \right| \phi_0 \right\rangle, \tag{132}$$

where

$$I_{so} = -2 \ln 2 \tag{133}$$

and

$$I_{ss} = \tfrac{103}{6} + 5 \ln 2 - 2 \ln B. \tag{134}$$

The logarithmic cutoff term in the spin-spin correction cancels that from the nonrelativistic contribution in Eq. (312) of Ref. [1]. Although individual terms in the spin-orbit correction are logarithmic singular, the sum is not. This agrees with a similar result from the nonrelativistic contribution in Eq. (311) of Ref. [1]. These results provide a good check for both relativistic and nonrelativistic contributions arising from the one-pair single transverse photon exchange.

C. Two pairs

Relativistic contribution to the helium fine-structure splittings of order $\alpha^7 mc^2$ also comes from two-pair single transverse photon exchange. For two distinct particles, the energy correction due to crossed-ladder diagrams arising from a relativistic momentum region may be calculated using the following four-dimensional formula deriving from Eq. (1):

$$\Delta E_{--}^{T\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k})}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega-i\delta}$$
$$\times\frac{\mathcal{L}_{2-}(\mathbf{p}_2+\mathbf{k}')}{\mu_2E+\varepsilon_2(\mathbf{p}_2+\mathbf{k}')-\epsilon+\omega'-i\delta}\alpha_2^i\psi(p'_{1\mu}p'_{2\mu})+\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})$$
$$\times\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k}')}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k}')+\epsilon-\omega'-i\delta}\alpha_1^i\alpha_2^i\frac{\mathcal{L}_{2-}(\mathbf{p}_2+\mathbf{k})}{\mu_2E+\varepsilon_2(\mathbf{p}_2+\mathbf{k})-\epsilon+\omega-i\delta}\psi(p'_{1\mu}p'_{2\mu}). \quad (135)$$

Performing integration over the energy variables and ignoring the external potentials yield

$$\Delta E_{--}^{T\times C}=\left(\frac{\alpha^2}{2\pi}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')$$
$$\times\alpha_1^i\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})I+\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\Lambda_{2-}$$
$$\times(\mathbf{p}_2+\mathbf{k}')\alpha_2^iI'|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle, \quad (136)$$

where

$$I=\frac{1}{E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+E(\mathbf{p}_1-\mathbf{k}')+k}\,\frac{1}{E(\mathbf{p}_2)+E(\mathbf{p}_2+\mathbf{k})+k} \quad (137)$$

and

$$I'=\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k})+k}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')}. \quad (138)$$

To lowest order, the energy correction reduces to

$$\Delta E_{--}^{T\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$$
$$\times|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \quad (139)$$

where

$$I=\frac{k}{6E_1E_2}\,\frac{1}{E_1+k+m_1}\,\frac{1}{E_2+k+m_2}. \quad (140)$$

The energy correction formula due to ladder diagrams is derived from Eq. (1). The relevant times-order formula becomes

$$\Delta E_{--}^{T\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i$$
$$\times\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k})}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega-i\delta}\alpha_2^i$$
$$\times\frac{\mathcal{L}_{2-}(\mathbf{p}_2+\mathbf{k})}{\mu_2E+\varepsilon_2(\mathbf{p}_2+\mathbf{k})-\epsilon+\omega-i\delta}\psi(p'_{1\mu}p'_{2\mu})$$
$$+\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{k'^2(\omega^2-k^2+i\delta)}$$
$$\times\tilde{\psi}(p_{1\mu}p_{2\mu})\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k}')}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k}')+\epsilon-\omega'-i\delta}\alpha_1^i$$
$$\times\frac{\mathcal{L}_{2-}(\mathbf{p}_2+\mathbf{k}')}{\mu_2E+\varepsilon_2(\mathbf{p}_2+\mathbf{k}')-\epsilon+\omega'-i\delta}\alpha_2^i\psi(p'_{1\mu}p'_{2\mu}). \quad (141)$$

After integration over the energy variables and dropping the external potentials, we arrive at

$$\Delta E_{--}^{T\cdot C}=\left(\frac{\alpha^2}{2\pi}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{k'^2}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})$$
$$\times\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})I+\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k}')$$
$$\times\alpha_2^iI'|\phi_c(\mathbf{p}_1-\mathbf{k}-\mathbf{k}',\mathbf{p}_2+\mathbf{k}+\mathbf{k}')\rangle, \quad (142)$$

where

$$I=\frac{1}{E+E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_2+\mathbf{k})}\left[\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k})+k}+\frac{1}{E(\mathbf{p}_2)+E(\mathbf{p}_2+\mathbf{k})+k}\right] \quad (143)$$

and

$$I'=\frac{1}{E+E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k}')}\times\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+k}+\frac{1}{E(\mathbf{p}_2+\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+k}\right]. \quad (144)$$

To lowest order, the ladder correction becomes

$$\Delta E_{--}^{T\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\times|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \quad (145)$$

where

$$I=\frac{k}{6E_1E_2}\frac{1}{E_1+E_2+m_1+m_2}\left[\frac{1}{E_1+k+m_1}+\frac{1}{E_2+k+m_2}\right]. \quad (146)$$

Combining the crossed-ladder and ladder corrections of order α^5mc^2, we get

$$\Delta E_{--}^{TC}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\times|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \quad (147)$$

where

$$I=\frac{k}{6E_1E_2}\left[\frac{1}{E_1+k+m_1}\frac{1}{E_2+k+m_2}+\frac{1}{E_1+E_2+m_1+m_2}\left(\frac{1}{E_1+k+m_1}+\frac{1}{E_2+k+m_2}\right)\right]. \quad (148)$$

For helium, the above energy correction of lowest order reduces to

$$\Delta E_{--}^{TC}=\frac{4\alpha^5mc^2}{3}\left[\frac{\pi}{2}-1-\frac{1}{2}\ln 2\right]\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle, \quad (149)$$

which reproduces that of Sucher.

For the $O(\alpha^7mc^2)$ fine structure of helium, the crossed-ladder correction in Eq. (136) becomes

$$\Delta E_{--}^{T\times C}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \quad (150)$$

where

$$I_{so}=\frac{1}{6k(E_k+k+m)^2}\left[\frac{1}{m^2}-\frac{1}{E_k^2}+\frac{k^2}{2m^2E_k^2}+\frac{k^2}{2E_k^3}\left(-\frac{1}{m}+\frac{2}{E_k}\right)-\frac{k^2}{E_k^2}\frac{1}{E_k+k+m}\left(\frac{1}{m}-\frac{1}{E_k}\right)\right]-\frac{1}{3kE_k}\frac{1}{(E_k+k+m)^2}\left(\frac{1}{m}-\frac{1}{E_k}\right) \quad (151)$$

and

$$I_{ss}=\frac{1}{6k(E_k+k+m)^2}\left\{-\frac{1}{2}\left[\left(\frac{1}{m}+\frac{1}{E_k}\right)^2+\frac{2k^2}{m^2E_k^2}-\frac{k^2}{E_k^3}\left(\frac{1}{m}+\frac{2}{E_k}\right)+\frac{2k^4}{5E_k^6}\right]+\frac{k^2}{E_k^2(E_k+k+m)}\times\left(\frac{1}{m}+\frac{1}{E_k}-\frac{2k^2}{5E_k^3}\right)-\frac{k^4}{5E_k^4(E_k+k+m)^2}\right\}+\frac{1}{3kE_k}\frac{1}{(E_k+k+m)^2}\times\left[\frac{1}{m}+\frac{1}{E_k}-\frac{2k^2}{5E_k^2}\left(\frac{1}{E_k}+\frac{1}{E_k+k+m}\right)\right]-\frac{k}{15E_k^2(E_k+k+m)^2}\left[\frac{4}{k^2}-\frac{1}{E_k(E_k+k+m)}\right]. \quad (152)$$

As observed, nonlogarithmic singularity does not occur in either spin-orbit or spin-spin correction. This is because there is no nonrelativistic contribution of lower order as the nominal order of two-pair single transverse photon corrections is α^7mc^2 in nonrelativistic approximation. Furthermore, there is no individual logarithmic singular term in the spin-orbit correction. This agrees with the result from the corresponding nonrelativistic contribution.

The ladder correction to $O(\alpha^7mc^2)$ fine-structure splittings in helium, given in Eq. (142), becomes

$$\Delta E_{--}^{T\cdot C}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \quad (153)$$

where

$$I_{so}=\frac{1}{6k}\frac{1}{E_k+m}\frac{1}{E_k+k+m}\times\left[\frac{1}{m^2}-\frac{1}{E_k^2}+\frac{k^2}{2m^2E_k^2}-\frac{k^2}{2E_k^3}\left(\frac{1}{m}+\frac{2}{E_k}\right)-\frac{k^2}{2E_k^2}\left(\frac{1}{m}+\frac{1}{E_k}\right)\left(\frac{1}{E_k+m}+\frac{1}{E_k+k+m}\right)\right]-\frac{1}{3kE_k}\frac{1}{E_k+m}\frac{1}{E_k+k+m}\left(\frac{1}{m}-\frac{1}{E_k}\right) \quad (154)$$

and

$$I_{ss}=\frac{1}{12k}\frac{1}{E_k+m}\frac{1}{E_k+k+m}\times\left[\left(\frac{1}{m}+\frac{1}{E_k}\right)^2+\frac{2k^2}{m^2E_k^2}-\frac{k^2}{E_k^3}\left(\frac{1}{m}+\frac{2}{E_k}\right)-\frac{k^2}{E_k^2}\left(\frac{1}{m}+\frac{1}{E_k}\right)\left(\frac{1}{E_k+m}+\frac{1}{E_k+k+m}\right)\right]+\frac{1}{3kE_k}\frac{1}{E_k+m}\frac{1}{E_k+k+m}\left(\frac{1}{m}+\frac{1}{E_k}-\frac{2k^2}{5E_k^3}\right)-\frac{k}{15E_k^3}\frac{1}{E_k+m}\frac{1}{E_k+k+m}\left(\frac{1}{E_k+m}+\frac{1}{E_k+k+m}\right)-\frac{k}{15E_k^2}\frac{1}{E_k+m}\frac{1}{E_k+k+m}\times\left[\frac{4}{k^2}+\frac{1}{2E_k}\left(\frac{1}{E_k+m}+\frac{1}{E_k+k+m}\right)\right]. \tag{155}$$

Again, there is no nonlogarithmic singularity in both spin-orbit and spin-spin corrections. No logarithmic singular term appears in the spin-orbit correction. Equations (150) and (153) give the relativistic contribution due to the two-pair diagrams. On computation, we obtain

$$\Delta E^{TC}_{--}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{156}$$

where

$$I_{so}=-\tfrac{1}{48}\pi-\tfrac{1}{8}\ln 2+\tfrac{11}{72} \tag{157}$$

and

$$I_{ss}=-\tfrac{41}{720}+\tfrac{19}{120}\ln 2-\tfrac{1}{30}\ln B-\tfrac{29}{960}\pi. \tag{158}$$

Taking Fourier transform, we get

$$\Delta E^{TC}_{--}=\alpha^7mc^2\left\langle\phi_0\left|I_{so}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)+I_{ss}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\right|\phi_0\right\rangle, \tag{159}$$

where

$$I_{so}=-\tfrac{1}{2}\pi-3\ln 2+\tfrac{11}{3} \tag{160}$$

and

$$I_{ss}=\tfrac{41}{12}-\tfrac{19}{2}\ln 2+2\ln B+\tfrac{29}{16}\pi. \tag{161}$$

No logarithmic cutoff term in spin-orbit correction is consistent with the absence of singular terms in nonrelativistic contribution in Eq. (313) of Ref. [1]. The logarithmic cutoff term in spin-spin correction cancels that of nonrelativistic contribution in Eq. (314) of Ref. [1]. These results provide a good check for both relativistic and nonrelativistic contributions due to the two-pair diagrams.

IV. DOUBLE TRANSVERSE PHOTON EXCHANGE

Like single transverse photon exchange, relativistic contributions to the helium fine structure splittings of order α^7mc^2 due to double transverse photon exchange arise from no-pair, one-pair, and two-pair diagrams. Since all numerators in the energy corrections of double transverse photon exchange have a similar structure in terms of ladder and crossed-ladder diagrams, we derive them in some common form. For crossed-ladder diagrams, we have

$$\begin{aligned}\langle\alpha_1^j\Lambda_{1\pm}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^j\rangle=\frac{1}{4}\Bigg\{&\left(1\mp\frac{m}{E(\mathbf{p}_1-\mathbf{k}')}\right)\sigma_1^j\sigma_1^i+\frac{1}{4m^2}\left(1\mp\frac{m}{E(\mathbf{p}_1-\mathbf{k}')}\right)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\sigma_1^j\sigma_1^i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')\\&\pm\frac{1}{2mE(\mathbf{p}_1-\mathbf{k}')}[-|\mathbf{p}_1-\mathbf{k}''|^2\sigma_1^j\sigma_1^i+2(p_1^i-k''^i)(2p_1^j-\sigma_1^j\boldsymbol{\sigma}_1\cdot\mathbf{k}'')\\&-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\sigma_1^j\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}-\sigma_1^j\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')]\Bigg\}\\&\times\Bigg\{\left(1-\frac{m}{E(\mathbf{p}_2+\mathbf{k})}\right)\sigma_2^i\sigma_2^j+\frac{1}{4m^2}\left(1-\frac{m}{E(\mathbf{p}_2+\mathbf{k})}\right)\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\sigma_2^i\sigma_2^j\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')\\&+\frac{1}{2mE(\mathbf{p}_2+\mathbf{k})}[-p_2^2\sigma_2^i\sigma_2^j+2p_2^i(2p_2^j+\sigma_2^j\boldsymbol{\sigma}_2\cdot\mathbf{k}'')\\&-\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\sigma_2^j-\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\sigma_2^j\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')]\Bigg\}\\=&J_0+J_1+J_2,\end{aligned} \tag{162}$$

where

$$J_0=\frac{1}{2}\left(1\mp\frac{m}{E_k}\right)\left(1-\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}},$$

$$J_1=-\frac{1}{2}\left(1\mp\frac{m}{E_k}\right)\left(1-\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\left[\boldsymbol{\sigma}_2\cdot\frac{\mathbf{k}''}{k}-\frac{\mathbf{k}\cdot\mathbf{k}''}{k^2}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}}\right]+\frac{m}{2E_k^3}\left[\pm\left(1-\frac{m}{E_k}\right)(\mathbf{p}_1-\mathbf{k}'')\cdot\mathbf{k}+\left(1\mp\frac{m}{E_k}\right)\mathbf{p}_2\cdot\mathbf{k}\right]\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}}$$
$$-\frac{1}{4mE_k}\left(1\mp\frac{m}{E_k}\right)[i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(2\mathbf{p}_2+\mathbf{k}'')]\mp\frac{1}{4mE_k}\left(1-\frac{m}{E_k}\right)[-i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(2\mathbf{p}_1-\mathbf{k}'')],$$

$$J_2=\frac{1}{6m^2}\left[-\left(1\mp\frac{m}{E_k}\right)\left(1-\frac{m}{E_k}\right)-\frac{m}{2E_k}\left(1\mp\frac{m}{E_k}\right)\mp\frac{m}{2E_k}\left(1-\frac{m}{E_k}\right)\pm\frac{k^2}{E_k^2}\right][i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']$$
$$+\frac{1}{3k^2}\left(1\mp\frac{m}{E_k}\right)\left(1-\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''+\frac{m}{6E_k^3}\left[\pm\left(1-\frac{m}{E_k}\right)+\left(1\mp\frac{m}{E_k}\right)\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''$$
$$+\frac{1}{6mE_k}\left[\left(1\mp\frac{m}{E_k}\right)\pm\left(1-\frac{m}{E_k}\right)\right][i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']+\frac{k^2}{12mE_k^3}(-1\mp1)i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')$$
$$+\frac{k^2}{12mE_k^3}\left[-1\mp1\pm\frac{4m^3}{5E_k^3}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''-\left[\frac{2}{5k^2}\left(1-\frac{m}{E_k}\right)\left(1\mp\frac{m}{E_k}\right)\pm\frac{m}{15E_k^3}\left(1-\frac{m}{E_k}\right)+\frac{m}{15E_k^3}\left(1\mp\frac{m}{E_k}\right)\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''. \tag{163}$$

Similarly, we obtain

$$\begin{aligned}\langle\alpha_1^j\Lambda_{1\pm}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j\rangle&=\frac{1}{4}\Bigg\{\left(1\mp\frac{m}{E(\mathbf{p}_1-\mathbf{k}')}\right)\sigma_1^j\sigma_1^i+\frac{1}{4m^2}\left(1\mp\frac{m}{E(\mathbf{p}_1-\mathbf{k}')}\right)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\sigma_1^j\sigma_1^i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')\\&\pm\frac{1}{2mE(\mathbf{p}_1-\mathbf{k}')}[-|\mathbf{p}_1-\mathbf{k}''|^2\sigma_1^j\sigma_1^i+2(p_1^i-k''^i)(2p_1^j-\sigma_1^j\boldsymbol{\sigma}_1\cdot\mathbf{k}'')\\&-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\sigma_1^j\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}-\sigma_1^j\sigma_1^i\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')]\Bigg\}\Bigg\{\left(1+\frac{m}{E(\mathbf{p}_2+\mathbf{k})}\right)\sigma_2^i\sigma_2^j\\&+\frac{1}{4m^2}\left(1+\frac{m}{E(\mathbf{p}_2+\mathbf{k})}\right)\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\sigma_2^i\sigma_2^j\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')\\&-\frac{1}{2mE(\mathbf{p}_2+\mathbf{k})}[-p_2^2\sigma_2^i\sigma_2^j+2p_2^i(2p_2^j+\sigma_2^j\boldsymbol{\sigma}_2\cdot\mathbf{k}'')-\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\sigma_2^j\\&-\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\sigma_2^j\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')]\Bigg\}\\&=J_0+J_1+J_2,\end{aligned} \tag{164}$$

where

$$J_0=\frac{1}{2}\left(1\mp\frac{m}{E_k}\right)\left(1+\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}},$$

$$J_1=-\frac{1}{2}\left(1\mp\frac{m}{E_k}\right)\left(1+\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\left[\boldsymbol{\sigma}_2\cdot\frac{\mathbf{k}''}{k}-\frac{\mathbf{k}\cdot\mathbf{k}''}{k^2}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}}\right]+\frac{m}{2E_k^3}\left[\pm\left(1+\frac{m}{E_k}\right)(\mathbf{p}_1-\mathbf{k}'')\cdot\mathbf{k}-\left(1\mp\frac{m}{E_k}\right)\mathbf{p}_2\cdot\mathbf{k}\right]\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}}$$
$$+\frac{1}{4mE_k}\left(1\mp\frac{m}{E_k}\right)[i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(2\mathbf{p}_2+\mathbf{k}'')]\mp\frac{1}{4mE_k}\left(1+\frac{m}{E_k}\right)[-i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(2\mathbf{p}_1-\mathbf{k}'')],$$

$$J_2=\frac{1}{6m^2}\left[-\left(1\mp\frac{m}{E_k}\right)\left(1+\frac{m}{E_k}\right)+\frac{m}{2E_k}\left(1\mp\frac{m}{E_k}\right)\mp\frac{m}{2E_k}\left(1+\frac{m}{E_k}\right)\mp\frac{k^2}{E_k^2}\right][i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']$$
$$+\frac{1}{3k^2}\left(1\mp\frac{m}{E_k}\right)\left(1+\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''+\frac{m}{6E_k^3}\left[\pm\left(1+\frac{m}{E_k}\right)-\left(1\mp\frac{m}{E_k}\right)\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''+\frac{1}{6mE_k}\left[-\left(1\mp\frac{m}{E_k}\right)\right.$$
$$\left.\pm\left(1+\frac{m}{E_k}\right)\right][i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']+\frac{k^2}{12mE_k^3}(1\mp1)i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+\frac{k^2}{12mE_k^3}\left[1\mp1\mp\frac{4m^3}{5E_k^3}\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''$$
$$-\left[\frac{2}{5k^2}\left(1+\frac{m}{E_k}\right)\left(1\mp\frac{m}{E_k}\right)\pm\frac{m}{15E_k^3}\left(1+\frac{m}{E_k}\right)-\frac{m}{15E_k^3}\left(1\mp\frac{m}{E_k}\right)\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''. \quad (165)$$

For ladder diagrams, all spin dependent numerators have the following structure:

$$\langle\alpha_1^i\Lambda_{1\pm}(\mathbf{p}_1-\mathbf{k})\alpha_1^j\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^j\rangle=\frac{1}{4}\left\{\left(1\mp\frac{m}{E(\mathbf{p}_1-\mathbf{k})}\right)\sigma_1^i\sigma_1^j+\frac{1}{4m^2}\left(1\mp\frac{m}{E(\mathbf{p}_1-\mathbf{k})}\right)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\sigma_1^i\sigma_1^j\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')\right.$$
$$\pm\frac{1}{2mE(\mathbf{p}_1-\mathbf{k})}[-p_1^2\sigma_1^i\sigma_1^j+2p_1^i(2p_1^j-\sigma_1^j\boldsymbol{\sigma}_1\cdot\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\sigma_1^j$$
$$\left.+\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\sigma_1^j\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')]\right\}\left\{\left(1-\frac{m}{E(\mathbf{p}_2+\mathbf{k})}\right)\sigma_2^i\sigma_2^j+\frac{1}{4m^2}\left(1-\frac{m}{E(\mathbf{p}_2+\mathbf{k})}\right)\right.$$
$$\times\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\sigma_2^i\sigma_2^j\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')+\frac{1}{2mE(\mathbf{p}_2+\mathbf{k})}[-p_2^2\sigma_2^i\sigma_2^j+2p_2^i(2p_2^j+\sigma_2^j\boldsymbol{\sigma}_2\cdot\mathbf{k}'')$$
$$\left.-\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\sigma_2^j-\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\sigma_2^j\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')]\right\}$$
$$=J_0+J_1+J_2, \quad (166)$$

where

$$J_0=-\frac{1}{2}\left(1\mp\frac{m}{E_k}\right)\left(1-\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}},$$

$$J_1=\frac{1}{2}\left(1\mp\frac{m}{E_k}\right)\left(1-\frac{m}{E_k}\right)\left\{-i\boldsymbol{\sigma}_1\cdot\left(\hat{\mathbf{k}}\times\frac{\mathbf{k}''}{k}\right)+\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\left[\boldsymbol{\sigma}_2\cdot\frac{\mathbf{k}''}{k}-\frac{\mathbf{k}\cdot\mathbf{k}''}{k^2}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}}\right]\right\}-\frac{m}{2E_k^3}\left[\mp\left(1-\frac{m}{E_k}\right)\mathbf{p}_1\cdot\mathbf{k}\right.$$
$$\left.+\left(1\mp\frac{m}{E_k}\right)\mathbf{p}_2\cdot\mathbf{k}\right]\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}}-\frac{1}{4mE_k}\left(1\mp\frac{m}{E_k}\right)[i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(2\mathbf{p}_2+\mathbf{k}'')]\mp\frac{1}{4mE_k}\left(1-\frac{m}{E_k}\right)$$
$$\times[i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(2\mathbf{p}_1-\mathbf{k}'')],$$

$$J_2=-\frac{1}{6m^2}\left(1\mp\frac{m}{E_k}\right)\left(1-\frac{m}{E_k}\right)[2i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']\mp\frac{k^2}{6m^2E_k^2}[i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-2\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']$$
$$+\frac{1}{12mE_k}\left[-\left(1\mp\frac{m}{E_k}\right)\mp\left(1-\frac{m}{E_k}\right)\right][3i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']-\frac{1}{3k^2}\left(1\mp\frac{m}{E_k}\right)\left(1-\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''$$
$$+\frac{1}{6mE_k}\left\{\frac{m^2}{E_k^2}\left[\pm\left(1-\frac{m}{E_k}\right)+\left(1\mp\frac{m}{E_k}\right)\right]-\left(1\mp\frac{m}{E_k}\right)\mp\left(1-\frac{m}{E_k}\right)\right\}[i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']$$
$$+\frac{k^2}{12mE_k^3}\left[-\left(1\mp\frac{2m}{E_k}\right)\mp\left(1-\frac{2m}{E_k}\right)\right][i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']$$
$$+\left[\frac{2}{5k^2}\left(1-\frac{m}{E_k}\right)\left(1\mp\frac{m}{E_k}\right)\pm\frac{m}{15E_k^3}\left(1-\frac{m}{E_k}\right)+\frac{m}{15E_k^3}\left(1\mp\frac{m}{E_k}\right)\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''. \quad (167)$$

Similarly, we have

$$\langle\alpha_1^i\Lambda_{1\pm}(\mathbf{p}_1-\mathbf{k})\alpha_1^j\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j\rangle=\frac{1}{4}\Bigg\{\left(1\mp\frac{m}{E(\mathbf{p}_1-\mathbf{k})}\right)\sigma_1^i\sigma_1^j+\frac{1}{4m^2}\left(1\mp\frac{m}{E(\mathbf{p}_1-\mathbf{k})}\right)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\sigma_1^i\sigma_1^j\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')$$
$$\pm\frac{1}{2mE(\mathbf{p}_1-\mathbf{k})}[-p_1^2\sigma_1^i\sigma_1^j+2p_1^i(2p_1^j-\sigma_1^j\boldsymbol{\sigma}_1\cdot\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\sigma_1^j$$
$$+\boldsymbol{\sigma}_1\cdot\mathbf{k}\sigma_1^i\sigma_1^j\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1-\mathbf{k}'')]\Bigg\}\Bigg\{\left(1+\frac{m}{E(\mathbf{p}_2+\mathbf{k})}\right)\sigma_2^i\sigma_2^j+\frac{1}{4m^2}\left(1+\frac{m}{E(\mathbf{p}_2+\mathbf{k})}\right)$$
$$\times\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\sigma_2^i\sigma_2^j\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')-\frac{1}{2mE(\mathbf{p}_2+\mathbf{k})}[-p_2^2\sigma_2^i\sigma_2^j+2p_2^i(2p_2^j+\sigma_2^j\boldsymbol{\sigma}_2\cdot\mathbf{k}'')$$
$$-\boldsymbol{\sigma}_2\cdot\mathbf{p}_2\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\sigma_2^j-\boldsymbol{\sigma}_2\cdot\mathbf{k}\sigma_2^i\sigma_2^j\boldsymbol{\sigma}_2\cdot(\mathbf{p}_2+\mathbf{k}'')]\Bigg\}$$
$$=J_0+J_1+J_2, \tag{168}$$

where

$$J_0=-\frac{1}{2}\left(1\mp\frac{m}{E_k}\right)\left(1+\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}},$$

$$J_1=\frac{1}{2}\left(1\mp\frac{m}{E_k}\right)\left(1+\frac{m}{E_k}\right)\left[-i\boldsymbol{\sigma}_1\cdot\left(\hat{\mathbf{k}}\times\frac{\mathbf{k}''}{k}\right)+\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\left[\boldsymbol{\sigma}_2\cdot\frac{\mathbf{k}''}{k}-\frac{\mathbf{k}\cdot\mathbf{k}''}{k^2}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}}\right]\right\}-\frac{m}{2E_k^3}\left[\mp\left(1+\frac{m}{E_k}\right)\mathbf{p}_1\cdot\mathbf{k}-\left(1\mp\frac{m}{E_k}\right)\mathbf{p}_2\cdot\mathbf{k}\right]$$
$$\times\boldsymbol{\sigma}_1\cdot\hat{\mathbf{k}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{k}}+\frac{1}{4mE_k}\left(1\mp\frac{m}{E_k}\right)[i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(2\mathbf{p}_2+\mathbf{k}'')]$$
$$\mp\frac{1}{4mE_k}\left(1+\frac{m}{E_k}\right)[i\boldsymbol{\sigma}_1\cdot(\mathbf{k}\times\mathbf{k}'')+\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot(2\mathbf{p}_1-\mathbf{k}'')],$$

$$J_2=-\frac{1}{6m^2}\left(1\mp\frac{m}{E_k}\right)\left(1+\frac{m}{E_k}\right)[2i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']\pm\frac{k^2}{6m^2E_k^2}[i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-2\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']$$
$$+\frac{1}{12mE_k}\left[\left(1\mp\frac{m}{E_k}\right)\mp\left(1+\frac{m}{E_k}\right)\right][3i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']-\frac{1}{3k^2}\left(1\mp\frac{m}{E_k}\right)\left(1+\frac{m}{E_k}\right)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''$$
$$+\frac{1}{6mE_k}\left\{\frac{m^2}{E_k^2}\left[\pm\left(1+\frac{m}{E_k}\right)-\left(1\mp\frac{m}{E_k}\right)\right]+\left(1\mp\frac{m}{E_k}\right)\mp\left(1+\frac{m}{E_k}\right)\right\}[i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']$$
$$+\frac{k^2}{12mE_k^3}\left[\left(1\mp\frac{2m}{E_k}\right)\mp\left(1+\frac{2m}{E_k}\right)\right][i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}'']$$
$$+\left[\frac{2}{5k^2}\left(1+\frac{m}{E_k}\right)\left(1\mp\frac{m}{E_k}\right)\pm\frac{m}{15E_k^3}\left(1+\frac{m}{E_k}\right)-\frac{m}{15E_k^3}\left(1\mp\frac{m}{E_k}\right)\right]\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''. \tag{169}$$

The above FW transformation for the double transverse photon corrections requires great care. The presentation of these results provides a very good check for any other independent calculation in case of disagreement.

A. No pair

Again, we start from the relativistic energy corrections of lowest order. For two distinct particles, the four-dimensional formula to derive the energy correction due to no-pair crossed-ladder double transverse photon exchange is

$$\Delta E_{++}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{(\omega^2-k^2+i\delta)(\omega'^2-k'^2+i\delta)}\widetilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i$$
$$\times\frac{\mathcal{L}_{1+}(\mathbf{p}_1-\mathbf{k})}{\mu_1E-\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega+i\delta}\alpha_1^j\alpha_2^j\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k}')}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k}')-\epsilon+\omega'+i\delta}\alpha_2^i\psi(p'_{1\mu}p'_{2\mu}). \tag{170}$$

Integrating over the energy variables and neglecting the external potentials give

$$\Delta E_{++}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^j\Lambda_{1+}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^j I|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{171}$$

where

$$\begin{aligned}I=&-\left[\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_2)-k'}+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}\right]\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k})+k+k'-E}\\&\times\left[\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k}+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})-k'}\right]\\&+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k-k'}\,\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k}\,\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}\\&+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2)-k-k'}\,\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}')-E(\mathbf{p}_2)-k'}\,\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})-k'},\end{aligned} \tag{172}$$

which reproduced that obtained in Ref. [1] using Sucher's times-order formulation. To lowest order, the relativistic energy correction from two no-pair crossed-ladder transverse photons exchanged is

$$\Delta E_{++}^{T\times T}=\alpha^5\mu^3c^2\langle\phi_0|I\delta(\mathbf{r})(1+\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle, \tag{173}$$

where

$$I=\int_0^\infty dk\left(1-\frac{m_1}{E_1}\right)\left(1-\frac{m_2}{E_2}\right)\left[\frac{1}{m_1+m_2-2k-E_1-E_2}\left(\frac{1}{E_1+k-m_1}+\frac{1}{E_2+k-m_2}\right)^2-\frac{1}{k}\,\frac{1}{(E_1+k-m_1)(E_2+k-m_2)}\right]. \tag{174}$$

The energy correction arising from the no-pair ladder double transverse photon exchange may be derived using the following four-dimensional formula:

$$\begin{aligned}\Delta E_{++}^{T\cdot T}=&\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{(\omega^2-k^2+i\delta)(\omega'^2-k'^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i\frac{\mathcal{L}_{1+}(\mathbf{p}_1-\mathbf{k})}{\mu_1E-\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega+i\delta}\alpha_1^j\alpha_2^i\\&\times\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k})}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k})-\epsilon+\omega+i\delta}\alpha_2^j\psi(p'_{1\mu}p'_{2\mu}).\end{aligned} \tag{175}$$

Integrating over the energy variables and dropping the external potentials lead to

$$\Delta E_{++}^{T\cdot T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\Lambda_{1+}(\mathbf{p}_1-\mathbf{k})\alpha_1^j\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^j I|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{176}$$

where

$$\begin{aligned}I=&-\left[\frac{1}{E-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2)-k}+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}\right]\frac{1}{E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_2+\mathbf{k})-E}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k'-m}\right.\\&\left.+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'-m}\right]+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2)-k-k'}\,\frac{1}{E-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2)-k}\,\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})-k'}\\&+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k-k'}\,\frac{1}{E-E(\mathbf{p}_1-\mathbf{k})-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k'}\,\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k},\end{aligned} \tag{177}$$

which reproduced that in Ref. [1], derived in Sucher's times-order formalism. To lowest order, the above correction becomes

$$\Delta E_{++}^{T\cdot T}=\alpha^5\mu^3c^2\langle\phi_0|I\delta(\mathbf{r})(1-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle, \tag{178}$$

where

$$I=\int_0^\infty dk\left(1-\frac{m_1}{E_1}\right)\left(1-\frac{m_2}{E_2}\right)\left[\frac{1}{m_1+m_2-E_1-E_2}\left(\frac{1}{E_1+k-m_1}+\frac{1}{E_2+k-m_2}\right)^2-\frac{1}{k}\,\frac{1}{(E_1+k-m_1)(E_2+k-m_2)}\right]. \tag{179}$$

For helium, the two corrections in Eqs. (173) and (178) reduce to

$$\Delta E_{++}^{T\times T}=\alpha^5 mc^2\left[-\frac{\pi}{2}+1+\frac{1}{2}\ln 2\right]\langle\phi_0|\delta(\mathbf{r})(1+\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle \tag{180}$$

and

$$\Delta E_{++}^{T\cdot T}=-\frac{\pi}{4}\alpha^5 mc^2\langle\phi_0|\delta(\mathbf{r})(1-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle, \tag{181}$$

which reproduce Sucher's results.

For the helium fine structure of order $\alpha^7 mc^2$, the relativistic correction due to the no-pair double crossed-ladder transverse photon exchange is given by

$$\Delta E_{++}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^j\Lambda_{1+}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^j I|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{182}$$

where

$$\begin{aligned}I=&-\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k'-m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k-m}\right]\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k})+k+k'-2m}\\&\times\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k-m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'-m}\right]\\&-\frac{1}{k+k'}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k-m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k-m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k'-m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'-m}\right].\end{aligned} \tag{183}$$

After nonrelativistic expansion of the denominators, we get

$$I=I_0+I_1+I_2, \tag{184}$$

where

$$\begin{aligned}I_0=&-\frac{1}{(E_k+k-m)^2}\left[\frac{1}{k}+\frac{2}{E_k+k-m}\right],\\I_1=&\frac{(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k}'')\cdot\mathbf{k}}{E_k(E_k+k-m)^3}\left[\frac{1}{k}+\frac{3}{E_k+k-m}\right]-\frac{3\mathbf{k}\cdot\mathbf{k}''}{k(E_k+k-m)^4}-\frac{\mathbf{k}\cdot\mathbf{k}''}{2k^2(E_k+k-m)^2}\left[\frac{1}{k}+\frac{2}{E_k+k-m}\right],\\I_2=&-\frac{\mathbf{p}_1\cdot\mathbf{k}\mathbf{k}\cdot\mathbf{k}''}{E_k^2(E_k+k-m)^4}\left[\frac{1}{k}+\frac{4}{E_k+k-m}\right]-\frac{3(\mathbf{k}\cdot\mathbf{k}'')^2}{2k(E_k+k-m)^4}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k-m)}+\frac{1}{k(E_k+k+m)}\right]\\&-\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{2k^2(E_k+k-m)^2}\left[\frac{1}{2k^3}+\frac{1}{E_k(E_k+k-m)^2}+\frac{2}{k^2(E_k+k+m)}+\frac{1}{k(E_k+k-m)^2}\right]-\frac{9(\mathbf{k}\cdot\mathbf{k}'')^2}{2kE_k(E_k+k-m)^5}\\&-\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{2k^2E_k(E_k+k-m)^3}\left[\frac{1}{k}+\frac{2}{E_k+k-m}\right].\end{aligned} \tag{185}$$

Combining with the nonrelativistic expansion of the numerator presented at the beginning of this section, we obtain

$$\Delta E_{++}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{186}$$

where

$$I_{so}=\frac{1}{24m^2}\frac{1}{(E_k+k-m)^2}\left[\frac{1}{k}+\frac{2}{E_k+k-m}\right]\left(1-\frac{m}{E_k}\right)\left[1-\frac{2m}{E_k}-\frac{k^2}{E_k^2}\right] \tag{187}$$

and

$$I_{ss}=\frac{1}{24m^2}\frac{1}{(E_k+k-m)^2}\left[\frac{1}{k}+\frac{2}{E_k+k-m}\right]\left\{\left(1-\frac{m}{E_k}\right)\left(1+\frac{2m}{E_k}-\frac{2m^3}{E_k^3}\right)-\frac{2m^2}{k^2}\left(1-\frac{m}{E_k}\right)^2-\frac{k^2}{E_k^2}\left[1-\frac{m}{E_k}\left(1-\frac{2m^3}{5E_k^3}\right)\right]\right\}$$
$$+\frac{1}{4(E_k+k-m)^2}\left(1-\frac{m}{E_k}\right)\left[1-\frac{m}{E_k}+\frac{2mk^2}{5E_k^3}\right]\left\{\frac{1}{E_k(E_k+k-m)}\left(\frac{1}{3k}+\frac{1}{E_k+k-m}\right)+\frac{1}{k}\left[\frac{1}{2k^2}+\frac{1}{k(E_k+k-m)}\right.\right.$$
$$\left.\left.+\frac{1}{(E_k+k-m)^2}\right]\right\}+\frac{1}{2(E_k+k-m)^2}\left(\frac{1}{k}+\frac{2}{E_k+k-m}\right)\left(1-\frac{m}{E_k}\right)\left[\frac{1}{5k^2}\left(1-\frac{m}{E_k}\right)+\frac{m}{15E_k^3}\right]+\frac{1}{60}\left(1-\frac{m}{E_k}\right)^2$$
$$\left\{-\frac{k^2}{E_k^2(E_k+k-m)^4}\left[\frac{1}{k}+\frac{4}{E_k+k-m}\right]-\frac{k}{(E_k+k-m)^4}\left[\frac{10}{k^2}+\frac{12}{E_k(E_k+k-m)}+\frac{3}{k(E_k+k-m)}\right]\right.$$
$$-\frac{1}{(E_k+k-m)^4}\left(\frac{2}{k}+\frac{5}{E_k+k-m}\right)-\frac{1}{(E_k+k-m)^2}\left[\frac{8}{k^3}+\frac{16}{k^2(E_k+k-m)}+\frac{5}{kE_k(E_k+k-m)}\right.$$
$$\left.\left.+\frac{15}{E_k(E_k+k-m)^2}+\frac{6}{k(E_k+k-m)^2}\right]\right\}. \tag{188}$$

It is seen that no nonlogarithmic cutoff term appears since nonrelativistic contributions are of nominal order $\alpha^7 mc^2$.

The ladder diagrams contribute

$$\Delta E_{++}^{T\cdot T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\Lambda_{1+}(\mathbf{p}_1-\mathbf{k})\alpha_1^j\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^j I|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{189}$$

where

$$I=-\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k-m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k-m}\right]\frac{1}{E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_2+\mathbf{k})-2m}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k'-m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'-m}\right]$$
$$-\frac{1}{k+k'}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k-m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'-m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k'-m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k-m}\right]. \tag{190}$$

After nonrelativistic expansion, we get

$$I=I_0+I_1+I_2, \tag{191}$$

where

$$I_0=-\frac{1}{(E_k+k-m)^2}\left[\frac{1}{k}+\frac{2}{E_k-m}\right],$$

$$I_1=-\frac{(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{k}}{E_k(E_k+k-m)^2}\left[\frac{1}{E_k+k-m}\left(\frac{1}{k}+\frac{2}{E_k-m}\right)+\frac{1}{(E_k-m)^2}\right]-\frac{\mathbf{k}\cdot\mathbf{k}''}{k(E_k+k-m)^2}\left[\frac{1}{E_k+k-m}\left(\frac{1}{k}+\frac{2}{E_k-m}\right)+\frac{1}{2k^2}\right],$$

$$I_2=-\frac{(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{k}\mathbf{k}\cdot\mathbf{k}''}{kE_k(E_k-m)(E_k+k-m)^3}\left[\frac{1}{E_k-m}+\frac{2}{E_k+k-m}\right]-\frac{1}{k}\frac{1}{(E_k+k-m)^2}\left[\frac{(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{k}}{E_k(E_k+k-m)}\frac{\mathbf{k}\cdot\mathbf{k}''}{k}\left(\frac{1}{2k}+\frac{1}{E_k+k-m}\right)\right.$$
$$\left.+\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{2k^3}\left(\frac{1}{2k}+\frac{1}{E_k+k-m}\right)\right]-\frac{1}{2k(E_k+k-m)^3}\left[\frac{1}{k}+\frac{2}{E_k-m}\right]\left[\frac{(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{k}\mathbf{k}\cdot\mathbf{k}''}{E_k(E_k+k-m)}+\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{k^2}\right]. \tag{192}$$

Reducing the numerator and denominators leads to

$$\Delta E_{++}^{T\cdot T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{193}$$

where

$$I_{so}=\frac{1}{24m^2}\frac{1}{(E_k+k-m)^2}\left[\frac{1}{k}+\frac{2}{E_k-m}\right]\left\{2\left(1-\frac{m}{E_k}\right)^2+\frac{3m}{E_k}\left(1-\frac{m}{E_k}\right)+\frac{2m}{E_k}\left(1-\frac{m}{E_k}\right)^2\left(1+\frac{m}{E_k}\right)+\frac{k^2}{E_k^2}\left[1+\frac{m}{E_k}\left(1-\frac{2m}{E_k}\right)\right]\right\}$$
$$+\frac{1}{12E_k(E_k+k-m)^2}\left[\frac{1}{E_k+k-m}\left(\frac{1}{k}+\frac{2}{E_k-m}\right)+\frac{1}{(E_k-m)^2}\right]\left(1-\frac{m}{E_k}\right)\left[1-\frac{m}{E_k}+\frac{k^2}{mE_k}\right]-\frac{1}{2k^2m^3} \tag{194}$$

and

$$I_{ss}=-\frac{1}{24m^2}\frac{1}{(E_k+k-m)^2}\left[\frac{1}{k}+\frac{2}{E_k-m}\right]\left\{\left(1-\frac{m}{E_k}\right)\left[1+\frac{2m}{E_k}\left(1-\frac{m^2}{E_k^2}\right)\right]-\frac{2m^2}{k^2}\left(1-\frac{m}{E_k}\right)^2+\frac{2k^2}{E_k^2}\left[1+\frac{m}{2E_k}\left(1-\frac{2m}{E_k}\right)\right]\right\}$$
$$-\frac{1}{12E_k(E_k+k-m)^2}\left[\frac{1}{E_k+k-m}\left(\frac{1}{k}+\frac{2}{E_k-m}\right)+\frac{1}{(E_k-m)^2}\right]\left(1-\frac{m}{E_k}\right)\left[1-\frac{m}{E_k}+\frac{k^2}{mE_k}\right]$$
$$-\frac{1}{12k(E_k+k-m)^2}\left[\frac{1}{E_k+k-m}\left(\frac{1}{k}+\frac{2}{E_k-m}\right)+\frac{3}{2k^2}+\frac{2}{k(E_k-m)}\right]\left(1-\frac{m}{E_k}\right)\left[1-\frac{m}{E_k}+\frac{2mk^2}{5E_k^3}\right]$$
$$-\frac{1}{2(E_k+k-m)^2}\left(\frac{1}{k}+\frac{2}{E_k-m}\right)\left(1-\frac{m}{E_k}\right)\left[\frac{1}{5k^2}\left(1-\frac{m}{E_k}\right)+\frac{m}{E_k^3}\right]+\frac{1}{30k(E_k+k-m)^2}\left(1-\frac{m}{E_k}\right)^2\left[\frac{1}{k}\left(\frac{4}{k}+\frac{5}{E_k-m}\right)\right.$$
$$\left.+\frac{1}{E_k+k-m}\left(\frac{2}{k}+\frac{5}{E_k-m}\right)+\frac{1}{k(E_k+k-m)}\right]+\frac{1}{30E_k(E_k+k-m)^2}\left(1-\frac{m}{E_k}\right)^2\left[\frac{2}{E_k+k-m}\left(\frac{5}{4k}+\frac{2}{E_k-m}\right)+\frac{2}{(E_k-m)^2}\right.$$
$$\left.+\frac{E_k}{(E_k+k-m)^2}\left(\frac{1}{k}+\frac{2}{E_k-m}\right)+\frac{k}{(E_k-m)(E_k+k-m)}\left(\frac{3}{E_k+k-m}+\frac{1}{E_k-m}\right)+\frac{3}{2(E_k+k-m)^2}\right]+\frac{31}{60k^2m^3}. \tag{195}$$

Here, the lower-order nonrelativistic contributions are subtracted. Combining Eqs. (186) and (193) on computation, we obtain the total relativistic contribution due to the no-pair diagrams. The contribution is

$$\Delta E_{++}^{TT}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')$$
$$+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{196}$$

where

$$I_{so}=-\tfrac{7}{24}-\tfrac{1}{12}\ln 2+\tfrac{1}{4}\ln B-\tfrac{1}{48}\pi \tag{197}$$

and

$$I_{ss}=\tfrac{71}{240}+\tfrac{1}{960}\pi-\tfrac{113}{240}\ln B+\tfrac{39}{240}\ln 2. \tag{198}$$

Taking Fourier transform, we get

$$\Delta E_{++}^{TT}=\alpha^7mc^2\langle\phi_0|I_{so}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)$$
$$+I_{ss}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle, \tag{199}$$

where

$$I_{so}=-7-2\ln 2+6\ln B-\tfrac{1}{2}\pi \tag{200}$$

and

$$I_{ss}=-\tfrac{71}{4}-\tfrac{1}{16}\pi+\tfrac{113}{4}\ln B-\tfrac{39}{4}\ln 2. \tag{201}$$

Again the logarithmic terms above do not cancel those in Eqs. (315)–(317) in Ref. [1]. Separating out ladder terms of nonrelativistic nominal order α^6mc^2 or the first two lines in Eq. (190) as pure ladder correction, the remaining logarithmic terms cancel those in Eqs. (316) and (317) in Ref. [1]. The singular terms in the pure ladder correction cancel those arising from Eq. (94) in Ref. [1] as we indicated earlier. Combining with corrections from no-pair single transverse photon diagrams, the logarithmic cutoff terms are seen to cancel the logarithmic singular terms in Eqs. (307)–(309) and (315)–(317). The cancellation provides a good check on both nonrelativistic and relativistic calculations due to the no-pair single and double transverse photon exchange.

B. One pair

For two distinct particles, the one-pair energy correction due to the times-order crossed-ladder diagrams can be derived from Eq. (1). The corresponding times-order formula is

$$\Delta E^{T\times T}_{-+}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{(\omega^2-k^2+i\delta)(\omega'^2-k'^2+i\delta)}\widetilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k})}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega-i\delta}\alpha_1^j\alpha_2^j$$
$$\times\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k}')}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k}')-\epsilon+\omega'+i\delta}\alpha_2^i\psi(p'_{1\mu}p'_{2\mu}). \tag{202}$$

Integrating over the energy variables and neglecting the external potentials give

$$\Delta E^{T\times T}_{-+}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^j\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^jI|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{203}$$

where

$$I=\left[\frac{-1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+k}+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}\right]\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k})-E}$$
$$\times\left[\frac{-1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k}')+k'}+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})-k'}\right]$$
$$+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2)-k-k'}\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+k}\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})+k'}$$
$$+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k-k'}\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k}')+k'+m}\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}, \tag{204}$$

which reproduces that in Ref. [1], derived in Sucher's times-order formalism. To lowest order, the crossed-ladder correction becomes

$$\Delta E^{T\times T}_{-+}=\alpha^5\mu^3c^2\langle\phi_0|I\delta(\mathbf{r})(1+\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle, \tag{205}$$

where

$$I=\int_0^\infty dk\left(1+\frac{m_1}{E_1}\right)\left(1-\frac{m_2}{E_2}\right)\left[\frac{1}{E_1+E_2}\left(\frac{1}{E_1+k+m_1}+\frac{1}{E_2+k-m_2}\right)^2+\frac{1}{k}\frac{1}{(E_1+k+m_1)(E_2+k-m_2)}\right]. \tag{206}$$

The ladder energy correction may be derived using the following formula:

$$\Delta E^{T\cdot T}_{-+}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1d\mathbf{p}_2d^4kd^4k'}{(\omega^2-k^2+i\delta)(\omega'^2-k'^2+i\delta)}\widetilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k})}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega-i\delta}\alpha_1^j\alpha_2^i$$
$$\times\frac{\mathcal{L}_{2+}(\mathbf{p}_2+\mathbf{k})}{\mu_2E-\varepsilon_2(\mathbf{p}_2+\mathbf{k})-\epsilon+\omega+i\delta}\alpha_2^j\psi(p'_{1\mu}p'_{2\mu}). \tag{207}$$

Integrating over the energy variables and neglecting the external field, we get

$$\Delta E^{T\cdot T}_{-+}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\alpha_1^j\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^jI|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{208}$$

where

$$I=\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k})+k}\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})-k'}\left[\frac{-1}{E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+k'}+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})-k}\right]$$
$$+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2)-k-k'}\frac{1}{E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+k'}\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2+\mathbf{k})-k'}$$
$$+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k-k'}\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k})+k}\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k})+k}, \tag{209}$$

which reproduces that in Ref. [1]. To lowest order, the ladder correction becomes

$$\Delta E^{T\cdot T}_{-+}=\alpha^5\mu^3c^2\langle\phi_0|I\delta(\mathbf{r})(1-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle, \tag{210}$$

where

$$I=\int_0^\infty dk\left(1+\frac{m_1}{E_1}\right)\left(1-\frac{m_2}{E_2}\right)\left[\frac{1}{E_1+k+m_1}\frac{1}{E_2+k-m_2}\left(\frac{1}{E_1+k+m_1}+\frac{1}{E_2+k-m_2}\right)+\frac{1}{k}\frac{1}{(E_1+k+m_1)(E_2+k-m_2)}\right]. \tag{211}$$

For helium, the two corrections in Eqs. (205) and (210) reduce to

$$\Delta E_{-+}^{T\times T}=\tfrac{1}{2}\,(1+\ln 2)\,\alpha^5 mc^2\langle\phi_0|\,\delta(\mathbf{r})(1+\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle \tag{212}$$

and

$$\Delta E_{-+}^{T\cdot T}=\ln 2\,\alpha^5 mc^2\langle\phi_0|\,\delta(\mathbf{r})(1-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle, \tag{213}$$

which reproduce Sucher's results.

For the helium fine structure of order $\alpha^7 mc^2$, the crossed-ladder correction reads

$$\Delta E_{-+}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^j\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^j I|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{214}$$

where

$$\begin{aligned}I=&\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k+m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k-m}\right]\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k})}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k'+m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'-m}\right]\\&+\frac{1}{k+k'}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k+m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'-m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k'+m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k-m}\right].\end{aligned} \tag{215}$$

After nonrelativistic expansion, we get

$$I=I_0+I_1+I_2, \tag{216}$$

where

$$\begin{aligned}I_0=&\frac{1}{2E_k}\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]^2+\frac{1}{k}\,\frac{1}{(E_k+k)^2-m^2},\\
I_1=&-\frac{1}{E_k^2}\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right)\left[\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right)\frac{(\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}'')\cdot\mathbf{k}}{4E_k}+\frac{(\mathbf{p}_1-\mathbf{k}'')\cdot\mathbf{k}}{(E_k+k+m)^2}+\frac{\mathbf{p}_2\cdot\mathbf{k}}{(E_k+k-m)^2}\right]\\
&-\frac{1}{kE_k}\,\frac{1}{(E_k+k)^2-m^2}\left[\frac{(\mathbf{p}_1-\mathbf{k}'')\cdot\mathbf{k}}{E_k+k+m}+\frac{\mathbf{p}_2\cdot\mathbf{k}}{E_k+k-m}\right]+\frac{\mathbf{k}\cdot\mathbf{k}''}{2k^2}\,\frac{1}{(E_k+k)^2-m^2}\left[\frac{1}{k}\,\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right]\\
&+\frac{\mathbf{k}\cdot\mathbf{k}''}{2kE_k}\left[\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right]\left[\frac{1}{(E_k+k+m)^2}+\frac{1}{(E_k+k-m)^2}\right]\\
I_2=&\left\{\frac{1}{2E_k^4}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\left\{\frac{1}{2E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\left[\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right]\right\}\right.\\
&\left.+\frac{1}{E_k^2}\,\frac{1}{[(E_k+k)^2-m^2]^2}\left(\frac{1}{k}+\frac{1}{E_k}\right)\right\}\mathbf{p}_1\cdot\mathbf{k}\mathbf{k}\cdot\mathbf{k}''+\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{4kE_k^2}\left[\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right]\\
&\times\left[\frac{1}{E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right]+\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{4kE_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\\
&\times\left\{\frac{1}{(E_k+k+m)^2}\left[\frac{1}{k^2}+\frac{2}{E_k(E_k+k+m)}\right]\frac{1}{(E_k+k-m)^2}\left[\frac{1}{k^2}+\frac{2}{E_k(E_k+k-m)}\right]\right\}\\
&+\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{4k^2}\,\frac{1}{(E_k+k)^2-m^2}\left\{\frac{1}{k^3}+\frac{1}{E_k(E_k+k+m)}\left(\frac{1}{k}+\frac{1}{E_k+k-m}\right)+\frac{1}{E_k(E_k+k-m)}\left(\frac{1}{k}+\frac{1}{E_k+k+m}\right)\right.\\
&\left.+\frac{2}{E_k+k+m}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k+m)}\right]+\frac{2}{E_k+k-m}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k-m)}\right]\right\}.\end{aligned} \tag{217}$$

Combining with the nonrelativistic expansion of the numerator, we obtain

$$\Delta E_{-+}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{4}\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+(I_{ss}^1+I_{ss}^2)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{218}$$

where

$$\begin{aligned}I_{so}=&\frac{1}{6E_k}\left[\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)^2+\frac{1}{k}\frac{2E_k}{(E_k+k)^2-m^2}\right]\left[-\frac{1}{2m^2}\left(1+\frac{k^2}{E_k^2}\right)+\frac{1}{E_k^2}\right]-\frac{k^2}{6mE_k^3}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\\&\times\left[\frac{1}{(E_k+k-m)^2}-\frac{1}{(E_k+k+m)^2}\right]-\frac{k}{6mE_k^2}\frac{1}{(E_k+k)^2-m^2}\left(\frac{1}{E_k+k-m}-\frac{1}{E_k+k+m}\right)-\frac{1}{6k^2m^3}\end{aligned} \tag{219}$$

and

$$\begin{aligned}I_{ss}^1=&\frac{1}{6E_k}\left[\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)^2+\frac{1}{k}\frac{2E_k}{(E_k+k)^2-m^2}\right]\left[-\frac{1}{2m^2}\left(1+\frac{k^2}{E_k^2}\right)+\left(\frac{1}{k^2}-\frac{1}{E_k^2}\right)\left(1-\frac{m^2}{E_k^2}\right)-\frac{m^2k^2}{5E_k^6}\right]\\&-\frac{1}{12E_k^3}\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]^2\left[1-\frac{m^2}{E_k^2}+\frac{2m^2k^2}{5E_k^4}\right]-\frac{1}{6E_k}\frac{1}{(E_k+k+m)^2}\\&\times\left[\frac{1}{E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{k}\frac{1}{E_k+k-m}\right]\left[1-\frac{m^2}{E_k^2}-\frac{k^2}{mE_k}+\frac{2mk^2}{5E_k^3}\left(1+\frac{m}{E_k}\right)\right]\\&-\frac{1}{6E_k}\frac{1}{(E_k+k-m)^2}\left[\frac{1}{E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{k}\frac{1}{E_k+k+m}\right]\left[1-\frac{m^2}{E_k^2}+\frac{k^2}{mE_k}-\frac{2mk^2}{5E_k^3}\left(1-\frac{m}{E_k}\right)\right]\\&-\frac{1}{6k}\left[1-\frac{m^2}{E_k^2}+\frac{2m^2k^2}{5E_k^4}\right]\left\{\frac{1}{k}\frac{1}{(E_k+k)^2-m^2}\left[\frac{3}{k}+\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right]\right.\\&\left.+\frac{1}{kE_k}\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]^2+\frac{1}{E_k}\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]\left[\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right]\right\}\\&+\frac{1}{15}\left(1-\frac{m^2}{E_k^2}\right)\left\{\frac{k^2}{E_k^2}\left(\frac{1}{k^2}+\frac{1}{2E_k^2}\right)\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]\left[\frac{1}{2E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{(E_k+k-m)^2}\right.\right.\\&\left.+\frac{1}{(E_k+k+m)^2}\right]+\frac{k^2}{E_k^2}\left(\frac{1}{k}+\frac{1}{E_k}\right)\frac{1}{(E_k+k)^2-m^2}+\frac{3}{2k^2E_k}\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]^2+\frac{1}{k}\frac{1}{(E_k+k)^2-m^2}\\&\times\left[\frac{4}{k^2}+\left(\frac{1}{k}+\frac{1}{E_k}\right)\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\right]+\frac{k}{E_k}\left[\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right]\\&\left.\times\left\{\left(\frac{1}{k^2}+\frac{1}{2E_k^2}\right)\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]+\frac{1}{2E_k}\left[\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right]\right\}\right\}\\&+\frac{k}{2E_k}\left[\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]\left\{\frac{1}{(E_k+k+m)^2}\left[\frac{1}{k^2}+\frac{2}{E_k(E_k+k+m)}\right]+\frac{1}{(E_k+k-m)^2}\left[\frac{1}{k^2}+\frac{2}{E_k(E_k+k-m)}\right]\right\}\\&+\frac{1}{2}\frac{1}{(E_k+k)^2-m^2}\left\{\frac{1}{k}\left[\frac{2}{k^2}+\frac{1}{E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\right]+\frac{2}{E_k}\frac{1}{(E_k+k)^2-m^2}+\frac{2}{E_k+k+m}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k+m)}\right]\right.\\&\left.+\frac{2}{E_k+k-m}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k-m)}\right]\right\}+\frac{1}{E_k}\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right)\left[\frac{1}{(E_k+k+m)^3}+\frac{1}{(E_k+k-m)^3}\right]\\&+\frac{1}{k}\frac{1}{(E_k+k)^2-m^2}\left[\frac{1}{(E_k+k+m)^2}+\frac{1}{(E_k+k-m)^2}\right]\right\}+\frac{1}{15E_k^3}\left(1+\frac{k^2}{2E_k^2}\right)\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right)^2\end{aligned} \tag{220}$$

and

$$I_{ss}^2=-\frac{2}{5E_k^2}\left(1+\frac{m^2}{3E^2}\right)\left[\frac{1}{2E_k}\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right)^2+\frac{1}{k[(E_k+k)^2-m^2]}\right]+\frac{1}{15E_k^2}\left\{\frac{k}{E_k}\left(1+\frac{k}{E_k}\right)\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right)\right.$$
$$\left.\times\left(\frac{1}{(E_k+k+m)^2}+\frac{1}{(E_k+k-m)^2}\right)+\frac{1}{(E_k+k)^2-m^2}\left[\frac{3}{k}+\left(1+\frac{k}{E_k}\right)\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right)\right]\right\}. \tag{221}$$

The inverse linear singularity in the spin-orbit correction corresponds to the subtraction of the lower-order nonrelativistic contribution. However, the inverse linear singularity in the spin-spin correction cancels that from the ladder diagrams to be analyzed in the following. Such cancellation shows that there is no lower-order nonrelativistic contribution of spin-spin type.

The ladder diagrams contribute to the helium fine structure with

$$\Delta E_{-+}^{T\cdot T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\alpha_1^j\alpha_2^i\Lambda_{2+}(\mathbf{p}_2+\mathbf{k})\alpha_2^jI|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{222}$$

where

$$I=\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k+m}\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'-m}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k'+m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k-m}\right]$$
$$+\frac{1}{k+k'}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k'+m}\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'-m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k+m}\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k-m}\right]. \tag{223}$$

After nonrelativistic expansion, we get

$$I=I_0+I_1+I_2, \tag{224}$$

where

$$I_0=\frac{1}{(E_k+k)^2-m^2}\left[\frac{1}{k}+\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right],$$

$$I_1=\frac{1}{(E_k+k)^2-m^2}\left[\frac{1}{E_k}\left(\frac{1}{k}+\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\left(\frac{\mathbf{p}_1\cdot\mathbf{k}}{E_k+k+m}-\frac{\mathbf{p}_2\cdot\mathbf{k}}{E_k+k-m}\right)+\frac{\mathbf{p}_1\cdot\mathbf{k}}{E_k(E_k+k+m)^2}-\frac{\mathbf{p}_2\cdot\mathbf{k}}{E_k(E_k+k-m)^2}\right]$$
$$+\frac{\mathbf{k}\cdot\mathbf{k}''}{k}\frac{1}{(E_k+k)^2-m^2}\left[\frac{1}{2k^2}+\left(\frac{1}{2k}+\frac{1}{E_k+k-m}\right)\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{(E_k+k+m)^2}\right],$$

$$I_2=\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{2k}\frac{1}{(E_k+k)^2-m^2}\left\{\frac{1}{E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\left[\frac{1}{E_k+k-m}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{(E_k+k+m)^2}\right.\right.$$
$$\left.\left.+\frac{1}{2k}\left(\frac{1}{k}+\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\right]+\frac{1}{E_k(E_k+k-m)(E_k+k+m)^2}+\frac{1}{E_k(E_k+k-m)^3}+\frac{1}{E_k(E_k+k+m)^3}\right\}$$
$$+\frac{(\mathbf{k}\cdot\mathbf{k})^2}{(E_k+k)^2-m^2}\left\{\frac{1}{2k(E_k+k-m)}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k-m)}\right]+\frac{1}{k^2(E_k+k+m)^2(E_k+k-m)}\right.$$
$$+\frac{1}{2k(E_k+k+m)^2}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k+m)}\right]+\frac{1}{4k^5}+\frac{1}{2k^3}\frac{1}{(E_k+k)^2-m^2}+\frac{1}{4k^4}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)$$
$$\left.+\frac{1}{4k^2}\frac{1}{E_k+k+m}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k+m)}\right]+\frac{1}{4k^2}\frac{1}{E_k+k-m}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k-m)}\right]\right\}. \tag{225}$$

Reducing the numerator and denominators leads to

$$\Delta E_{-+}^{T\cdot T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{4}\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+(I_{ss}^1+I_{ss}^2)\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{226}$$

where

$$I_{so}=\frac{1}{(E_k+k)^2-m^2}\left[\frac{1}{k}+\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]\left[-\frac{1}{3}\left(\frac{1}{m^2}+\frac{1}{E_k^2}\right)\left(1-\frac{m^2}{E_k^2}\right)-\frac{1}{2E_k^2}+\frac{k^2}{3E_k^2}\left(\frac{1}{2m^2}-\frac{1}{E_k^2}\right)\right]$$
$$+\frac{1}{2k^2m^3}-\frac{1}{6E_k}\frac{1}{(E_k+k)^2-m^2}\left[1-\frac{m^2}{E_k^2}+\frac{k^2}{E_k^2}\right]\left[\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}+\frac{1}{k}\right)\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\right.$$
$$\left.+\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right] \tag{227}$$

and

$$I_{ss}^1=\frac{1}{(E_k+k)^2-m^2}\left[\frac{1}{k}+\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right]\left[\frac{1}{3}\left(\frac{1}{2m^2}+\frac{1}{E_k^2}-\frac{1}{k^2}\right)\left(1-\frac{m^2}{E_k^2}\right)+\frac{1}{6E_k^2}-\frac{k^2}{3E_k^2}\left(\frac{1}{m^2}-\frac{1}{E_k^2}\right)\right]$$
$$+\frac{1}{6E_k}\frac{1}{(E_k+k)^2-m^2}\left[1-\frac{m^2}{E_k^2}+\frac{k^2}{E_k^2}\right]\left[\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}+\frac{1}{k}\right)\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{(E_k+k-m)^2}\right.$$
$$\left.+\frac{1}{(E_k+k+m)^2}\right]+\frac{1}{3k}\frac{1}{(E_k+k)^2-m^2}\left[1-\frac{m^2}{E_k^2}+\frac{2m^2k^2}{5E_k^4}\right]\left[\frac{1}{E_k+k-m}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{(E_k+k+m)^2}\right.$$
$$\left.+\frac{3}{2k}\left(\frac{1}{k}+\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\right]-\frac{1}{15}\frac{1}{(E_k+k)^2-m^2}\left(1-\frac{m^2}{E_k^2}\right)\left\{\frac{4}{k^2}\left(\frac{1}{k}+\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\right.$$
$$+\frac{3}{2E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\left(\frac{1}{k}+\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{E_k}\left[\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right]$$
$$+\frac{2}{k}\frac{1}{E_k+k-m}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{2}{k(E_k+k+m)^2}+\frac{2}{(E_k+k+m)^2(E_k+k-m)}+\frac{k}{(E_k+k+m)^2}$$
$$\times\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k+m)}\right]+\frac{k}{E_k+k-m}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k-m)}\right]+\frac{1}{k^3}+\frac{1}{k}\frac{1}{(E_k+k)^2-m^2}$$
$$+\frac{1}{2k^2}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{2}\frac{1}{E_k+k+m}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k+m)}\right]+\frac{1}{2}\frac{1}{E_k+k-m}\left[\frac{1}{k^2}+\frac{1}{E_k(E_k+k-m)}\right]$$
$$+\frac{k}{E_k}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)\left[\frac{1}{E_k-m}\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{(E_k+k+m)^2}\right]$$
$$+\frac{k}{E_k(E_k+k-m)(E_k+k+m)^2}+\frac{k}{E_k(E_k+k+m)^3}+\frac{k}{E_k(E_k+k-m)^3}+\frac{2}{(E_k+k+m)^3}+\frac{2}{(E_k+k-m)^2}$$
$$\left.\times\left(\frac{1}{E_k+k-m}+\frac{1}{E_k+k+m}\right)+\frac{1}{k}\left[\frac{1}{(E_k+k-m)^2}+\frac{1}{(E_k+k+m)^2}\right]\right\} \tag{228}$$

and

$$I_{ss}^2=+\frac{2}{5E_k^2}\left(1+\frac{m^2}{3E^2}\right)\left[\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}+\frac{1}{k}\right]\frac{1}{(E_k+k)^2-m^2}-\frac{1}{15E_k^2}\frac{1}{(E_k+k)^2-m^2}$$
$$\times\left\{\left[3+\frac{k^2}{E_k}\left(\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right)\right]\left(\frac{1}{k}+\frac{1}{E_k+k+m}+\frac{1}{E_k+k-m}\right)+\frac{2k}{(E_k+k)^2-m^2}\right.$$
$$\left.+k\left(2+\frac{k}{E_k}\right)\left[\frac{1}{(E_k+k+m)^2}+\frac{1}{(E_k+k-m)^2}\right]\right\}. \tag{229}$$

The pure singular spin-orbit terms $1/(2k^2)-1/(6k^2)$ in Eqs. (218) and (226) correspond to the subtraction of the lower-order terms in Eq. (6.8) of Ref. [4]. In Eq. (6.8) of Ref. [4], the first term gives $1/(2k^2)$ and the second term gives $-1/(3k^2)$. The total becomes $1/(6k^2)$ which is $2\frac{1}{4}(1/2k^2)-1/6k^2)$. All the logarithmic singular spin-spin terms in Eqs. (219) and (227) cancel out. The cancellation indicates that there is no nonrelativistic spin-spin contribution of order $\alpha^6 mc^2$, which agrees with the absence of spin-spin terms in Eq. (6.8) of Ref. [4]. Doubling the corrections in Eqs. (218) and (226), we obtain the total contribution due to the one pair. On computation, we get

$$\Delta E_{-+}^{TT}+\Delta E_{+-}^{TT} = \left(\frac{\alpha}{2\pi^2}\right)^2 \frac{1}{4}\int \frac{d\mathbf{k}}{k^2} d\mathbf{k}'' \langle \phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so} i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'') + I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{230}$$

where

$$I_{so}=\tfrac{1}{12}\ln 2+\tfrac{1}{6}\ln B+\tfrac{1}{8} \tag{231}$$

and

$$I_{ss}=\tfrac{13}{120}+\tfrac{2}{15}\ln B-\tfrac{1}{6}\ln 2. \tag{232}$$

Performing Fourier transform, we obtain

$$\Delta E_{-+}^{TT}+\Delta E_{+-}^{TT}=\alpha^7 mc^2\langle\phi_0|I_{so}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1) + I_{ss}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle, \tag{233}$$

where

$$I_{so}=2\ln 2+4\ln B+3 \tag{234}$$

and

$$I_{ss}=-\tfrac{13}{2}-8\ln B+10\ln 2. \tag{235}$$

Both logarithmic cutoff terms here cancel those from nonrelativistic contributions [Eqs. (319) and (320) in Ref. [1]] due to the one-pair double transverse photon exchange, although there are a great many individual terms contributing to the logarithmic cutoff. In fact, all terms in the nonrelativistic contribution in Eqs. (242) and (244) of Ref. [1] contribute to logarithmic singularity. The cancellation of the singularities provides an excellent check.

C. Two pairs

For two distinct particles, the energy correction due to crossed-ladder diagrams is derived also from Eq. (1). The corresponding times-order formula becomes

$$\Delta E_{--}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1 d\mathbf{p}_2 d^4k d^4k'}{(\omega^2-k^2+i\delta)(\omega'^2-k'^2+i\delta)}\tilde{\psi}(p_{1\mu}p_{2\mu})\alpha_1^i\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k})}{\mu_1 E+\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega-i\delta}\alpha_1^j\alpha_2^j \times\frac{\mathcal{L}_{2-}(\mathbf{p}_2+\mathbf{k}')}{\mu_2 E+\varepsilon_2(\mathbf{p}_2+\mathbf{k}')-\epsilon+\omega'-i\delta}\alpha_2^i\psi(p'_{1\mu}p'_{2\mu}). \tag{236}$$

Integration over the energy variables and neglecting the external potentials, we obtain

$$\Delta E_{--}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^j\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j I|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{237}$$

where

$$\begin{aligned}I=&-\left[\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k}')+k'+m}+\frac{1}{E(\mathbf{p}_2)+E(\mathbf{p}_2+\mathbf{k})+k+m}\right]\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k})+k+k'+E}\\&\times\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+k}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+k'}\right]+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k-k'}\\&\times\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+k}\frac{1}{E(\mathbf{p}_2)+E(\mathbf{p}_2+\mathbf{k})+k}+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2)-k-k'}\\&\times\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k}')+k'}\frac{1}{E(\mathbf{p}_2+\mathbf{k})+E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+k'},\end{aligned} \tag{238}$$

which reproduces that in Ref. [1]. To lowest order, the relativistic energy correction

$$\Delta E_{--}^{T\times T}=\alpha^5\mu^3c^2\langle\phi_0|I\delta(\mathbf{r})(1+\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle, \tag{239}$$

where

$$I=-\int_0^\infty dk\left(1+\frac{m_1}{E_1}\right)\left(1+\frac{m_2}{E_2}\right)\left[\frac{1}{m_1+m_2+2k+E_1+E_2}\left(\frac{1}{E_1+k+m_1}+\frac{1}{E_2+k+m_2}\right)^2+\frac{1}{k}\frac{1}{(E_1+k+m_1)(E_2+k+m_2)}\right]. \tag{240}$$

The times-order formula for the ladder energy correction is

$$\Delta E_{--}^{T\cdot T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{-2\pi i}\int\frac{d^4p_1 d\mathbf{p}_2 d^4k d^4k'}{(\omega^2-k^2+i\delta)(\omega'^2-k'^2+i\delta)}\widetilde{\psi}(p_{1\mu}p_{2\mu})$$
$$\times\alpha_1^i\frac{\mathcal{L}_{1-}(\mathbf{p}_1-\mathbf{k})}{\mu_1E+\varepsilon_1(\mathbf{p}_1-\mathbf{k})+\epsilon-\omega-i\delta}\alpha_1^j\alpha_2^i\frac{\mathcal{L}_{2-}(\mathbf{p}_2+\mathbf{k})}{\mu_2E+\varepsilon_2(\mathbf{p}_2+\mathbf{k})-\epsilon+\omega-i\delta}\alpha_2^j\psi(p'_{1\mu}p'_{2\mu}). \tag{241}$$

Performing integration over the energy variables and ignoring the external potentials, we obtain

$$\Delta E_{--}^{T\cdot T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\alpha_1^j\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^jI|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{242}$$

where

$$I=-\left[\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k})+k}+\frac{1}{E+(\mathbf{p}_2)+E(\mathbf{p}_2+\mathbf{k})+k}\right]\frac{1}{E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_2+\mathbf{k})+E}$$
$$\times\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+k'}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+k'+m}\right]+\frac{1}{E-E(\mathbf{p}_1)-E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')-k-k'}$$
$$\times\frac{1}{E(\mathbf{p}_1)+E(\mathbf{p}_1-\mathbf{k})+k}\frac{1}{E(\mathbf{p}_2+\mathbf{k})+E(\mathbf{p}_2+\mathbf{k}+\mathbf{k}')+k'}+\frac{1}{E-E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')-E(\mathbf{p}_2)-k-k'}$$
$$\times\frac{1}{E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_1-\mathbf{k}-\mathbf{k}')+k'}\frac{1}{E(\mathbf{p}_2)+E(\mathbf{p}_2+\mathbf{k})+k}, \tag{243}$$

which reproduces that in Ref. [1]. To lowest order, the ladder correction becomes

$$\Delta E_{--}^{T\cdot T}=\alpha^5\mu^3c^2\langle\phi_0|I\delta(\mathbf{r})(1-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle, \tag{244}$$

where

$$I=-\int_0^\infty dk\left(1+\frac{m_1}{E_1}\right)\left(1+\frac{m_2}{E_2}\right)\left[\frac{1}{m_1+m_2+E_1+E_2}\left(\frac{1}{E_1+k+m_1}+\frac{1}{E_2+k+m_2}\right)^2+\frac{1}{k}\frac{1}{(E_1+k+m_1)(E_2+k+m_2)}\right]. \tag{245}$$

In the case of helium, the two-pair corrections in Eqs. (240) and (245) reduce to

$$\Delta E_{--}^{T\times T}=\alpha^5mc^2\left[\frac{\pi}{2}-1-\frac{1}{2}\ln2+\ln\frac{B}{m}\right]\langle\phi_0|\delta(\mathbf{r})(1+\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle \tag{246}$$

and

$$\Delta E_{--}^{T\cdot T}=\alpha^5mc^2\left[\frac{\pi}{4}-\ln2+\ln\frac{B}{m}\right]\langle\phi_0|\delta(\mathbf{r})(1-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)|\phi_0\rangle, \tag{247}$$

which reproduce Sucher's results.

For the helium fine structure of order α^7mc^2, the crossed-ladder correction becomes

$$\Delta E_{--}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^j\Lambda_{1-}(\mathbf{p}_1-\mathbf{k}')\alpha_1^i\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^jI|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{248}$$

where

$$I=-\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k'+m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k+m}\right]\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+E(\mathbf{p}_2+\mathbf{k})+k+k'+2m}$$
$$\times\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k+m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'+m}\right]-\frac{1}{k+k'}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k+m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k+m}\right.$$
$$\left.+\frac{1}{E(\mathbf{p}_1-\mathbf{k}')+k'+m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'+m}\right]. \tag{249}$$

After nonrelativistic expansion, we get

$$I=I_0+I_1+I_2, \tag{250}$$

where

$$I_0=-\frac{1}{(E_k+k+m)^2}\left[\frac{1}{k}+\frac{2}{E_k+k+m}\right],$$
$$I_1=\frac{(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k}'')\cdot\mathbf{k}}{E_k(E_k+k+m)^3}\left[\frac{1}{k}+\frac{3}{E_k+k+m}\right]-\frac{3\mathbf{k}\cdot\mathbf{k}''}{k(E_k+k+m)^4}-\frac{\mathbf{k}\cdot\mathbf{k}''}{2k^2(E_k+k+m)^2}\left[\frac{1}{k}+\frac{2}{E_k+k+m}\right],$$
$$I_2=-\frac{\mathbf{p}_1\cdot\mathbf{k}\mathbf{k}\cdot\mathbf{k}''}{E_k^2(E_k+k+m)^4}\left[\frac{1}{k}+\frac{4}{E_k+k+m}\right]-\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{k^2E_k}\,\frac{1}{(E_k+k+m)^3}\left(\frac{1}{2k}+\frac{1}{E_k+k+m}\right)-\frac{3(\mathbf{k}\cdot\mathbf{k}'')^2}{2k(E_k+k+m)^4}$$
$$\times\left[\frac{1}{k^2}+\left(\frac{1}{k}+\frac{1}{E_k}\right)\frac{1}{E_k+k+m}\right]-\frac{9(\mathbf{k}\cdot\mathbf{k}'')^2}{2kE_k}\,\frac{1}{(E_k+k+m)^5}-\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{2k^2(E_k+k+m)^2}$$
$$\times\left[\frac{1}{2k^3}+\frac{2}{k^2(E_k+k+m)}+\left(\frac{1}{k}+\frac{1}{E_k}\right)\frac{1}{(E_k+k+m)^2}\right]. \tag{251}$$

Combining with nonrelativistic expansion of the numerator, we obtain

$$\Delta E_{--}^{T\times T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{4}\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{252}$$

where

$$I_{so}=\frac{1}{6m^2}\,\frac{1}{(E_k+k+m)^2}\left[\frac{1}{k}+\frac{2}{E_k+k+m}\right]\left(1+\frac{m}{E_k}\right)\left[1+\frac{2m}{E_k}-\frac{k^2}{E_k^2}\right] \tag{253}$$

and

$$I_{ss}=-\frac{1}{6m^2}\,\frac{1}{(E_k+k+m)^2}\left[\frac{1}{k}+\frac{2}{E_k+k+m}\right]\left\{\left(1+\frac{m}{E_k}\right)\left(-1+\frac{2m}{E_k}-\frac{2m^3}{E_k^3}\right)+\frac{2m^2}{k^2}\left(1+\frac{m}{E_k}\right)^2+\frac{k^2}{E_k^2}\left[1+\frac{m}{E_k}\left(1+\frac{2m^3}{5E_k^3}\right)\right]\right\}$$
$$+\frac{1}{(E_k+k+m)^2}\left(1+\frac{m}{E_k}\right)\left[1+\frac{m}{E_k}-\frac{2mk^2}{5E_k^3}\right]\left\{\frac{1}{E_k(E_k+k+m)}\left(\frac{1}{3k}+\frac{1}{E_k+k+m}\right)\right.$$
$$\left.+\frac{1}{k}\left[\frac{1}{2k^2}+\frac{1}{k(E_k+k+m)}+\frac{1}{(E_k+k+m)^2}\right]\right\}+2\frac{1}{(E_k+k+m)^2}\left(1+\frac{m}{E_k}\right)\left(\frac{1}{k}+\frac{2}{E_k+k+m}\right)\left[\frac{1}{5k^2}\left(1+\frac{m}{E_k}\right)-\frac{m}{15E_k^3}\right]$$
$$-\frac{1}{15(E_k+k+m)^2}\left(1+\frac{m}{E_k}\right)^2\left\{\frac{k^2}{E_k^2(E_k+k+m)^2}\left[\frac{1}{k}+\frac{4}{E_k+k+m}\right]+\frac{k}{(E_k+k+m)^2}\left[\frac{1}{k}\left(\frac{10}{k}+\frac{15}{E_k}\right)\right.\right.$$
$$\left.\left.+\left(\frac{1}{k}+\frac{4}{E_k}\right)\frac{3}{E_k+k+m}\right]+\left[\frac{8}{k^3}+\frac{1}{k}\left(\frac{16}{k}+\frac{5}{E_k}\right)\frac{1}{E_k+k+m}\right]+\frac{1}{(E_k+k+m)^2}\left(\frac{8}{k}+\frac{5}{E_k+k+m}\right)\right\}. \tag{254}$$

It is interesting to note that the inverse square and inverse linear singularities in the spin-spin correction imply nonrelativistic contributions of orders α^5mc^2 and α^6mc^2. On the other hand, we know that no spin-dependent nonrelativistic correction appears to order α^5mc^2 or α^6mc^2 arising from the two-pair diagrams. Indeed, these singularities cancel out those from the ladder diagrams to be discussed in the following.

The ladder diagrams contribute to the fine structure with

$$\Delta E_{--}^{T\cdot T}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{2k}\frac{d\mathbf{k}'}{2k'}\langle\phi_c(\mathbf{p}_1,\mathbf{p}_2)|\alpha_1^i\Lambda_{1-}(\mathbf{p}_1-\mathbf{k})\alpha_1^j\alpha_2^i\Lambda_{2-}(\mathbf{p}_2+\mathbf{k})\alpha_2^j I|\phi_c(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{255}$$

where

$$I=-\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k+m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k+m}\right]\frac{1}{E(\mathbf{p}_1-\mathbf{k})+E(\mathbf{p}_2+\mathbf{k})+2m}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k'+m}+\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'+m}\right]$$
$$-\frac{1}{k+k'}\left[\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k+m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k'+m}+\frac{1}{E(\mathbf{p}_1-\mathbf{k})+k'+m}\,\frac{1}{E(\mathbf{p}_2+\mathbf{k})+k+m}\right]. \tag{256}$$

After nonrelativistic expansion, we get

$$I=I_0+I_1+I_2, \tag{257}$$

where

$$I_0=-\frac{1}{(E_k+k+m)^2}\left[\frac{1}{k}+\frac{2}{E_k+m}\right],$$

$$I_1=-\frac{(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{k}}{E_k(E_k+k+m)^2}\left[\frac{1}{E_k+k+m}\left(\frac{1}{k}+\frac{2}{E_k+m}\right)+\frac{1}{(E_k+m)^2}\right]-\frac{\mathbf{k}\cdot\mathbf{k}''}{k(E_k+k+m)^2}\left[\frac{1}{E_k+k+m}\left(\frac{1}{k}+\frac{2}{E_k+m}\right)+\frac{1}{2k^2}\right],$$

$$I_2=-\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{k^3(E_k+k+m)^3}\left(\frac{1}{k}+\frac{1}{E_k+m}\right)-\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{4k^5(E_k+k+m)^2}-\frac{(\mathbf{k}\cdot\mathbf{k}'')^2}{kE_k(E_k+k+m)^3}\left[\frac{1}{2k^2}+\frac{3}{E_k+k+m}\left(\frac{1}{2k}+\frac{1}{E_k+m}\right)+\frac{1}{(E_k+m)^2}\right]. \tag{258}$$

Reducing the numerator and denominators leads to

$$\Delta E_{--}^{T\cdot T}=\left(\frac{\alpha}{2\pi^2}\right)^2\frac{1}{4}\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_2\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{259}$$

where

$$I_{so}=-\frac{1}{6m^2}\frac{1}{(E_k+k+m)^2}\left[\frac{1}{k}+\frac{2}{E_k+m}\right]\left\{\left(1+\frac{m}{E_k}\right)\left(-2+\frac{3m}{E_k}-\frac{2m^3}{E_k^3}\right)-\frac{k^2}{E_k^2}\left[1-\frac{m}{E_k}\left(1+\frac{2m}{E_k}\right)\right]\right\}$$
$$+\frac{1}{3E_k(E_k+k+m)^2}\left[\frac{1}{E_k+k+m}\left(\frac{1}{k}+\frac{2}{E_k+m}\right)+\frac{1}{(E_k+m)^2}\right]\left(1+\frac{m}{E_k}\right)\left[1+\frac{m}{E_k}-\frac{k^2}{mE_k}\right] \tag{260}$$

and

$$I_{ss}=-\frac{1}{6m^2}\frac{1}{(E_k+k+m)^2}\left[\frac{1}{k}+\frac{2}{E_k+m}\right]\left\{\left(1+\frac{m}{E_k}\right)\left[1-\frac{2m}{E_k}\left(1-\frac{m^2}{E_k^2}\right)\right]-\frac{2m^2}{k^2}\left(1+\frac{m}{E_k}\right)^2+\frac{k^2}{E_k^2}\left[2-\frac{m}{E_k}\left(1+\frac{2m}{E_k}\right)\right]\right\}$$
$$-\frac{1}{3E_k(E_k+k+m)^2}\left[\frac{1}{E_k+k+m}\left(\frac{1}{k}+\frac{2}{E_k+m}\right)+\frac{1}{(E_k+m)^2}\right]\left(1+\frac{m}{E_k}\right)\left[1+\frac{m}{E_k}-\frac{k^2}{mE_k}\right]$$
$$-\frac{1}{3k(E_k+k+m)^2}\left[\frac{1}{E_k+k+m}\left(\frac{1}{k}+\frac{2}{E_k+m}\right)+\frac{3}{2k^2}+\frac{2}{k(E_k+m)}\right]\left(1+\frac{m}{E_k}\right)\left[1+\frac{m}{E_k}-\frac{2mk^2}{5E_k^3}\right]-2\frac{1}{(E_k+k+m)^2}\left(1+\frac{m}{E_k}\right)$$
$$\times\left(\frac{1}{k}+\frac{2}{E_k+m}\right)\left[\frac{1}{5k^2}\left(1+\frac{m}{E_k}\right)-\frac{m}{15E_k^3}\right]+\frac{2}{15k(E_k+k+m)^2}\left(1+\frac{m}{E_k}\right)^2\left[\frac{1}{k}\left(\frac{4}{k}+\frac{5}{E_k+m}\right)+\frac{1}{E_k+k+m}\left(\frac{3}{k}+\frac{5}{E_k+m}\right)\right]$$
$$+\frac{2}{15E_k(E_k+k+m)^2}\left(1+\frac{m}{E_k}\right)^2\left[\frac{2}{E_k+k+m}\left(\frac{5}{4k}+\frac{2}{E_k+m}\right)+\frac{E_k}{(E_k+k+m)^2}\left(\frac{1}{k}+\frac{2}{E_k+m}\right)+\frac{2}{(E_k+m)^2}\right.$$
$$\left.+\frac{k}{(E_k+m)(E_k+k+m)}\left(\frac{3}{E_k+k+m}+\frac{1}{E_k+m}\right)+\frac{3}{2(E_k+k+m)^2}\right]. \tag{261}$$

As we noted earlier, nonlogarithmic singularities in the above spin-spin correction cancel those from the crossed-ladder diagrams. Computing and combining the corrections in Eqs. (252) and (259), we obtain

$$\Delta E^{TT}_{--}=\left(\frac{\alpha}{2\pi^2}\right)^2\int\frac{d\mathbf{k}}{k^2}d\mathbf{k}''\langle\phi_0(\mathbf{p}_1,\mathbf{p}_2)|I_{so}i\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1\times\mathbf{k}'')+I_{ss}\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\boldsymbol{\sigma}_1\cdot\mathbf{k}''|\phi_0(\mathbf{p}_1-\mathbf{k}'',\mathbf{p}_2+\mathbf{k}'')\rangle, \tag{262}$$

where

$$I_{so}=-\tfrac{1}{3}-\tfrac{1}{8}\ln B+\tfrac{1}{48}\pi+\tfrac{7}{24}\ln 2 \tag{263}$$

and

$$I_{ss}=\tfrac{55}{240}-\tfrac{1}{960}\pi+\tfrac{3}{240}\ln B-\tfrac{77}{240}\ln 2. \tag{264}$$

Taking Fourier transform, we get

$$\Delta E^{TT}_{--}=\alpha^7mc^2\langle\phi_0|I_{so}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)+I_{ss}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle, \tag{265}$$

where

$$I_{so}=-8-3\ln B+\tfrac{1}{2}\pi+7\ln 2 \tag{266}$$

and

$$I_{ss}=-\tfrac{55}{4}+\tfrac{1}{16}\pi-\tfrac{3}{4}\ln B+\tfrac{77}{4}\ln 2. \tag{267}$$

The logarithmic cutoff term in the spin-orbit correction cancels that in Eq. (322) of Ref. [1] due to nonrelativistic approximation. The spin-spin logarithmic cutoff term cancels that from the nonrelativistic contribution in Eq. (323) in Ref. [1]. An additional check is provided between the no-pair and two-pair double transverse photon corrections. Replacing m by $-m$ in the no-pair kernel reproduces the correct result for the two-pair diagrams and *vice versa*.

V. RESULTS AND CONCLUSIONS

Summing all the corrections given in Eqs. (58), (97), (132), (159), (199), (233), and (265), we obtain

$$\Delta E=\alpha^7mc^2\langle\phi_0|I_{so}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)+I_{ss}\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}|\phi_0\rangle, \tag{268}$$

where

$$I_{so}=-\tfrac{3}{4}\pi-\tfrac{11}{3}+3\ln B+3\ln 2 \tag{269}$$

and

$$I_{ss}=-\tfrac{5}{2}\pi-\tfrac{80}{3}+\tfrac{27}{2}\ln 2+\tfrac{35}{2}\ln B. \tag{270}$$

The nonrelativistic contribution presented in Ref. [1] is

$$\Delta E=\alpha^7mc^2\left[9\left(\frac{R_{so}}{4\pi}+L_{so}\right)-\frac{15}{2}\left(\frac{R_{ss}}{4\pi}+L_{ss}\right)\right]+\alpha^7mc^2\langle\phi_0|O_{so}+O_{ss}|\phi_0\rangle, \tag{271}$$

where

$$O_{so}=-2Z\ln(Z\alpha)^{-2}\delta(\mathbf{r}_1)\frac{1}{r_1^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1\times\mathbf{p}_1)+\left(-9\ln\alpha-3\ln B+9\ln 2-\frac{221}{12}\right)\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)+\frac{8i}{9}\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\mathbf{p}_1)\mathbf{p}_2] \tag{272}$$

and

$$O_{ss}=\left(\frac{15}{2}\ln\alpha-\frac{35}{2}\ln B-\frac{9}{2}\ln 2-\frac{1555}{96}\right)\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+\frac{2i}{9}\frac{\delta(\mathbf{r})}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}(\boldsymbol{\sigma}_2-3\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\cdot\mathbf{p}_1. \tag{273}$$

The logarithmic cutoff terms are seen to cancel out between the relativistic and nonrelativistic contributions as they must. In fact, there are 14 individual cancellations as we demonstrated earlier. These cancellations provide a good check for the calculation. In addition, nonlogarithmic singularities are shown to cancel out the corresponding nonrelativistic contributions of lower order. Such systematic cancellations of singularities at operator level (not numerical level) have never been demonstrated explicitly in a high-order calculation for any bound-state system. This is an important procedure for minimizing the calculational error in increasingly sophisticated higher- and higher-order QED calculations.

Combining the relativistic contributions with the nonrelativistic contributions, we obtain

$$\Delta E=\alpha^7mc^2\left[9\left(\frac{R_{so}}{4\pi}+L_{so}\right)-\frac{15}{2}\left(\frac{R_{ss}}{4\pi}+L_{ss}\right)\right]+\alpha^7mc^2\langle\phi_0|O_{so}+O_{ss}|\phi_0\rangle, \tag{274}$$

where

$$O_{so}=-2Z\ln(Z\alpha)^{-2}\delta(\mathbf{r}_1)\frac{1}{r_1^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1\times\mathbf{p}_1)+\left(-9\ln\alpha+12\ln 2-\frac{265}{12}-\frac{3}{4}\pi\right)\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\mathbf{p}_1)+\frac{8i}{9}\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\mathbf{p}_1)\mathbf{p}_2] \tag{275}$$

and

$$O_{ss}=\left(\frac{15}{2}\ln\alpha-\frac{5}{2}\pi+9\ln 2-\frac{4115}{96}\right)\delta(\mathbf{r})\frac{1}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+\frac{2i}{9}\frac{\delta(\mathbf{r})}{r^2}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}(\boldsymbol{\sigma}_2-3\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})\cdot\mathbf{p}_1. \tag{276}$$

The first two spin-orbit and first spin-spin operators above have been calculated by Drake and Yan [13,14]. The electron-electron part gives $\nu_{01}=40.6$ kHz and $\nu_{12}=-16.3$ kHz. The revised helium fine structure splittings of $1s2p$ state [13,14] then become $\nu_{01}=296\,169\,74.1$ kHz and $\nu_{12}=229\,117\,9.9$ kHz. Calculation of the other nonrelativistic operators is in progress. However, the additional contribution is expected to be small. For the spin-orbit part, the last correction in Eq. (275) would be less than 0.5 kHz if the expectation value of its operator were the same as that of the second operator. For the spin-spin terms, the last two corrections in Eq. (276) would be no more than 0.2 kHz assuming their operators are identical to the first one in Eq. (276). A recent experiment by Shiner's group [7] gives $\nu_{01}=296\,169\,62(3)$ kHz and $\nu_{12}=229\,117\,4(3)$ kHz. Reasonable good agreement between theory and experiment is found, given the magnitude of order for uncalculated corrections to the order of interest. A precise comparison with experiments cannot be made until all corrections are included. First, nonlogarithmic terms of second order also need to be included. Second, nonlogarithmic terms in radiative corrections [8] need to be calculated numerically, which are expected to be around 10 kHz. Finally, effects of the nuclear motion corrections need to be included. The two-body part of the nuclear motion effects was derived and presented in Ref. [1]. However, our analysis indicates that some three-body terms of order $\alpha^6m^2c^2/M$ might give a contribution of a few kHz. These three-body terms cannot be calculated in the current two-body external-field formalism. Recently, a three-body formalism has been developed as an extension of the two-body Bethe-Salpeter formalism with which three-body corrections of lowest order are derived. The details of this three-body formalism and the calculation of the true three-body corrections will be presented elsewhere, along with recalculation of the two-body-plus-Coulomb-field corrections evaluated previously [1] using the external-potential Bethe-Salpeter two-body formalism. This recalculation not only verifies the correctness of our three-body formalism and of the external-potential two-body formalism to the order of interest, but also provides a beautiful physical picture or interpretation for those two-body-plus-Coulomb-field terms, which is totally unclear within the external-potential two-body formalism.

In this paper, we presented the calculation of the off-leading-order relativistic contributions in helium. Such calculation has not been carried out in any other bound-state system. A test of these corrections is very important because the off-leading-order effects are characteristic of bound-state physics in comparison with free particle systems. It is even more interesting that such tests can only be provided by helium at the moment. Tests of the off-leading-order corrections arising from the relativistic momentum region cannot be carried out in hydrogen, positronium, or muonium due to the lack of experimental accuracy.

We have reformulated the times-order external-potential Bethe-Salpeter formalism in a form more suitable for calculation of energy corrections arising from the relativistic momentum region. The essential difference between the current formalism and Sucher's formalism is that we obtain all relativistic kernels directly from the scattering theory and the Brillouin-Wigner perturbation method is used in Sucher's formalism to derive both nonrelativistic and relativistic kernels. In order to compare the Brillouin-Wigner perturbation theory with our method, we recalculated the helium energy levels of order α^5mc^2 as well as the $O(\alpha^7mc^2)$ no-pair Coulomb correction to helium fine structure. The calculation shows that the current method is much more convenient for the calculation of corrections arising from the relativistic momentum region. The calculation is carried out in times order. An explicitly covariant calculation of the leading-order corrections arising from the relativistic momentum region is also possible as presented in Ref. [16]. This formalism is similar to the one presented here since all kernels are obtained directly from the scattering amplitudes for free particle systems. In this calculation, the explicit covariance of all propagators is kept throughout the calculation. Coulomb and transverse photons are treated on equal footing. In the calculation, one needs to sandwich the covariant scattering amplitudes between the three-dimensional wave functions. The wave function may be written as

$$\psi(\mathbf{p})=\overline{u(\mathbf{p})}\Gamma v(\mathbf{p})f(p), \tag{277}$$

where u and v are the Dirac spinors and Γ corresponds to the angular part of the wave function. $f(p)$ is the radial wave function. Note that in either the Brillouin-Wigner or the Salpeter perturbation theory, the Breit corrections have to be subtracted in order to prevent singularity. On the other hand, the $O(\alpha^5mc^2)$ corrections to the hyperfine structure in positronium due to two covariant photons exchanged were calculated in the explicitly covariant approach in Ref. [16] without subtracting the Breit corrections. The result is

$$\Delta E=-\frac{16\alpha^2}{\pi Nm^2}\left[\int_0^\infty p^2f(p)dp\right]^2=-4\alpha^5mc^2\phi_0^2(0). \tag{278}$$

This result is reproduced by summing all the relevant corrections calculated by Karplus and Klein [17] and by Fulton and Martin [12], and is given by

$$\Delta E=-\alpha^5mc^2\langle\phi_0|\delta(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi_0\rangle, \tag{279}$$

which is the same as that in Eq. (278) for hyperfine structure. In deriving the above result by Sucher [3] as shown in this paper, photon propagators are expressed in Coulomb gauge and fermion propagators are expressed in terms of positive- and negative-energy projection operators. This times-order calculation is the least covariant one. The more covariant method is the one used by Karplus and Klein [17], by Fulton and Martin [12], and by Araki [2], in which the fermion propagators are kept covariant instead of times order. The result in Eq. (278) is obtained without breaking explicit covariance of both fermion and photon propagators. This most explicitly covariant method may only be applied to the calculation of the leading-order relativistic contributions such as contributions of orders α^5mc^2, α^6mc^2, and α^7mc^2, arising from two-, three-, and four-photon exchange diagrams, respectively, when all nonrelativistic variables can be neglected. So far, the two-photon diagrams are best understood in two-body bound-state study beyond tree level. This is because the two-photon corrections are the only ones that have been calculated in all three possible approaches. Since nonrelativistic contributions are calculated in times order within any two-body formalism, the difference between various formalisms is the calculation of relativistic contributions. The calculation of the leading-order relativistic contributions from two-photon diagrams has been carried out, as shown above, in all three possible ways from the most covariant one to the least covariant times-order form. The next-to-leading-order relativistic contributions from two-photon diagrams are calculated in the times-order manner as presented in this paper. It would be interesting to calculate these contributions using the other two methods. However, such calculation could be difficult since the fermion propagators are expressed covariantly while the four-dimensional wave functions are expressed in terms of positive or negative poles. To leading order, such calculation is relatively easy since all nonrelativistic variables are neglected. To higher order, one needs to separate effectively relativistic variables from nonrelativistic variables. In addition, there are two overall spin-orbit and spin-spin cancellations of logarithmic singularity to check the correctness of the calculation. In the times-order calculation, there are 14 additional individual cancellations to pinpoint possible missing terms in terms of the times-order diagrams for no-pair pure single transverse photon exchange; no-pair, one-pair, and two-pair single transverse photon exchange plus a Coulomb photon; and no-pair, one-pair, and two-pair double transverse photon exchange.

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