# Quantum tunneling in the Wigner representation

M. S. Marinov and Bilha Segev
*Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel*
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Time dependence for barrier penetration is considered in the phase space. An asymptotic phase-space propagator for nonrelativistic scattering on a one-dimensional barrier is constructed. The propagator has a form universal for various initial state preparations and local potential barriers. It is manifestly causal and includes time-lag effects and quantum spreading. Specific features of quantum dynamics which disappear in the standard semiclassical approximation are revealed. The propagator may be applied to calculation of the final momentum and coordinate distributions, for particles transmitted through or reflected from the potential barrier, as well as for elucidating the tunneling time problem. [S1050-2947(96)04312-0]

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## I. INTRODUCTION

Observable properties of quantal systems, such as energy levels and transition probabilities, are mostly related to stationary states. Meanwhile, the time dependence of physical processes is also described by quantum theory and may be of considerable interest. An important class of effects is various barrier penetration (tunneling) processes. The transition probabilities are usually obtained by means of the time-independent (energy) methods, in particular, in the semiclassical approximation (see, e.g., in Refs. [1,2]). As soon as one gets the complete solution in the energy representation, the time evolution is obtained straightforwardly, in principle, in terms of the inverse Laplace transform. However, the evaluation of the large-time asymptotics may be an intricate job. A source of the trouble is in the very statement of the problem. If we insist that the particle was on one side of the barrier in the beginning, the state cannot be described by the plane wave, which is the eigenstate of the momentum operator. Thus the initial energy is never free of an uncertainty. The uncertainty may be made smaller if the particle is delocalized in space, so one has to start far enough from the barrier, and to detect the result long enough after the start in order to be sure that the particle has left the potential domain completely. It is clear, however, that the problem needs a special theoretical analysis.

The time dependence of tunneling processes has been attracting attention for decades. A controversial question is that of the tunneling time and the effect of causality on the particle propagation [3–12]. New experimental techniques enable detailed measurements performed on electromagnetic wave packets transmitted through optical or microwave analogs of quantum potential barriers [13–18] There are some theoretical concerns about the validity of the semiclassical approximation in processes, such as tunneling, which have no classical counterparts, because the effect of quantal fluctuations must get a proper account. Therefore a consistent time-dependent formalism for tunneling is hardly redundant.

Barrier penetration processes and their time dependence were investigated by numerical, experimental, and analytic methods in a number of works, e.g., [19–30]. The literature contains specific examples of barriers and wave-packet shapes. Our purpose was to consider a general case, with no assumptions on the initial state preparation or on the form of the (local) potential barrier. Barrier penetration is described sometimes by means of the imaginary time method [31–36] In the present approach, the process is described in real space-time, even though we exploit analytical properties in the complex energy plane, especially for causality arguments.

The method applied here is an investigation of the time evolution of the Wigner phase-space distribution [37]. In this way, we can consider any initial state, not necessarily pure, which is important for applications to experiments. Besides, cumbersome oscillations of the wave functions are not involved. The Wigner function was used successfully in many problems of quantum theory, and its properties were considered, e.g., in Refs. [38–47]. Carruthers and Zachariasen, in their review of quantum collision theory with phase-space distributions [38], considered cross sections for scattering processes in three dimensions, inclusive reactions within a second quantization approach, inclusive multiparticle production processes in the ultrarelativistic domain, and many other processes but discussed neither quantum jumps nor barrier penetration. Various other approaches have been tried recently [48–52] to describe quantum dynamics in terms of the Wigner function. The asymptotic analysis of Wigner functions in scattering is of particular importance [53]. The time evolution of the Wigner function was considered previously mainly within the semiclassical approximation [54–56]. As was shown by Berry [57], the semiclassical approximation for stationary Wigner functions describing bound states is given by the Airy function, its spread from the classical $\delta$ function being the first-order quantum effect. Propagation of wave packets was discussed in a number of works; it was noted in particular that the spreading in the coordinate is not a specific quantal effect [58]. Within the semiclassical approximation the quantal features of the long-time evolution were attributed to the interference between amplitudes corresponding to different classical paths [59]. Our point is to emphasize the difference between the genuine quantum dynamics and the semiclassical approximation for classically forbidden processes where no classical paths exist, so quantum dynamics is not reduced just to a smearing around classical paths, or to an interference between them.

The Wigner function was applied to tunneling by Balazs and Voros [60] for parabolic potential barrier. The equivalence of Wigner's integro-differential equation to Liouville's classical equation is in an apparent conflict to tunneling for that potential. The puzzle is solved as soon as it is realized that the initial Wigner function cannot be chosen arbitrarily, if the potential does not vanish asymptotically. Physically available states correspond to Wigner distributions extended in the momentum, so real classical trajectories transport the particle above the barrier. The Wigner function for the parabolic potential was constructed explicitly, and the transmission and reflection coefficients were obtained, within a qualitative picture of tunneling where the difference between classical and quantum mechanics lies mainly in the initial state preparation.

For sufficiently broad incident momentum distributions, the classical paths enable the particle to overcome the barrier, and the semiclassical approach can indeed be used [61,62]. However, if the barrier potential is localized, the incident wave packet (or the Wigner function) can be prepared with an arbitrarily definite energy, and no classical paths would be responsible for the tunneling. In that situation the barrier penetration must be a result of an essential difference between the quantum and classical dynamics.

Purely quantal effects and their role in scattering processes have also been considered in the phase-space formalism. In the ''Wigner-trajectory'' approach [8,63–65] each phase-space point in the initial Wigner distribution is propagating along a definite trajectory. If the third- and higher-order derivatives of the potential vanish, the Wigner trajectories coincide with the classical paths. Otherwise, the Wigner trajectories are defined with a modified ''quantum'' potential and are not classical. For a system in an energy eigenstate (i.e., in the stationary barrier problem), the time-shift invariance implies that the trajectories are the ''equi-Wigner curves'' which are lines of constant values of the Wigner function. That approach seems rather problematic [66,67]. The effective ''potential'' may be singular, the Liouville theorem is violated, and quantum jumps [68,69] can hardly be included. In another approach, quantum corrections to classical dynamics are interpreted as finite momentum jumps between classical paths [68]. Negative quasi-probabilities appear in the the calculation which distinguish the quantum treatment from the classical theory. The phase-space points are smeared to finite domains and do not propagate along continuous trajectories.

We consider the phase-space evolution kernel [70], which is the fundamental solution of the dynamical equation for the Wigner function. In classical theory, the evolution kernel is the fundamental solution of the Liouville equation and equals the $\delta$ function restricted to classical trajectories. In the semiclassical approximation in classically allowed regions one can get the Airy function. For the barrier penetration, an explicit expression is obtained and it is shown that it is not reduced to the semiclassical approximation. From that point of view, the barrier penetration is an essentially quantal process.

In Sec. II the phase-space propagator (the evolution kernel) is defined, and some of its properties are given. Section III shows the relation between the time evolution and the $S$-matrix formalism in the momentum representation. The large-time asymptotics for the space-time propagator is derived in Sec. IV. The result is an integral representation in terms of scattering amplitudes. The exact result for the narrow potential barrier is presented in Sec. V. The semiclassical approximation is considered in Sec. VI, and its validity is discussed. Some technical aspects are considered in the Appendixes: the accuracy of the large-time asymptotics, the projections into coordinate and into momentum space, and the exact result for the $\cosh^{-2}$ potential barrier. These exact results are compared to the various approximations, each of which is shown to be adequate for different parameters.

The units used in the paper are $\hbar=1$, and $2m=1$ for the particle mass.

## II. THE PHASE-SPACE PROPAGATOR

In general, any quantum state is described by a density matrix $\hat{\rho}$, which can be represented by its matrix elements, say, in the coordinate representation, $\langle q'|\hat{\rho}|q\rangle$, or by its Weyl symbol (the Wigner function [37]) $\rho(x)$, where $x\equiv(q,p)$,

$$\rho(q,p)\equiv\int_{-\infty}^{\infty}\left\langle q+\frac{\eta}{2}\right|\hat{\rho}\left|q-\frac{\eta}{2}\right\rangle e^{-ip\eta}d\eta. \tag{1}$$

For a given Hamiltonian $\hat{H}$, the evolution operator $\hat{U}(t)\equiv\exp(-i\hat{H}t)$ determines the time evolution of the state: $\hat{\rho}_t=\hat{U}(t)\hat{\rho}_0\hat{U}^{\dagger}(t)$. Respectively, the time evolution of the Wigner function is given by an integral kernel, i.e., the phase-space propagator,

$$\rho_t(x)=\int L_t(x,x_0)\rho_0(x_0)dx_0. \tag{2}$$

Here $dx=dq\,dp$, and $L_t$ is a real function which satisfies the following identities:

$$L_t(q,p;q_0,p_0)=L_{-t}(q_0,-p_0;q,-p),$$

$$\int dxL_t(x,x_0)=1=\int dx_0L_t(x,x_0),$$

$$L_{t_1+t_2}(x,x_0)=\int dx'L_{t_2}(x,x')L_{t_1}(x',x_0). \tag{3}$$

For example, the probabilities to detect the system in the coordinate $q$ or in the momentum $p$ at a time $t$ after it was prepared in the state given by $\rho_0(x)$ are given, respectively, by the integrals

$$\mathcal{P}_t(q)=\int_{-\infty}^{\infty}dp\rho_t(q,p)$$
$$=\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dq_0\int_{-\infty}^{\infty}dp_0L_t(q,p;q_0,p_0)\rho_0(q_0,p_0), \tag{4}$$

$$\mathcal{P}_t(p)=\int_{-\infty}^{\infty}dq\rho_t(q,p)$$
$$=\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}dq_0\int_{-\infty}^{\infty}dp_0L_t(q,p;q_0,p_0)\rho_0(q_0,p_0). \tag{5}$$

The quasidistribution $\rho(x)$ is normalizable and satisfies a set of conditions owing to the positive definiteness of the density matrix, for instance,

$$\int [\rho(x)]^2 dx \leq \frac{1}{2\pi}\left[\int \rho(x)dx\right]^2. \qquad (6)$$

(The equality takes place if the state is pure.) Other conditions are more intricate. Qualitatively, the distribution cannot be localized to domains of areas less than $2\pi\hbar$ by the order of magnitude. The function is not necessarily positive everywhere, but the domains of negativity must be small enough.

The *phase-space propagator* $L_t$ is expressed in terms of the matrix elements of $\hat{U}(t)$, e.g., the coordinate (or momentum) propagators, as follows from (1),

$$\begin{aligned} L_t(q,p;q_0,p_0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\eta_0 e^{i(p\eta - p_0\eta_0)} \\ &\quad\times\left\langle q-\frac{\eta}{2}\right|\hat{U}(t)\left|q_0-\frac{\eta_0}{2}\right\rangle \\ &\quad\times\overline{\left\langle q+\frac{\eta}{2}\right|\hat{U}(t)\left|q_0+\frac{\eta_0}{2}\right\rangle} \\ &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\sigma_0 e^{i(q\sigma - q_0\sigma_0)} \\ &\quad\times\left\langle p+\frac{\sigma}{2}\right|\hat{U}(t)\left|p_0+\frac{\sigma_0}{2}\right\rangle \\ &\quad\times\overline{\left\langle p-\frac{\sigma}{2}\right|\hat{U}(t)\left|p_0-\frac{\sigma_0}{2}\right\rangle}. \end{aligned} \qquad (7)$$

For any Hamiltonian, quadratic in $x$, $\rho_t(x)$ is the solution of the *classical* Liouville equation. In that case the classical equations of motion are linear, as well as the Heisenberg equations. The solution is linear: $x = R_t x_0$, where $R_t$ is an $x$-independent matrix, and $L_t(x;x_0) = \delta(x - R_t x_0)$. In particular, for the nonrelativistic free motion one has

$$L_t(q,p;q_0,p_0) = \delta(p-p_0)\,\delta(q - vt - q_0), \qquad (8)$$

where $v = p/m$ is the particle velocity.

For nonlinear systems, effects specific for quantum dynamics result in deviations of $L_t$ from the $\delta$ function.

## III. TIME DEPENDENCE IN THE MOMENTUM REPRESENTATION

As soon as the Hamiltonian $\hat{H}$ has no explicit time dependence, the evolution operator can be written as the Laplace transform of the resolvent:

$$\hat{U}(t) \equiv e^{-it\hat{H}} = \frac{1}{2\pi i}\int_{\Gamma_\infty} \hat{G}_\varepsilon e^{-it\varepsilon} d\varepsilon, \quad \hat{G}_\varepsilon \equiv (\hat{H} - \varepsilon)^{-1} \qquad (9)$$

where $\Gamma_\infty$ is the usual integration contour in the complex $\varepsilon$ plane, running above the real axis.

We shall consider a nonrelativistic particle of mass $m = 1/2$ scattered from a localized one-dimensional and time-independent potential barrier. The Hamiltonian is $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0 = \hat{p}^2$ is the kinetic energy operator. The basis of the normalized momentum eigenstates $|k\rangle$ will be used, so

$$\hat{H}_0|k\rangle = k^2|k\rangle, \quad \langle k|k_0\rangle = \delta(k - k_0). \qquad (10)$$

Introducing the transition operator $\hat{T}_\varepsilon$, one has

$$\hat{G}_\varepsilon = \hat{G}_\varepsilon^{(0)} - \hat{G}_\varepsilon^{(0)}\hat{T}_\varepsilon\hat{G}_\varepsilon^{(0)}, \quad \hat{G}_\varepsilon^{(0)} \equiv (\hat{H}_0 - \varepsilon)^{-1}. \qquad (11)$$

The momentum propagator is now given by

$$\begin{aligned} \langle k|\hat{U}(t)|k_0\rangle &= \frac{1}{2\pi i}\int_{\Gamma_\infty} \langle k|\hat{G}_\varepsilon|k_0\rangle e^{-it\varepsilon} d\varepsilon \\ &= \delta(k-k_0)e^{-itk^2} - \frac{1}{2\pi i}\int_{\Gamma_\infty} d\varepsilon e^{-it\varepsilon} \\ &\quad\times\frac{\langle k|\hat{T}_\varepsilon|k_0\rangle}{(k^2 - i\gamma - \varepsilon)(k_0^2 - i\gamma - \varepsilon)}. \end{aligned} \qquad (12)$$

The infinitesimal positive quantity $\gamma$ is introduced here to specify the integral near the poles due to the free propagators. The integration contour $\Gamma_\infty$ may be deformed to run around the positive real axis, as the exponential vanishes in the lower half plane and $\langle k|\hat{T}_\varepsilon|k_0\rangle$ has singularities only for the real values of $\varepsilon$ corresponding to physical energy values. The kinematic poles are isolated, leaving an integral along the real positive axis,

$$\begin{aligned} \langle k|\hat{U}(t)|k_0\rangle &= \exp\left(-\frac{i}{2}t(k^2 + k_0^2)\right) \\ &\quad\times\left[\delta(k-k_0) - \frac{e^{it\xi}}{2\xi}\langle k|\hat{T}_\varepsilon|k_0\rangle|_{\varepsilon=k^2}\right. \\ &\quad\left. + \frac{e^{-it\xi}}{2\xi}\langle k|\hat{T}_\varepsilon|k_0\rangle|_{\varepsilon=k_0^2} - J_t(k,k_0)\right], \end{aligned} \qquad (13)$$

where $\xi = \frac{1}{2}(k_0^2 - k^2)$ and

$$J_t(k,k_0) = \frac{1}{\pi}\mathrm{P}\int_{\varepsilon_0}^{\infty} d\varepsilon e^{-it[\varepsilon - (1/2)(k^2 + k_0^2)]}\frac{\mathrm{Im}[\langle k|\hat{T}_\varepsilon|k_0\rangle]}{(\varepsilon - k^2)(\varepsilon - k_0^2)}. \qquad (14)$$

At two zeroes of the integrand denominator the integral is taken in the sense of its principal value. (Here $\varepsilon_0$ is a threshold energy value. With no bound states $\varepsilon_0 = 0$, otherwise it is the lowest boundstate energy.)

In order to calculate explicitly the time-dependent propagator for a given potential, the transition operator matrix elements for this potential should be known on and off the energy shell $k^2 = \varepsilon = k_0^2$. For scattering problems, one needs the large-time asymptotics, where the matrix elements are reduced to the energy shell because of the known fact of the theory of distributions,

$$\lim_{t\to\infty} e^{i\xi t}/\xi = i\pi\delta(\xi). \qquad (15)$$

The integral in (14) vanishes, as $t\to\infty$, because $\mathrm{Im}[\langle k|\hat{T}_\varepsilon|k_0\rangle]$ is smooth and the integral is converging. In general, $J_t = O(t^{-1/2})$. (The proof is given in Appendix A.)

The result is expressed in terms of a unitary $2\times 2$ matrix $S$,

$$\langle k|\hat{U}(t)|k_0\rangle \asymp e^{-i\kappa^2 t}2\kappa\delta(k^2-k_0^2)S_{\nu\nu_0}, \quad SS^\dagger = I, \quad (16)$$

where $\kappa=|k|=|k_0|$, $I$ is the unit $2\times 2$ matrix, and $\nu\equiv k/\kappa=\pm 1$. The $S$-matrix elements are related to elements of the $\hat{T}$ operator on the energy shell $\kappa^2\equiv\varepsilon$,

$$S_{\nu\nu_0}(\kappa)=\delta_{\nu\nu_0}-\frac{i\pi}{\kappa}\langle\nu\kappa|\hat{T}|\nu_0\kappa\rangle. \quad (17)$$

These are the probability amplitudes for transmission through and reflection from the potential region. The amplitudes can be expressed in terms of two analytical functions $a(\kappa)$ and $b(\kappa)$,

$$S(\kappa)=\frac{1}{a}\begin{pmatrix} 1 & b \\ -\overline{b} & 1\end{pmatrix}, \quad |a|^2-|b|^2=1. \quad (18)$$

These functions are defined for $\mathrm{Re}\kappa>0$ by the asymptotics of the solution of the stationary Schrödinger equation,

$$\hat{p}^2 y+Vy=\kappa^2 y,$$

$$y_-(q)\asymp\begin{cases} e^{-i\kappa q}, & q\to-\infty,\\ ae^{-i\kappa q}+be^{i\kappa q}, & q\to\infty.\end{cases} \quad (19)$$

The functions $a(\kappa)$ and $b(\kappa)$ have the analytical continuation to the left half plane by

$$a(-\overline{\kappa})=\overline{a(\kappa)}, \quad b(-\overline{\kappa})=\overline{b(\kappa)}. \quad (20)$$

The analytical properties of these functions in the complex $\kappa$ plane have been investigated previously [71]. It was shown, in particular, that for any finite-range and positive potential they can be expressed in terms of two *entire* functions $\alpha(\varepsilon)$ and $\beta(\varepsilon)$,

$$a(\kappa)\equiv 1-\alpha(\varepsilon)/2i\kappa, \quad b(\kappa)\equiv\beta(\varepsilon)/2i\kappa. \quad (21)$$

Thus the only singularities of $S(\kappa)$ are poles due to zeroes of $a(\kappa)$ which are all in the lower half of the $\kappa$ plane. Moreover, $\alpha$ and $\beta$ are bounded for $\mathrm{Im}\kappa>0$, so that $\lim a=1$ and $\lim b=0$, as $|\kappa|\to\infty$ in the upper half plane. If the potential has an exponential decrease as $q\to\pm\infty$, the functions $\alpha$ and $\beta$ may have infinite series of poles. Besides, for symmetric barriers, where $V(-q)=V(q)$, one has a real $\beta(\varepsilon)$ and purely imaginary $b(\kappa)=-b(-\kappa)$.

## IV. THE LARGE-TIME ASYMPTOTICS

In order to get the large-time asymptotics for the phase-space propagator, we set the amplitudes from (16) into Eq. (7). Using the following property of the $\delta$ function:

$$2|k|\delta(k^2-k_0^2)=\delta(k-k_0)+\delta(k+k_0), \quad (22)$$

one gets the result

$$L_t(q,p;q_0,p_0)\asymp\delta(p-p_0)\mathcal{T}(p_0,r_+)+\delta(p+p_0)\mathcal{R}(p_0,r_-) + \frac{2}{\pi}\mathrm{Re}\left[\frac{b(p_0-p)}{a(p_0-p)a(p_0+p)} \times e^{2i(q_0p-qp_0)+4ipp_0t}\right]. \quad (23)$$

Here $r_\pm=q_0+2p_0t\mp q$; i.e., the differences between the free classical trajectory and actual positions of the transmitted particle $(r_+)$ and the particle reflected from the origin $(r_-)$. The first two terms describe transmission and reflection from the potential barrier. The third term represents an interference between transmitted and reflected waves and is responsible for quantum fluctuations at the barrier region. It is irrelevant for large time $t$, if the wave packet was prepared in free space with a narrow momentum distribution [72].

The functions representing the transmission and reflection probabilities are given by the Fourier integrals

$$\mathcal{T}(p_0,r_+)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\sigma r_+}d\sigma}{a(\frac{1}{2}\sigma+p_0)a(\frac{1}{2}\sigma-p_0)}, \quad (24)$$

$$\mathcal{R}(p_0,r_-)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{b(\frac{1}{2}\sigma+p_0)b(\frac{1}{2}\sigma-p_0)e^{-i\sigma r_-}d\sigma}{a(\frac{1}{2}\sigma+p_0)a(\frac{1}{2}\sigma-p_0)} + \frac{1}{2\pi}\int_{2p_0}^{+\infty}\frac{B(p_0,\sigma)d\sigma}{a(\frac{1}{2}\sigma+p_0)a(\frac{1}{2}\sigma-p_0)}, \quad (25)$$

where

$$B(p_0,\sigma)\equiv[b(\tfrac{1}{2}\sigma+p_0)+b(-\tfrac{1}{2}\sigma-p_0)] \times[b(\tfrac{1}{2}\sigma-p)e^{-i\sigma r_-}+b(-\tfrac{1}{2}\sigma+p)e^{i\sigma r_-}]. \quad (26)$$

The result is obtained in the following way. We assume that $p_0>0$, so $L_t$ in Eq. (7) gets contributions from three segments in the $\sigma$ axis, which are proportional to products of the $S$-matrix elements, as follows:

$$\sigma\in(-\infty,-2p_0), \quad (-2p_0,2p_0), \quad (2p_0,+\infty),$$

$$\mathcal{T}\propto S_{--}\overline{S_{++}}, \quad S_{++}\overline{S_{++}}, \quad S_{++}\overline{S_{--}},$$

$$\mathcal{R}\propto, S_{-+}\overline{S_{+-}}, \quad S_{+-}\overline{S_{+-}}, \quad S_{+-}\overline{S_{-+}}. \quad (27)$$

The arguments are $p+\frac{1}{2}\sigma$ in the first factor and $p-\frac{1}{2}\sigma$ in the second factor. The $S$ matrix is given in Eq. (18), and the complex conjugation is taken on the real axis by means of Eq. (20), inverting the signs of the arguments. Thus the integral is given in terms of analytical functions with the general properties determined by the Schrödinger equation.

Evidently, $L_t$ is real and respects the reciprocity principle, due to the time-inversion invariance [cf. also Eq. (3)]:

$$L_t(x,x_0)=L_{-t}(x_0,x). \quad (28)$$

Note that the result is translationally invariant; namely, for $q\to q-c$ the quantities $r_+$ and $a(\kappa)$ remain invariant, while

$r_- \to r_- - 2c$ and $b(\kappa) \to b(\kappa)e^{2i\kappa c}$. The second integral in Eq. (25) vanishes, if $V(q) \equiv V(-q)$, since in that case $b(\kappa) = -b(-\kappa)$ and $B(p,\sigma) \equiv 0$. The *total* transmission and reflection probabilities (for a given initial momentum $p$) are given by integration,

$$T(p) = \int_{-\infty}^{+\infty} \mathcal{T}(p,r)dr = |a(p)|^{-2},$$

$$R(p) = \int_{-\infty}^{+\infty} \mathcal{R}(p,r)dr = |b(p)/a(p)|^2. \quad (29)$$

The integral representations in Eqs. (24) and (25) are manifestly causal. For $V(q) \geqslant 0$, it was proven [71] that the only singularities of the transmission amplitudes, and therefore of the integrand, are due to zeroes of $a(\kappa)$, which are all in the lower complex half plane. If $r_+ < 0$, i.e., if $q$ is ahead of the free propagation coordinate, the exponent for $\mathrm{Im}\sigma > 0$ is decreasing, so the integration contour may be deformed to the upper half plane and the only contribution would be from $\infty$ where $a \to 1$, leading to $\mathcal{T} = \delta(r_+)$, as in free motion (8). For $r_+ \geqslant 0$, the integral can be evaluated by means of residues in the lower half plane. Each zero of $a(\kappa)$ gives rise to a pair of poles in the integrand of (24) and (25). Thus $\mathcal{T}$ and $\mathcal{R}$ can be expressed as a sum over the $S$-matrix singularities. The transmission propagator is given by

$$\mathcal{T}(p,r_+) = \delta(r_+) - \theta(r_+) \sum_n \mathrm{Re}\{C_n(p) \exp[i2r_+(p-\kappa_n)]\}, \quad (30)$$

where $\kappa_n$ are zeroes of $a(\kappa)$ and $C_n(p)$ are determined by the corresponding residues, and $\theta(r)$ is the step function ($=0$ for $r<0$ and $=1$ for $r>0$). Two examples are given below: the $\delta$-potential barrier (Sec. V) and the modified Pöschl-Teller potential (Appendix C). In general, contributions from purely imaginary poles have the form of that for the $\delta$ barrier, Eq. (33). The functional dependence on the lag distance $r_+$ is universal, though $C_n$ and $\kappa_n$ depend on the potential. As $\mathrm{Im}\kappa_n < 0$, the second term in $\mathcal{T}(p,r_+)$ is an exponentially decreasing and oscillating function of $r_+$. Thus the transmitted Wigner function equals a freely propagating Wigner function modulated by exponentially decreasing and oscillating deformations. Due to the oscillations the deformations do not have a definite sign. However, their dominant contribution is negative because clearly the transmitted wave packet must be smaller than the freely propagating one. The result of superimposing slowly propagating destructive deformations on a freely propagating wave packet is primarily to remove from the wave packet portions of its delayed part.

Note that Eq. (30) was derived under the assumption $V(q) \geqslant 0$. In order to consider the case where $V(q) \leqslant 0$, in some regions, the influence of bound states must be considered.

## V. NARROW POTENTIAL BARRIER

Let us consider the $\delta$-potential barrier

$$V^\delta(q) = v_0 \delta(q), \quad (31)$$

for which the $S$-matrix elements are given by

$$a^\delta(\kappa) = 1 - v_0/2i\kappa, \quad b^\delta(\kappa) = v_0/2i\kappa. \quad (32)$$

There is one pair of poles in the integrands in (24) and (25), and the integrals are calculated by the residue theorem, leading to

$$\mathcal{T}^\delta(p,r_+) = \delta(r_+) - \theta(r_+) 2v_0 \sqrt{1+(v_0/4p)^2} e^{-v_0 r_+} \times \cos(2pr_+ + \gamma),$$

$$\mathcal{R}^\delta(p,r_-) = \theta(r_-) \frac{v_0^2}{2p} e^{-v_0 r_-} \sin(2pr_-), \quad (33)$$

where $\gamma = \arctan(v_0/4p)$. The total transmission probability, given by Eq. (29), is

$$T(p) = 1 - \frac{v_0^2}{v_0^2 + 4p^2}.$$

The $\delta$ potential has a simple explicit solution also off the energy shell,

$$\langle k | \hat{T}^\delta | k_0 \rangle = \frac{v_0}{2\pi(1 + v_0/2\sqrt{-\varepsilon})}, \quad (34)$$

independently of $k$ and $k_0$. Thus one can evaluate the nonasymptotic time dependence of Eq. (14),

$$J_t^\delta(k,k_0) = \frac{v_0^2}{4\pi\sqrt{-it}} e^{(i/2)t(k^2+k_0^2)} \left[ \frac{W(\frac{1}{2}v_0\sqrt{-it})}{(k^2+\frac{1}{4}v_0^2)(k_0^2+\frac{1}{4}v_0^2)} + \frac{1}{k^2-k_0^2} \left( \frac{W(\sqrt{itk^2})}{k^2+\frac{1}{4}v_0^2} - \frac{W(\sqrt{itk_0^2})}{k_0^2+\frac{1}{4}v_0^2} \right) \right], \quad (35)$$

where $W(z)$ is a function related to the probability integral [73],

$$W(z) = zw(z) \equiv ze^{-z^2}[1 - \mathrm{erf}(-iz)] \asymp \frac{i}{\sqrt{\pi}} \left( 1 + \frac{1}{2z^2} + O(z^{-4}) \right). \quad (36)$$

In the large-time asymptotics, for $t \gg \max\{v_0^{-2}, k_0^{-2}, k^{-2}\}$, $J_t^\delta$ vanishes as $t^{-3/2}$, more rapidly than in the general case since the transition matrix elements in Eq. (34) are insensitive to separation from the energy shell (cf. Appendix A). The result is

$$J_t^\delta = \frac{\exp[(i/2)(k^2+k_0^2)t]}{2(i\pi t)^{3/2} k^2 k_0^2} + O(t^{-5/2}). \quad (37)$$

As can be shown by means of the double Fourier transform in $k$ and $k_0$, the expression in Eq. (35) agrees with the evolution kernel for the $\delta$ potential in the coordinate representation, obtained previously [21].

## VI. SEMICLASSICAL APPROXIMATION

The transmission amplitude can be expressed as an exponent: $a(\kappa)\equiv\exp[iS(\kappa)]$. The transmission propagator of Eq. (24) is then given by

$$\mathcal{T}(p,r)=\frac{1}{\pi}\int_0^\infty d\sigma\cos\left[\sigma r+S\left(p+\frac{\sigma}{2}\right)+S\left(-p+\frac{\sigma}{2}\right)\right]. \tag{38}$$

Note that $S(-\kappa)=-S(\kappa)$, by virtue of the analytical continuation.

In the semiclassical approximation one has the eikonal formula,

$$S(\kappa)=\kappa\int_{-\infty}^{+\infty}[1-\sqrt{1-\kappa^{-2}V(q)}]dq. \tag{39}$$

Is this approximation adequate for the calculation of $\mathcal{T}(p,r)$? We will now show that while it is adequate for the calculation of the total transmission probability and for transmission above the barrier, it cannot be applied uniformly for all values of $r$. In particular, it fails to give the correct space-time dependence in the deep-tunneling regime.

The total transmission probability is obtained by integrating $\mathcal{T}(p,r)$ in $r$. At large $r$ the vicinity of $\sigma=0$ is dominating in the integral, so one can use Eqs. (38) and (39) and apply a series expansion in the exponent. For $p^2<V_0\equiv\max[V(q)]$,

$$S(p)+S(-p)=-2iI,\quad I\equiv\int\sqrt{V(q)-p^2}dq. \tag{40}$$

The integral is on the segment where $V(q)>p^2$, and $I$ is real. That leads to the familiar semiclassical expression for the transmission probability, $T=\exp(-2I)$.

In classically allowed processes, i.e., for transmission above the barrier, $p^2>V_0$. Applying Eq. (39) for these cases gives an approximation where $S$ is real and $|a|=1$, so the transmission is complete. Expanding the exponent around $\sigma=0$ and retaining terms up to the order of $\sigma^3$, one gets an Airy function for $\mathcal{T}(p,r)$. This is consistent with the semiclassical approximation for Wigner functions describing bound states which are also Airy functions [57].

The integral in Eq. (38) can sometimes be calculated by the stationary-phase method. Positions of the critical points of the integrand are given by the equation

$$2r_++S'(\tfrac{1}{2}\sigma+p)+S'(\tfrac{1}{2}\sigma-p)=0.$$

For positive $r_+$, the critical points of the exponent may take place at real $\sigma$, and their positions depend on the magnitudes of $r_+$ and $p$. If these critical points appear in the region where the semiclassical approximation is valid, one can make use of Eq. (39),

$$S'(\kappa)=\int_{-\infty}^{+\infty}\left[1-\frac{\kappa}{\sqrt{\kappa^2-V(q)}}\right]dq. \tag{41}$$

Fairly accurate results in some domains in the $(p_0,r_+)$ plane are then obtained in the framework of the WKB approach.

For $\kappa^2>V_0$ Eq. (41) has a classical meaning; namely, $S'(\kappa)=-v(\tau_V-\tau_0)$, where $v\equiv2\kappa$ is the initial particle velocity, and $\tau_V$ and $\tau_0$ are the times of flight in the presence of the potential barrier and in free space. Hence in cases of transmission above the barrier Eq. (38) can be interpreted in the following way. Treating $\sigma$ as a quantum fluctuation of the incident momentum, one can note that $r_+$ equals the lag due to the barrier, averaged between two classical trajectories, their mean momentum being equal to $p$.

This approximation scheme fails for negative values, as well as for positive but small values of $r$.

It was shown in Sec. IV that for positive potentials the principle of causality manifests itself in the fact that $\mathcal{T}(p,r)\equiv0$ for $r<0$. This property, which was proven by the exact quantum treatment, is violated when Eq. (38) is substituted in Eq. (39) as demonstrated for an example in Appendix C. The discrepancy is due to the fact that the analytical properties of the quantum transmission amplitude are not preserved by the eikonal formula. Unlike the exact solution, the approximate function $a(\kappa)$ of Eq. (39) is not meromorphic, since $S(\kappa)/\kappa$ has a branch point at $\kappa^2=V_0$. The violation of causality here is an artifact of a wrong semiclassical approximation.

At positive, small $r_+$ the region of large $|\sigma|$ contributes substantially, and one may not expand around $\sigma=0$. Here too an Airy function approximation would fail. However, the high-energy asymptotics can be used, where $\kappa\to\infty$, $a(\kappa)\sim1$, and

$$S(\kappa)\asymp\frac{v_0}{2\kappa},\quad v_0\equiv\int_{-\infty}^{+\infty}V(q)dq. \tag{42}$$

Consider, for example, the domain where

$$r_+p\gg1,\quad p/v_0\ll1,\quad r_+p^2/v_0\ll1. \tag{43}$$

The inequalities mean that (i) the domain is consistent with the uncertainty relation, (ii) the process is a deep under-barrier tunneling, and (iii) the asymptotics of Eq. (42) can be used. The corresponding critical points are given by

$$\sigma_0^2=2v_0/r_+\gg p^2. \tag{44}$$

The contribution from the critical points $\sigma=\pm\sigma_0$ is

$$\mathcal{T}(p,r_+)=\frac{1}{\pi}\int_0^\infty\cos\left(r_+\sigma+\frac{2v_0}{\sigma}\right)d\sigma$$
$$\asymp\frac{(2v_0r_+)^{1/4}}{r_+\sqrt{\pi}}\cos\left(2\sqrt{2v_0r_+}+\frac{\pi}{4}\right). \tag{45}$$

This result is an alternative semiclassical approximation which is adequate for small positive lag distances in the deep-tunneling regime. It is quite different from the Airy function. Furthermore, the standard semiclassical approximation for the total transmission of Eq. (40) cannot be recovered from it by integration, which is hardly surprising, since the domain given by inequalities (43) is not dominating in tunneling as a whole. Rather, small $r_+$ represent the most rapid signal transport.

## VII. SUMMARY AND CONCLUSION

The phase-space propagator for a general local one-dimensional potential barrier has a universal form; it is given by a sum over singularities of the $S$ matrix. It conserves energy, is manifestly causal, and shows certain features specific for quantum theory.

At large times, the initial Wigner function is split into three parts: the reflected bundle, the transmitted bundle, and a transient group which can be neglected if the incident particle was prepared with a narrow momentum distribution. The resulting probabilities have a momentum dependence, which causes the known effects, such as dispersion and forward attenuation. Asymptotically, after the particle leaves the interaction region, the time evolution of the Wigner functions is just the coordinate translation with constant classical velocities. As different parts of the Wigner function are translated with different velocities, the evolution of the phase-space distribution goes on, and the actual experiment results may depend on the detector position, in particular, on its distance from the barrier region.

The coordinate dependence of the propagators is universal for any local potential. The propagators are functions of the lag distance, which is the difference between the free motion and the current coordinate. At large lag distances, the probability has an exponential decrease and is oscillating as a result of quantum interference effects. The decrease rate and the oscillation frequency are determined by positions of poles of the transition amplitudes in the complex energy plane. The coefficients and the phase shifts depend on details of the barrier shape and on the initial momentum. Of course, the phase-space propagator is not a $\delta$ function. Each point in the initial Wigner function gives rise to disconnected domains in the final distribution. This is a purely quantal effect in the barrier penetration dynamics.

The standard semiclassical approximation can be used for evaluation of the total transmission probability, obtained by integration of the wave packet outside the potential domain. It is hardly adequate, however, for a description of the time dependence of the process, since the analytical properties representing the causality in the energy representation are not maintained properly.

In contrast to classical theory, individual phase-space trajectories cannot be traced in quantum theory. The principle of causality is not violated by quantum theory, and no information can be transported faster because of the barrier. The problem of the signal transport and the time delay is planned to be the subject of a future paper [74].

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## APPENDIX A: OFF-ENERGY-SHELL CORRECTIONS IN THE LARGE-$t$ ASYMPTOTICS

The transition operator of Eq. (11) has the following general property:

$$\hat{T}-\hat{T}^{\dagger}=\hat{T}[\hat{G}_{\varepsilon}^{(0)}-(\hat{G}_{\varepsilon}^{(0)})^{\dagger}]\hat{T}^{\dagger}, \tag{A1}$$

which leads, in particular, to the $S$ matrix unitarity. Hence one has for the discontinuity of the matrix element in Eq. (14)

$$\mathrm{Im}\langle k|\hat{T}|k_0\rangle=\frac{\pi}{2\kappa}D(\kappa;k,k_0),$$

$$D(\kappa;k,k_0)=\sum_{\nu=\pm1}\langle k|\hat{T}|\nu\kappa\rangle\overline{\langle k_0|\hat{T}|\nu\kappa\rangle}, \tag{A2}$$

where $\kappa=\varepsilon^{1/2}$. As a function of $\kappa$, $D$ is regular. Besides, it has the following general property of positive definiteness:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}D(\kappa;k,k_0)f(k)\overline{f(k_0)}dk_0dk\geqslant 0, \tag{A3}$$

for any integrable complex function $f(k)$. In particular, $D(\kappa;k,k)\geqslant 0$ for all $k$.

The ''half-on-shell'' transition amplitudes present in (A2) are expressed in terms of solutions $y_{\pm}(x)$ of the Schrödinger equation (19), satisfying the complementary asymptotical conditions, as given in Ref. [71],

$$\begin{aligned}\langle k|\hat{T}|\nu\kappa\rangle=&\frac{1}{8\pi\kappa a}\int_{-\infty}^{\infty}dx e^{-i(k-\nu\kappa)x}((\kappa^2-k^2)\\&\times(\eta_+'\eta_--\eta_+\eta_-')+V(x)\\&\times\{[(2+\nu)\kappa-k]y_+(x)e^{-i\kappa x}\\&+[(2-\nu)\kappa+k]y_-(x)e^{+i\kappa x}\}),\end{aligned} \tag{A4}$$

where $\eta_{\pm}(x)\equiv y_{\pm}(x)-e^{\pm i\kappa x}\rightarrow 0$ for $x\rightarrow\pm\infty$. It is evident from the integral representation that the transition amplitudes are regular functions of $\kappa$, and they are proportional to $\kappa$ at $k^2=\kappa^2$. (Note that $\kappa a$ is finite at $\kappa=0$.)

Finally, the off-energy-shell correction is given by the following equality (we assume, for simplicity, that there are no bound states):

$$\begin{aligned}J_t(k,k_0)=&\exp\left(\frac{i}{2}t(k^2+k_0^2)\right)\mathrm{P}\int_0^{\infty}e^{-it\kappa^2}\frac{D(\kappa;k,k_0)d\kappa}{(\kappa^2-k^2)(\kappa^2-k_0)}\\&\asymp O(t^{-1/2}).\end{aligned} \tag{A5}$$

The large-$t$ asymptotics of the integral is determined by the behavior of $D$ at $\kappa\rightarrow 0$.

## APPENDIX B: PROJECTIONS TO MOMENTUM SPACE AND TO COORDINATE SPACE

One can get a better understanding of the phase-space propagator by integrating in order to return either to the coordinate or the momentum representations. In particular, we will show that neglecting the interference term in the large-time asymptotics is consistent with the results of these projections.

Consider the phase-space propagator for a symmetric barrier, including the third term. [In this appendix $A(p)\equiv 1/a(p)$ and $B(p)\equiv b(p)/a(p)$]:

$$L_t(q,p;q_0,p_0)\asymp\delta(p-p_0)\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma e^{i\sigma(q_0+tp_0/m-q)}A\left(p_0-\frac{\sigma}{2}\right)A^*\left(p_0+\frac{\sigma}{2}\right)+\delta(p+p_0)\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma e^{i\sigma(q_0+tp_0/m+q)}$$
$$\times B\left(p_0-\frac{\sigma}{2}\right)B^*\left(p_0+\frac{\sigma}{2}\right)+\frac{1}{\pi}[e^{i2[qp_0-(q_0+tp_0/m)p]}B^*(p_0-p)A(p+p_0)+\text{c.c.}]. \tag{B1}$$

The probability of having a final momentum $p$ is

$$\mathcal{P}_t(p)=\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}dq_0\int_{-\infty}^{\infty}dp_0L_t(q,p;q_0,p_0)\rho_0(q_0,p_0)$$
$$=|A(p)|^2\int_{-\infty}^{\infty}dq_0\rho_0(q_0,p)+|B(-p)|^2\int_{-\infty}^{\infty}dq_0\rho_0(q_0,-p)+[B^*(-p)A(p)+B(-p)A^*(p)]\int_{-\infty}^{\infty}dq_0\rho_0(q_0,0), \tag{B2}$$

where the order of integration was changed, and use was made of the identity $\int dqe^{i\sigma q}=2\pi\delta(\sigma)$. If the initial Wigner function has a well defined momentum, then $\rho_0(q_0,0)=0$, and the last term vanishes, giving

$$\mathcal{P}_t(p)=|A(p)|^2\mathcal{P}_0(p)+|B(-p)|^2\mathcal{P}_0(-p). \tag{B3}$$

The probability of finding the particle in the final coordinate $q$ is given by

$$\mathcal{P}_t(q)=\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dq_0\int_{-\infty}^{\infty}dp_0L_t(q,p;q_0,p_0)\rho_0(q_0,p_0)$$
$$=\int dq_0dp_0\rho_0(q_0,p_0)\frac{1}{2\pi}\int d\sigma e^{i\sigma(q_f-q)}A\left(p_0-\frac{\sigma}{2}\right)A^*\left(p_0+\frac{\sigma}{2}\right)+\int dq_0dp_0\rho_0(q_0,p_0)\frac{1}{2\pi}\int d\sigma e^{i\sigma(q_f+q)}$$
$$\times B\left(p_0-\frac{\sigma}{2}\right)B^*\left(p_0+\frac{\sigma}{2}\right)+\int dq_0dp_0\rho_0(q_0,p_0)\left(\frac{e^{i2qp_0}}{2\pi}\int d(2p)e^{-i2pq_f}B^*(p_0-p)A(p+p_0)+\text{c.c.}\right),$$

$$q_f\equiv q_0+\frac{p_0}{m}t. \tag{B4}$$

If the time is large enough so that $q_f\to\infty$ for every $(q_0,p_0)$ for which $\rho_0(q_0,p_0)\neq 0$, the integrands oscillate rapidly, giving $\mathcal{P}_t(q)\to 0$ except for $q\sim q_f$ or $q\sim -q_f$. Again, the last term does not contribute.

## APPENDIX C: MODIFIED PÖSCHL-TELLER POTENTIAL BARRIER

The modified Pöschl-Teller (PT) potential barrier,

$$V^{\text{PT}}(q)=\frac{v_0/2s}{\cosh^2(q/s)}, \tag{C1}$$

where $v_0$ and $s$ are constants, is an example for a local potential barrier which enables an explicit solution and a test of the various approximations. The known solution [1] is

$$a^{\text{PT}}(\kappa)=i\frac{[\Gamma(1-is\kappa)]^2}{s\kappa\Gamma\left(\frac{1}{2}+\omega-is\kappa\right)\Gamma\left(\frac{1}{2}-\omega-is\kappa\right)},\quad b^{\text{PT}}(\kappa)=-i\frac{\cos\pi\omega}{\sinh\pi s\kappa}, \tag{C2}$$

where $\omega=\frac{1}{2}\sqrt{1-2sv_0}$; it is real if the barrier is narrow.

The singularities of the transmission amplitude are simple poles at $s\kappa=-i(n+\frac{1}{2}\pm\omega)$, $n=0,1,\ldots$ . The phase-space propagator is given by an infinite sum of the residues. For large time $t$ the result is

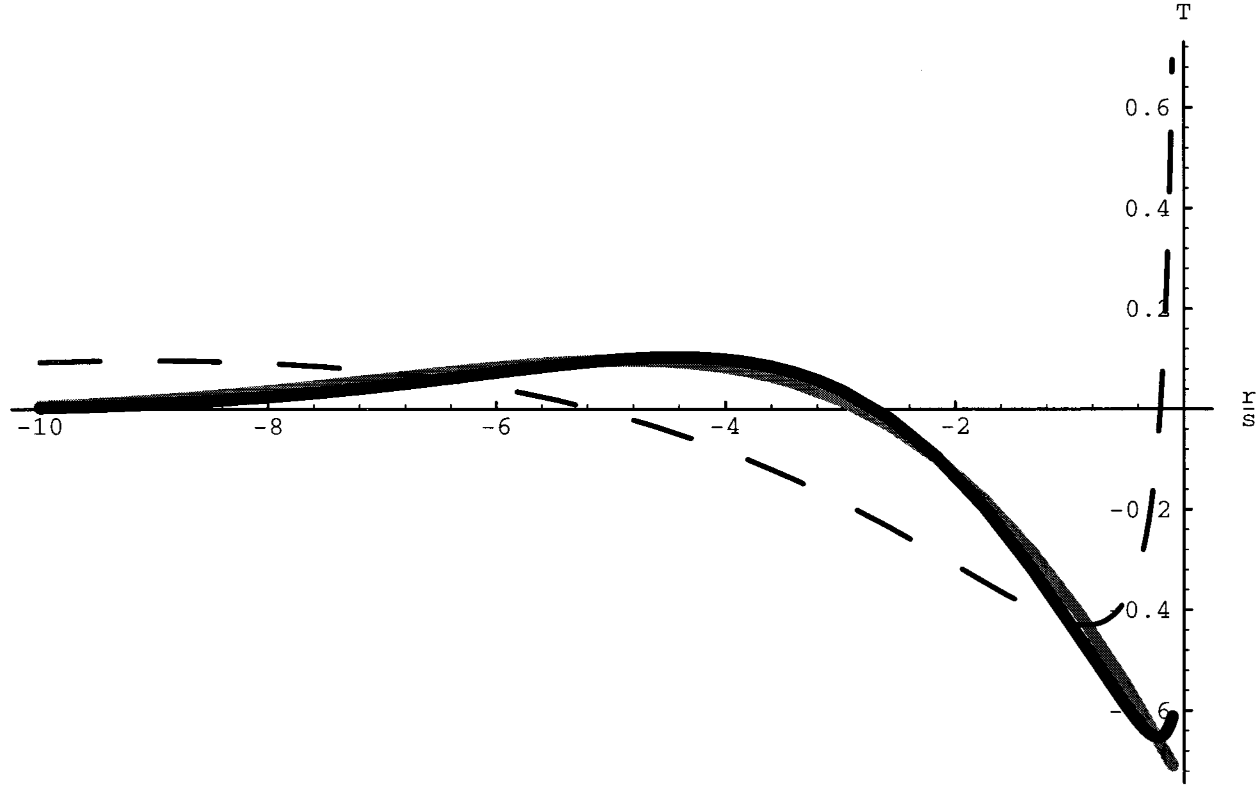


FIG. 1. The transmission propagator $\mathcal{T}$ as a function of the (finite and positive) lag distance, $-r/s \equiv (q-q_0-tp_0/m)/s$: (i) the solid line: exact propagator for the PT barrier, Eq. (55) (for $sp_0=0.2$ and $sv_0=0.375$); (ii) the gray line: the propagator for the $\delta$ barrier, Eq. (31) (for $v_0/p=1.875$); (iii) the dashed line: the semiclassical approximation of Eq. (45). *Note:* The $\delta$ term present at the front in the exact propagators, Eq. (30), is not shown in curves (i) and (ii).

$$
\begin{aligned}
L_t^{\rm PT}(q,p;q_0,p_0) &\asymp \delta(p-p_0)\mathcal{T}^{\rm PT}(r_+,p_0)+\delta(p+p_0)\mathcal{R}^{\rm PT}(r_-,p_0),\\
\mathcal{T}^{\rm PT}(r_+,p_0) &= \delta(r_+)+\theta(r_+)[\mathcal{F}_t(\nu,\omega)+\mathcal{F}_t(-\nu,\omega)+\mathcal{F}_t(\nu,-\omega)+\mathcal{F}_t(-\nu,-\omega)],\\
\mathcal{R}^{\rm PT}(r_-,p_0) &= \theta(r_-)[\mathcal{F}_r(\nu,\omega)+\mathcal{F}_r(-\nu,\omega)+\mathcal{F}_r(\nu,-\omega)+\mathcal{F}_r(-\nu,-\omega)]+\theta(-r_-)[\mathcal{F}_s(\nu,\omega)+\mathcal{F}_s(-\nu,-\omega)],
\end{aligned}
\tag{C3}
$$

where $\nu \equiv 2p_0 s$ and

$$
\begin{aligned}
\mathcal{F}_t(\nu,\omega) &= \frac{2\Gamma(2\omega)\Gamma(i\nu)\Gamma(i\nu+2\omega)}{s\Gamma(-\frac{1}{2}+\omega)\Gamma(\frac{1}{2}+\omega)\Gamma(-\frac{1}{2}+i\nu+\omega)\Gamma(\frac{1}{2}+i\nu+\omega)}\exp\left((i\nu+2\omega-1)\frac{r_+}{s}\right)\\
&\quad\times {}_4F_3\left(\begin{matrix}\frac{1}{2}-\omega,\frac{3}{2}-\omega,\frac{3}{2}-i\nu-\omega,\frac{1}{2}-i\nu-\omega\\ 1-2\omega,1-i\nu,1-i\nu-2\omega\end{matrix};\exp\left(-2\frac{r_+}{s}\right)\right),\\
\mathcal{F}_r(\nu,\omega) &= \frac{-2\Gamma(2\omega)\Gamma(i\nu)\Gamma(i\nu+2\omega)\Gamma(\frac{1}{2}-i\nu-\omega)}{s[\Gamma(\frac{1}{2}+\omega)]^2\Gamma(\frac{1}{2}-\omega)\Gamma(-\frac{1}{2}+\omega)\Gamma(-\frac{1}{2}+i\nu+\omega)}\\
&\quad\times\exp\left((i\nu+2\omega-1)\frac{r_-}{s}\right){}_4F_3\left(\begin{matrix}\frac{3}{2}-\omega,\frac{1}{2}-\omega,\frac{3}{2}-i\nu-\omega,\frac{1}{2}-i\nu-\omega\\ 1-2\omega,1-i\nu,1-i\nu-2\omega\end{matrix};\exp\left(-2\frac{r_-}{s}\right)\right),\\
\mathcal{F}_s(\nu,\omega) &= 2v_0^2 s\frac{\Gamma(\frac{3}{2}+i\nu+\omega)\Gamma(\frac{3}{2}+i\nu-\omega)\Gamma(-1-i\nu)}{\Gamma(\frac{1}{2}+\omega)\Gamma(\frac{1}{2}-\omega)\Gamma(1+i\nu)}\\
&\quad\times\exp\left((i\nu+2)\frac{r_-}{s}\right){}_4F_3\left(\begin{matrix}\frac{3}{2}+\omega,\frac{3}{2}-\omega,\frac{3}{2}+i\nu+\omega,\frac{1}{2}+i\nu-\omega\\ 2,1+i\nu,2+i\nu\end{matrix};\exp\left(\frac{2r_-}{s}\right)\right).
\end{aligned}
\tag{C4}
$$

Here ${}_4F_3$ are the generalized hypergeometric functions [75],

$$ {}_4F_3\left(\begin{matrix}\xi_1,\xi_2,\xi_3,\xi_4\\ \lambda_1,\lambda_2,\lambda_3\end{matrix};\zeta\right)=\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}\frac{(\xi_1)_n(\xi_2)_n(\xi_3)_n(\xi_4)_n}{(\lambda_1)_n(\lambda_2)_n(\lambda_3)_n}, \tag{C5}$$

and $(\xi)_n\equiv\Gamma(\xi+n)/\Gamma(\xi)$. The series are convergent, as long as $|\zeta|<1$.

Two complementary approximations were considered in Secs. V and VI. In the limit of a narrow barrier $s\to 0$ while $v_0=\text{const}$, and the $\delta$-function barrier disscused in Sec. V is reproduced. In this limit the exact propagator, given by the series in Eqs. (C4) reduces to the $\delta$ propagator of Eq. (33). On the other hand, in the domains where the inequalities (43) hold the deep-tunneling semiclassical approximation of Eq. (45) can be applied. In Fig. 1 we present an example for a transmission propagator in the deep-tunneling regime. Namely, a PT barrier with $sp_0=0.2$ and $sv_0=0.375$. the exact propagator is calculated and compared to the two approximations. As long as $r$ is not too small the propagator for the $\delta$ barrier is an excellent approximation. For $r=0$ the exact propagators are singular. This singularity is smeared by the approximation of Eq. (45).

For small momenta, $\kappa\ll\kappa_0\equiv\sqrt{v_0/2s}$, the WKB semiclassical approximation for the amplitude gives

$$ a_{\rm sc}^{\rm PT}(\kappa)\approx\exp\left[-\pi s(\kappa_0-\kappa)+is\kappa\ln\left(\frac{\kappa_0^2}{\kappa^2}-1\right)+is\kappa_0\ln\frac{\kappa_0+\kappa}{\kappa_0-\kappa}\right]. \tag{C6}$$

An Airy function is obtained by substituting this expression in the integral for $\mathcal{T}$ and expanding the exponent in the integrand to a power series in $\sigma$,

$$ \mathcal{T}_{\rm sc}^{\rm PT}\approx e^{-2\pi s(\kappa_0-p_0)}(3\varphi_3)^{-1/3}\mathrm{Ai}[(3\varphi_3)^{-1/3}(r_+-\varphi_1)],$$

$$ \varphi_1=-s\ln\left(\frac{\kappa_0^2}{p_0^2}-1\right),\quad \varphi_3=\frac{s\kappa_0^2(\kappa_0^2-3p_0^2)}{12p_0^2(\kappa_0^2-p_0^2)^2}. \tag{C7}$$

As $s\to\infty$, the Airy function approaches the $\delta$ function,

$$ \mathcal{T}_{\rm sc}^{\rm PT}\sim e^{-2\pi s(v-p_0)}\delta\left[q_0+2p_0t+s\ln\left(\frac{\kappa_0^2}{p_0^2}-1\right)-q\right]. \tag{C8}$$

The result looks like an advance in time, violating causality. This is an artifact of a wrong semiclassical approximation.


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